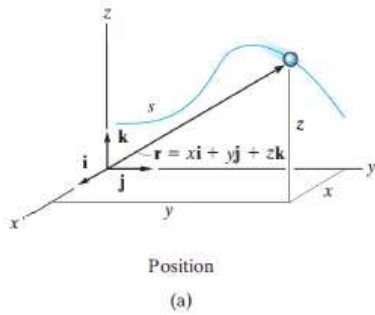


Curvilinear Motion: Rectangular Components

Occasionally the motion of a particle can best be described along a path that can be expressed in terms of its x, y, z coordinates.



Position. If the particle is at point (x, y, z) on the curved path s shown in Fig. 12-17a, then its location is defined by the *position vector*

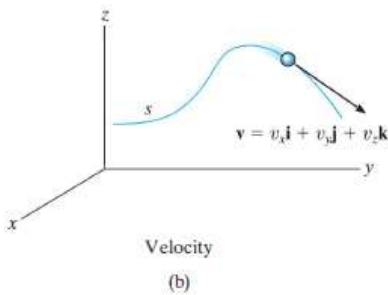
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (12-10)$$

When the particle moves, the x, y, z components of \mathbf{r} will be functions of time; i.e., $x = x(t), y = y(t), z = z(t)$, so that $\mathbf{r} = \mathbf{r}(t)$.

At any instant the *magnitude* of \mathbf{r} is defined from Eq. C-3 in Appendix C as

$$r = \sqrt{x^2 + y^2 + z^2}$$

And the *direction* of \mathbf{r} is specified by the unit vector $\mathbf{u}_r = \mathbf{r}/r$.



Velocity. The first time derivative of \mathbf{r} yields the velocity of the particle. Hence,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k})$$

When taking this derivative, it is necessary to account for changes in *both* the magnitude and direction of each of the vector's components. For example, the derivative of the \mathbf{i} component of \mathbf{r} is

$$\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt}$$

The second term on the right side is zero, provided the x, y, z reference frame is *fixed*, and therefore the *direction* (and the *magnitude*) of \mathbf{i} does not change with time. Differentiation of the \mathbf{j} and \mathbf{k} components may be carried out in a similar manner, which yields the final result,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \quad (12-11)$$

where

$$v_x = \dot{x} \quad v_y = \dot{y} \quad v_z = \dot{z} \quad (12-12)$$

Fig. 12-17

The “dot” notation \dot{x} , \dot{y} , \dot{z} represents the first time derivatives of $x = x(t)$, $y = y(t)$, $z = z(t)$, respectively.

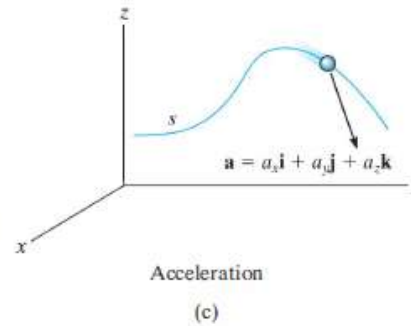
The velocity has a *magnitude* that is found from

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

and a *direction* that is specified by the unit vector $\mathbf{u}_v = \mathbf{v}/v$. As discussed in Sec. 12-4, this direction is *always tangent to the path*, as shown in Fig. 12-17b.

Acceleration. The acceleration of the particle is obtained by taking the first time derivative of Eq. 12-11 (or the second time derivative of Eq. 12-10). We have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \quad (12-13)$$



where

$$\begin{aligned} a_x &= \dot{v}_x = \ddot{x} \\ a_y &= \dot{v}_y = \ddot{y} \\ a_z &= \dot{v}_z = \ddot{z} \end{aligned} \quad (12-14)$$

Here a_x , a_y , a_z represent, respectively, the first time derivatives of $v_x = v_x(t)$, $v_y = v_y(t)$, $v_z = v_z(t)$, or the second time derivatives of the functions $x = x(t)$, $y = y(t)$, $z = z(t)$.

The acceleration has a *magnitude*

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

and a *direction* specified by the unit vector $\mathbf{u}_a = \mathbf{a}/a$. Since \mathbf{a} represents the time rate of *change* in both the magnitude and direction of the velocity, in general \mathbf{a} will *not* be tangent to the path, Fig. 12-17c.

Example

At any instant the horizontal position of the weather balloon in Fig. 12–18a is defined by $x = (8t)$ ft, where t is in seconds. If the equation of the path is $y = x^2/10$, determine the magnitude and direction of the velocity and the acceleration when $t = 2$ s.

SOLUTION

Velocity. The velocity component in the x direction is

$$v_x = \dot{x} = \frac{d}{dt}(8t) = 8 \text{ ft/s} \rightarrow$$

To find the relationship between the velocity components we will use the chain rule of calculus. (See Appendix A for a full explanation.)

$$v_y = \dot{y} = \frac{d}{dt}(x^2/10) = 2x\dot{x}/10 = 2(16)(8)/10 = 25.6 \text{ ft/s} \uparrow$$

When $t = 2$ s, the magnitude of velocity is therefore

$$v = \sqrt{(8 \text{ ft/s})^2 + (25.6 \text{ ft/s})^2} = 26.8 \text{ ft/s} \quad \text{Ans.}$$

The direction is tangent to the path, Fig. 12–18b, where

$$\theta_v = \tan^{-1} \frac{v_y}{v_x} = \tan^{-1} \frac{25.6}{8} = 72.6^\circ \quad \text{Ans.}$$

Acceleration. The relationship between the acceleration components is determined using the chain rule. (See Appendix C.) We have

$$a_x = \dot{v}_x = \frac{d}{dt}(8) = 0$$

$$\begin{aligned} a_y = \dot{v}_y &= \frac{d}{dt}(2x\dot{x}/10) = 2(\dot{x})\dot{x}/10 + 2x(\ddot{x})/10 \\ &= 2(8)^2/10 + 2(16)(0)/10 = 12.8 \text{ ft/s}^2 \uparrow \end{aligned}$$

Thus,

$$a = \sqrt{(0)^2 + (12.8)^2} = 12.8 \text{ ft/s}^2 \quad \text{Ans.}$$

The direction of \mathbf{a} , as shown in Fig. 12–18c, is

$$\theta_a = \tan^{-1} \frac{12.8}{0} = 90^\circ \quad \text{Ans.}$$

NOTE: It is also possible to obtain v_y and a_y by first expressing $y = f(t) = (8t)^2/10 = 6.4t^2$ and then taking successive time derivatives.

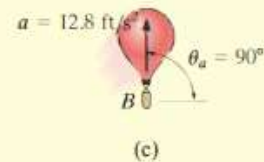
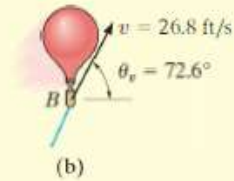
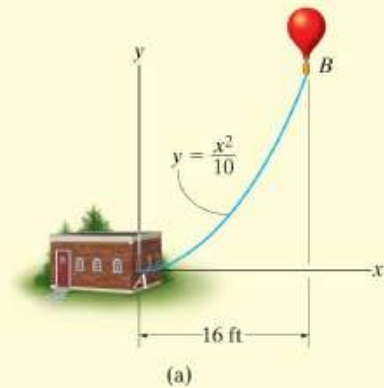


Fig. 12–18