Relative-Motion Analysis: Velocity

The general plane motion of a rigid body can be described as a *combination* of translation and rotation. To view these "component" motions *separately* we will use a *relative-motion analysis* involving two sets of coordinate axes. The x, y coordinate system is fixed and measures the *absolute* position of two points A and B on the body, here represented as a bar, Fig. 16–10a. The origin of the x', y' coordinate system will be attached to the selected "base point" A, which generally has a *known* motion. The axes of this coordinate system *translate* with respect to the fixed frame but do not rotate with the bar.

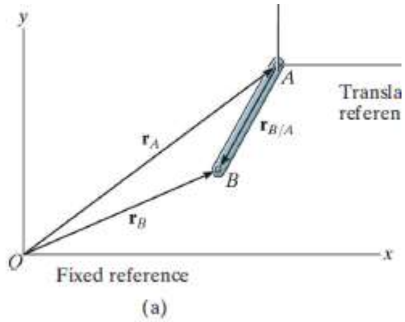


Fig. 16–10

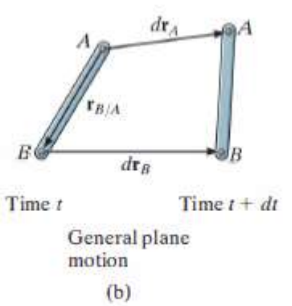
Position The position vector $\mathbf{r}_A$ in Fig. 16–10a specifies the location of the "base point" A, and the relative-position vector $\mathbf{r}_{B/A}$ locates point B with respect to point A. By vector addition, the *position* of B is then

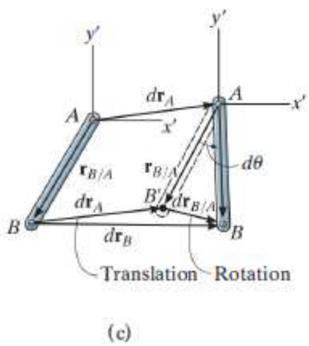
$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Displacement During an instant of time dt, points A and B undergo displacements $d\mathbf{r}_A$ and $d\mathbf{r}_B$ as shown in Fig. 16–10b. If we consider the general plane motion by its component parts then the *entire bar* first *translates* by an amount $d\mathbf{r}_A$ so that A, the base point, moves to its *final position* and point B moves to B', Fig. 16–10c. The bar is then *rotated* about A by an amount $d\theta$ so that B' undergoes a *relative displacement* $d\mathbf{r}_{B/A}$ and thus moves to its final position B. Due to the rotation about A, $dr_{B/A} = r_{B/A}\,d\theta$, and the displacement of B is

$$d\mathbf{r}_B = d\mathbf{r}_A + d\mathbf{r}_{B/A}$$

- $d\mathbf{r}_{B/A}$: due to rotation about A
- $d\mathbf{r}_A$: due to translation of A
- $d\mathbf{r}_B$: due to translation and rotation





Velocity To determine the relation between the velocities of points A and B, it is necessary to take the time derivative of the position equation, or simply divide the displacement equation by dt. This yields

$$\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt}$$

The terms $d\mathbf{r}_B/dt = \mathbf{v}_B$ and $d\mathbf{r}_A/dt = \mathbf{v}_A$ are measured with respect to the fixed x, y axes and represent the *absolute velocities* of points A and B, respectively. Since the relative displacement is caused by a rotation, the magnitude of the third term is $dr_{B/A}/dt = r_{B/A}\, d\theta/dt = r_{B/A}\dot{\theta} = r_{B/A}\omega$, where ω is the angular velocity of the body at the instant considered. We will denote this term as the *relative velocity* $\mathbf{v}_{B/A}$, since it represents the velocity of B with respect to A as measured by an observer fixed to the translating x', y' axes. In other words, *the bar appears to move as if it were rotating with an angular velocity* $\boldsymbol{\omega}$ *about the* z' *axis passing through* A. Consequently, $\mathbf{v}_{B/A}$ has a magnitude of $v_{B/A} = \omega r_{B/A}$ and a *direction* which is perpendicular to $\mathbf{r}_{B/A}$. We therefore have

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad (16\text{–}15)$$

where

$\mathbf{v}_B =$ velocity of point B
$\mathbf{v}_A =$ velocity of the base point A
$\mathbf{v}_{B/A} =$ velocity of B with respect to A

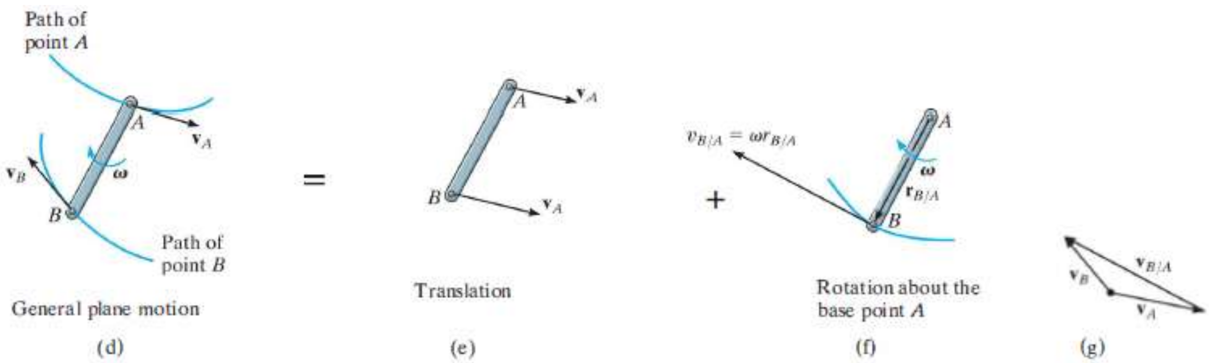


Fig. 16–10 (cont.)

