Curvilinear Motion: Normal and Tangential Components

When the path along which a particle travels is *known*, then it is often convenient to describe the motion using n and t coordinate axes which act normal and tangent to the path, respectively, and at the instant considered have their *origin located at the particle*.

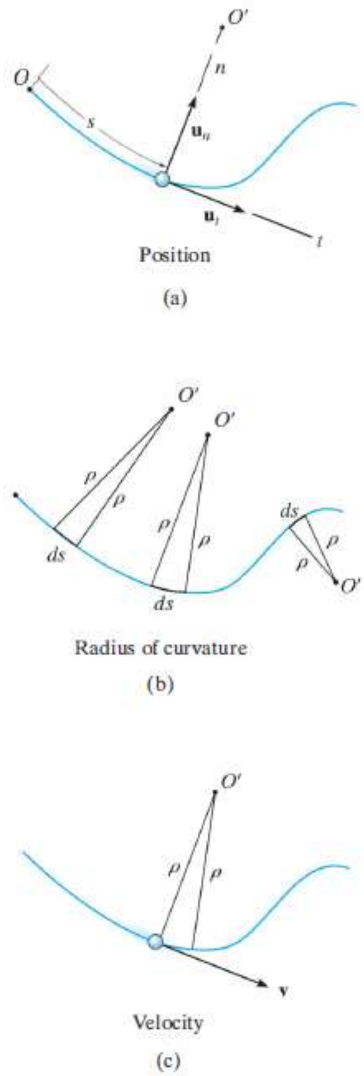


Fig. 12–24

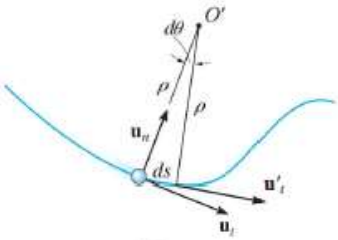
Planar Motion. Consider the particle shown in Fig. 12–24*a*, which moves in a plane along a fixed curve, such that at a given instant it is at position s, measured from point O. We will now consider a coordinate system that has its origin at a *fixed point* on the curve, and at the instant considered this origin happens to *coincide* with the location of the particle. The t axis is *tangent* to the curve at the point and is positive in the direction of *increasing* s. We will designate this positive direction with the unit vector $\mathbf{u}_t$. A unique choice for the *normal axis* can be made by noting that geometrically the curve is constructed from a series of differential arc segments ds, Fig. 12–24*b*. Each segment ds is formed from the arc of an associated circle having a *radius of curvature* ρ (rho) and *center of curvature* O'. The normal axis n is perpendicular to the t axis with its positive sense directed *toward* the center of curvature O', Fig. 12–24*a*. This positive direction, which is *always* on the concave side of the curve, will be designated by the unit vector $\mathbf{u}_n$. The plane which contains the n and t axes is referred to as the embracing or *osculating plane*, and in this case it is fixed in the plane of motion.*

Velocity. Since the particle moves, s is a function of time. As indicated in Sec. 12.4, the particle's velocity $\mathbf{v}$ has a *direction* that is *always tangent to the path*, Fig. 12–24*c*, and a *magnitude* that is determined by taking the time derivative of the path function $s = s(t)$, i.e., $v = ds/dt$ (Eq. 12–8). Hence

$$\mathbf{v} = v\mathbf{u}_t \qquad (12\text{–}15)$$

where

$$v = \dot{s} \qquad (12\text{–}16)$$



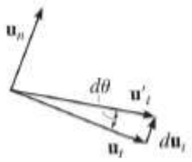
(d)

Acceleration. The acceleration of the particle is the time rate of change of the velocity. Thus,

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{v}\mathbf{u}_t + v\dot{\mathbf{u}}_t \qquad (12\text{–}17)$$

In order to determine the time derivative $\dot{\mathbf{u}}_t$, note that as the particle moves along the arc ds in time dt, $\mathbf{u}_t$ preserves its magnitude of unity; however, its *direction* changes, and becomes $\mathbf{u}'_t$, Fig. 12–24*d*. As shown in Fig. 12–24*e*, we require $\mathbf{u}'_t = \mathbf{u}_t + d\mathbf{u}_t$. Here $d\mathbf{u}_t$ stretches between the arrowheads of $\mathbf{u}_t$ and $\mathbf{u}'_t$, which lie on an infinitesimal arc of radius $u_t = 1$. Hence, $d\mathbf{u}_t$ has a *magnitude* of $du_t = (1)\,d\theta$, and its *direction* is defined by $\mathbf{u}_n$. Consequently, $d\mathbf{u}_t = d\theta\mathbf{u}_n$, and therefore the time derivative becomes $\dot{\mathbf{u}}_t = \dot{\theta}\mathbf{u}_n$. Since $ds = \rho d\theta$, Fig. 12–24*d*, then $\dot{\theta} = \dot{s}/\rho$, and therefore

$$\dot{\mathbf{u}}_t = \dot{\theta}\mathbf{u}_n = \frac{\dot{s}}{\rho}\mathbf{u}_n = \frac{v}{\rho}\mathbf{u}_n$$



