# Angular Momentum

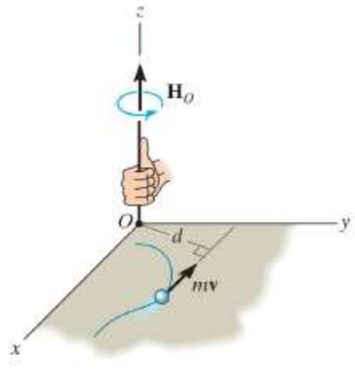



Fig. 15–19

The *angular momentum* of a particle about point $O$ is defined as the "moment" of the particle's linear momentum about $O$. Since this concept is analogous to finding the moment of a force about a point, the angular momentum, $\mathbf{H}_O$, is sometimes referred to as the *moment of momentum.*

**Scalar Formulation.** If a particle moves along a curve lying in the $x$–$y$ plane, Fig. 15–19, the angular momentum at any instant can be determined about point $O$ (actually the $z$ axis) by using a scalar formulation. The *magnitude* of $\mathbf{H}_O$ is

$$(H_O)_z = (d)(mv) \qquad (15\text{–}12)$$

Here $d$ is the moment arm or perpendicular distance from $O$ to the line of action of $m\mathbf{v}$. Common units for $(H_O)_z$ are $\text{kg}\cdot\text{m}^2/\text{s}$ or $\text{slug}\cdot\text{ft}^2/\text{s}$. The *direction* of $\mathbf{H}_O$ is defined by the right-hand rule. As shown, the curl of the fingers of the right hand indicates the sense of rotation of $m\mathbf{v}$ about $O$, so that in this case the thumb (or $\mathbf{H}_O$) is directed perpendicular to the $x$–$y$ plane along the $+z$ axis.

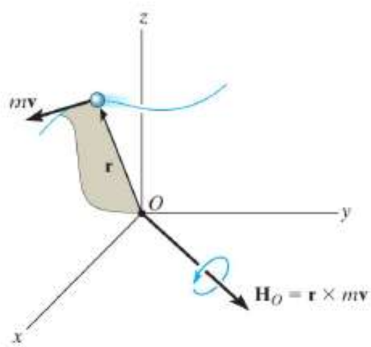



Fig. 15–20

**Vector Formulation.** If the particle moves along a space curve, Fig. 15–20, the vector cross product can be used to determine the *angular momentum* about $O$. In this case

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \qquad (15\text{–}13)$$

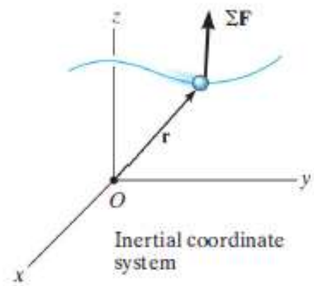
Here $\mathbf{r}$ denotes a position vector drawn from point $O$ to the particle. As shown in the figure, $\mathbf{H}_O$ is *perpendicular* to the shaded plane containing $\mathbf{r}$ and $m\mathbf{v}$.

In order to evaluate the cross product, $\mathbf{r}$ and $m\mathbf{v}$ should be expressed in terms of their Cartesian components, so that the angular momentum can be determined by evaluating the determinant:

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix} \qquad (15\text{–}14)$$

The moments about point $O$ of all the forces acting on the particle in Fig. 15–21$a$ can be related to the particle's angular momentum by applying the equation of motion. If the mass of the particle is constant, we may write

$$\Sigma \mathbf{F} = m\dot{\mathbf{v}}$$



(a)

**Fig. 15–21**

The moments of the forces about point $O$ can be obtained by performing a cross-product multiplication of each side of this equation by the position vector $\mathbf{r}$, which is measured from the $x$, $y$, $z$ inertial frame of reference. We have

$$\Sigma \mathbf{M}_O = \mathbf{r} \times \Sigma \mathbf{F} = \mathbf{r} \times m\dot{\mathbf{v}}$$

From Appendix B, the derivative of $\mathbf{r} \times m\mathbf{v}$ can be written as

$$\dot{\mathbf{H}}_O = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}}$$

The first term on the right side, $\dot{\mathbf{r}} \times m\mathbf{v} = m(\dot{\mathbf{r}} \times \dot{\mathbf{r}}) = \mathbf{0}$, since the cross product of a vector with itself is zero. Hence, the above equation becomes

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \qquad (15\text{–}15)$$

which states that *the resultant moment about point O of all the forces acting on the particle is equal to the time rate of change of the particle's angular momentum about point O.* This result is similar to Eq. 15–1, i.e.,

$$\Sigma \mathbf{F} = \dot{\mathbf{L}} \qquad (15\text{–}16)$$

Here $\mathbf{L} = m\mathbf{v}$, so that *the resultant force acting on the particle is equal to the time rate of change of the particle's linear momentum.*

From the derivations, it is seen that Eqs. 15–15 and 15–16 are actually another way of stating Newton's second law of motion. In other sections of this book it will be shown that these equations have many practical applications when extended and applied to problems involving either a system of particles or a rigid body.

## *Example*

The box shown in Fig. 15–22*a* has a mass $m$ and travels down the smooth circular ramp such that when it is at the angle $\theta$ it has a speed $v$. Determine its angular momentum about point $O$ at this instant and the rate of increase in its speed, i.e., $a_t$.


**Fig. 15–22**

### SOLUTION

Since **v** is tangent to the path, applying Eq. 15–12 the angular momentum is

$$H_O = rmv \circlearrowright \qquad \textit{Ans.}$$

The rate of increase in its speed $(dv/dt)$ can be found by applying Eq. 15–15. From the free-body diagram of the box, Fig. 15–22*b*, it can be seen that only the weight $W = mg$ contributes a moment about point $O$. We have

$$\circlearrowleft + \Sigma M_O = \dot{H}_O; \qquad mg(r \sin\theta) = \frac{d}{dt}(rmv)$$

Since $r$ and $m$ are constant,

$$mgr \sin\theta = rm\frac{dv}{dt}$$

$$\frac{dv}{dt} = g \sin\theta \qquad \textit{Ans.}$$

**NOTE:** This same result can, of course, be obtained from the equation of motion applied in the tangential direction, Fig. 15–22*b*, i.e.,

$$+\swarrow \Sigma F_t = ma_t; \qquad mg \sin\theta = m\left(\frac{dv}{dt}\right)$$

$$\frac{dv}{dt} = g \sin\theta \qquad \textit{Ans.}$$