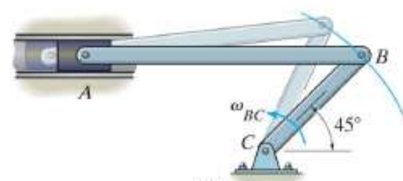
What this equation states is that the velocity of B, Fig. 16–10*d*, is determined by considering the entire bar to translate with a velocity of $\mathbf{v}_A$, Fig. 16–10*e*, and rotate about A with an angular velocity $\boldsymbol{\omega}$, Fig. 16–10*f*. Vector addition of these two effects, applied to B, yields $\mathbf{v}_B$, as shown in Fig. 16–10*g*.

Since the relative velocity $\mathbf{v}_{B/A}$ represents the effect of *circular motion*, about A, this term can be expressed by the cross product $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, Eq. 16–9. Hence, for application using Cartesian vector analysis, we can also write Eq. 16–15 as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \qquad (16\text{–}16)$$

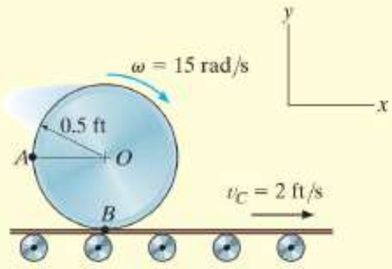
where

$\mathbf{v}_B$ = velocity of B
$\mathbf{v}_A$ = velocity of the base point A
$\boldsymbol{\omega}$ = angular velocity of the body
$\mathbf{r}_{B/A}$ = position vector directed from A to B

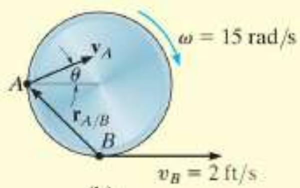


(a)

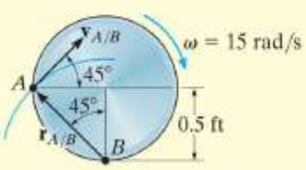
Example



(a)



(b)



Relative motion
(c)

Fig. 16–14

The cylinder shown in Fig. 16–14*a* rolls without slipping on the surface of a conveyor belt which is moving at 2 ft/s. Determine the velocity of point *A*. The cylinder has a clockwise angular velocity $\omega = 15$ rad/s at the instant shown.

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since no slipping occurs, point *B* on the cylinder has the same velocity as the conveyor, Fig. 16–14*b*. Also, the angular velocity of the cylinder is known, so we can apply the velocity equation to *B*, the base point, and *A* to determine $\mathbf{v}_A$.

Velocity Equation.

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 2\mathbf{i} + (-15\mathbf{k}) \times (-0.5\mathbf{i} + 0.5\mathbf{j})$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 2\mathbf{i} + 7.50\mathbf{j} + 7.50\mathbf{i}$$

so that

$$(v_A)_x = 2 + 7.50 = 9.50 \text{ ft/s} \qquad (1)$$

$$(v_A)_y = 7.50 \text{ ft/s} \qquad (2)$$

Thus,

$$v_A = \sqrt{(9.50)^2 + (7.50)^2} = 12.1 \text{ ft/s} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{7.50}{9.50} = 38.3^\circ \ \measuredangle \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

As an alternative procedure, the scalar components of $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ can be obtained directly. From the kinematic diagram showing the relative "circular" motion which produces $\mathbf{v}_{A/B}$, Fig. 16–14*c*, we have

$$v_{A/B} = \omega r_{A/B} = (15 \text{ rad/s})\left(\frac{0.5 \text{ ft}}{\cos 45^\circ}\right) = 10.6 \text{ ft/s}$$

Thus,

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\begin{bmatrix}(v_A)_x \\ \rightarrow\end{bmatrix} + \begin{bmatrix}(v_A)_y \\ \uparrow\end{bmatrix} = \begin{bmatrix}2 \text{ ft/s} \\ \rightarrow\end{bmatrix} + \begin{bmatrix}10.6 \text{ ft/s} \\ \measuredangle\ 45^\circ\end{bmatrix}$$

Equating the *x* and *y* components gives the same results as before, namely,

$$(\xrightarrow{+}) \qquad (v_A)_x = 2 + 10.6 \cos 45^\circ = 9.50 \text{ ft/s}$$

$$(+\uparrow) \qquad (v_A)_y = 0 + 10.6 \sin 45^\circ = 7.50 \text{ ft/s}$$