(e)

Substituting into Eq. 12–17, $\mathbf{a}$ can be written as the sum of its two components,

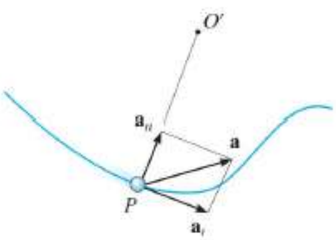
$$\mathbf{a} = a_t\mathbf{u}_t + a_n\mathbf{u}_n \qquad (12\text{–}18)$$

where

$$a_t = \dot{v} \quad \text{or} \quad a_t ds = v\,dv \qquad (12\text{–}19)$$

and

$$a_n = \frac{v^2}{\rho} \qquad (12\text{–}20)$$



Acceleration

(f)

Fig. 12–24 (cont.)

These two mutually perpendicular components are shown in Fig. 12–24*f*. Therefore, the *magnitude* of acceleration is the positive value of

$$a = \sqrt{a_t^2 + a_n^2} \qquad (12\text{–}21)$$

Example

When the skier reaches point A along the parabolic path in Fig. 12–27a, he has a speed of 6 m/s which is increasing at 2 m/s^2. Determine the direction of his velocity and the direction and magnitude of his acceleration at this instant. Neglect the size of the skier in the calculation.

SOLUTION

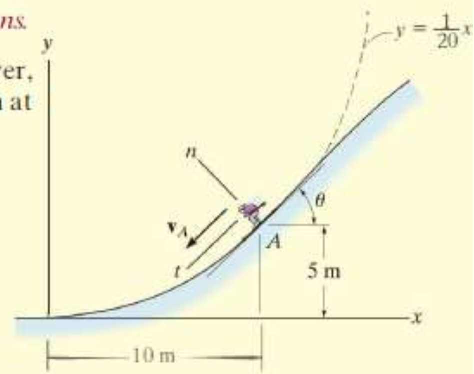
Coordinate System. Although the path has been expressed in terms of its x and y coordinates, we can still establish the origin of the n, t axes at the fixed point A on the path and determine the components of $\mathbf{v}$ and $\mathbf{a}$ along these axes, Fig. 12–27a.

Velocity. By definition, the velocity is always directed tangent to the path. Since $y = \frac{1}{20}x^2$, $dy/dx = \frac{1}{10}x$, then at $x = 10$ m, $dy/dx = 1$. Hence, at A, $\mathbf{v}$ makes an angle of $\theta = \tan^{-1}1 = 45^\circ$ with the x axis, Fig. 12–27a. Therefore,

$$v_A = 6 \text{ m/s} \qquad 45^\circ ⦩ \qquad \textit{Ans.}$$

The acceleration is determined from $\mathbf{a} = \dot{v}\mathbf{u}_t + (v^2/\rho)\mathbf{u}_n$. However, it is first necessary to determine the radius of curvature of the path at A (10 m, 5 m). Since $d^2y/dx^2 = \frac{1}{10}$, then

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} = \left.\frac{\left[1 + \left(\frac{1}{10}x\right)^2\right]^{3/2}}{\left|\frac{1}{10}\right|}\right|_{x=10\text{ m}} = 28.28 \text{ m}$$



(a)

The acceleration becomes

$$\begin{aligned} \mathbf{a}_A &= \dot{v}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n \\ &= 2\mathbf{u}_t + \frac{(6 \text{ m/s})^2}{28.28 \text{ m}}\mathbf{u}_n \\ &= \{2\mathbf{u}_t + 1.273\mathbf{u}_n\} \text{m/s}^2 \end{aligned}$$

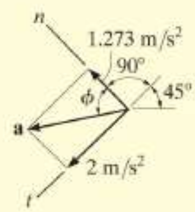
As shown in Fig. 12–27b,

$$a = \sqrt{(2 \text{ m/s}^2)^2 + (1.273 \text{ m/s}^2)^2} = 2.37 \text{ m/s}^2$$

$$\phi = \tan^{-1}\frac{2}{1.273} = 57.5^\circ$$

Thus, $45^\circ + 90^\circ + 57.5^\circ - 180^\circ = 12.5^\circ$ so that,

$$a = 2.37 \text{ m/s}^2 \qquad 12.5^\circ ⦩ \qquad \textit{Ans.}$$



(b)