Principle of Angular Impulse and Momentum

Principle of Angular Impulse and Momentum. If Eq. 15–15 is rewritten in the form $\Sigma \mathbf{M}_O\, dt = d\mathbf{H}_O$ and integrated, assuming that at time $t = t_1$, $\mathbf{H}_O = (\mathbf{H}_O)_1$ and at time $t = t_2$, $\mathbf{H}_O = (\mathbf{H}_O)_2$, we have

$$\Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

or

$$(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 \qquad (15\text{–}18)$$

This equation is referred to as the *principle of angular impulse and momentum*. The initial and final angular momenta $(\mathbf{H}_O)_1$ and $(\mathbf{H}_O)_2$ are defined as the moment of the linear momentum of the particle ($\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$) at the instants t_1 and t_2, respectively. The second term on the left side, $\Sigma \int \mathbf{M}_O\, dt$, is called the *angular impulse*. It is determined by integrating, with respect to time, the moments of all the forces acting on the particle over the time period t_1 to t_2. Since the moment of a force about point O is $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$, the angular impulse may be expressed in vector form as

$$\text{angular impulse} = \int_{t_1}^{t_2} \mathbf{M}_O\, dt = \int_{t_1}^{t_2} (\mathbf{r} \times \mathbf{F})\, dt \qquad (15\text{–}19)$$

Here $\mathbf{r}$ is a position vector which extends from point O to any point on the line of action of $\mathbf{F}$.

In a similar manner, using Eq. 15–18, the principle of angular impulse and momentum for a system of particles may be written as

$$\Sigma(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = \Sigma(\mathbf{H}_O)_2 \qquad (15\text{–}20)$$

Here the first and third terms represent the angular momenta of all the particles $[\Sigma \mathbf{H}_O = \Sigma(\mathbf{r}_i \times m\mathbf{v}_i)]$ at the instants t_1 and t_2. The second term is the sum of the angular impulses given to all the particles from t_1 to t_2. Recall that these impulses are created only by the moments of the external forces acting on the system where, for the ith particle, $\mathbf{M}_O = \mathbf{r}_i \times \mathbf{F}_i$.

Vector Formulation. Using impulse and momentum principles, it is therefore possible to write two equations which define the particle's motion, namely, Eqs. 15–3 and Eqs. 15–18, restated as

$$\begin{aligned} m\mathbf{v}_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt &= m\mathbf{v}_2 \\ (\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt &= (\mathbf{H}_O)_2 \end{aligned} \qquad (15\text{–}21)$$

Scalar Formulation. In general, the above equations can be expressed in x, y, z component form, yielding a total of six scalar equations. If the particle is confined to move in the x–y plane, three scalar equations can be written to express the motion, namely,

$$\begin{aligned} m(v_x)_1 + \Sigma \int_{t_1}^{t_2} F_x\, dt &= m(v_x)_2 \\ m(v_y)_1 + \Sigma \int_{t_1}^{t_2} F_y\, dt &= m(v_y)_2 \\ (H_O)_1 + \Sigma \int_{t_1}^{t_2} M_O\, dt &= (H_O)_2 \end{aligned} \qquad (15\text{–}22)$$

The first two of these equations represent the principle of linear impulse and momentum in the x and y directions, which has been discussed in Sec. 15.1, and the third equation represents the principle of angular impulse and momentum about the z axis.

Example

The 2-kg disk shown in Fig. 15–26*a* rests on a smooth horizontal surface and is attached to an elastic cord that has a stiffness $k_c = 20\ \text{N/m}$ and is initially unstretched. If the disk is given a velocity $(v_D)_1 = 1.5\ \text{m/s}$, perpendicular to the cord, determine the rate at which the cord is being stretched and the speed of the disk at the instant the cord is stretched 0.2 m.

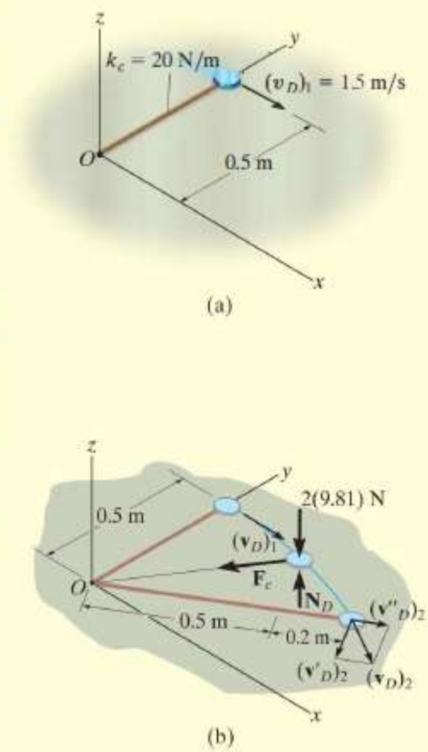

Fig. 15–26

SOLUTION

Free-Body Diagram. After the disk has been launched, it slides along the path shown in Fig. 15–26*b*. By inspection, angular momentum about point *O* (or the *z* axis) is *conserved*, since none of the forces produce an angular impulse about this axis. Also, when the distance is 0.7 m, only the transverse component $(\mathbf{v}'_D)_2$ produces angular momentum of the disk about *O*.

Conservation of Angular Momentum. The component $(\mathbf{v}'_D)_2$ can be obtained by applying the conservation of angular momentum about *O* (the *z* axis).

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

$$r_1 m_D (v_D)_1 = r_2 m_D (v'_D)_2$$

$$0.5\ \text{m}\ (2\ \text{kg})(1.5\ \text{m/s}) = 0.7\ \text{m}(2\ \text{kg})(v'_D)_2$$

$$(v'_D)_2 = 1.071\ \text{m/s}$$

Conservation of Energy. The speed of the disk can be obtained by applying the conservation of energy equation at the point where the disk was launched and at the point where the cord is stretched 0.2 m.

$$T_1 + V_1 = T_2 + V_2$$

$$\tfrac{1}{2} m_D (v_D)_1^2 + \tfrac{1}{2} k x_1^2 = \tfrac{1}{2} m_D (v_D)_2^2 + \tfrac{1}{2} k x_2^2$$

$$\tfrac{1}{2}(2\ \text{kg})(1.5\ \text{m/s})^2 + 0 = \tfrac{1}{2}(2\ \text{kg})(v_D)_2^2 + \tfrac{1}{2}(20\ \text{N/m})(0.2\ \text{m})^2$$

$$(v_D)_2 = 1.360\ \text{m/s} = 1.36\ \text{m/s} \qquad \textit{Ans.}$$

Having determined $(v_D)_2$ and its component $(v'_D)_2$, the rate of stretch of the cord, or radial component, $(v''_D)_2$ is determined from the Pythagorean theorem,

$$(v''_D)_2 = \sqrt{(v_D)_2^2 - (v'_D)_2^2}$$

$$= \sqrt{(1.360\ \text{m/s})^2 - (1.071\ \text{m/s})^2}$$

$$= 0.838\ \text{m/s} \qquad \textit{Ans.}$$