Chapter 7

Numerical Solution of Partial Differential Equations

Partial Differential Equations :

The Most commonly used PDE's in Engineering and Science are the second order P.D.E's, which have the following general Formulation :

$$A\, U_{xx} + B\, U_{xy} + C\, U_{yy} + D\, U_x + E\, U_y + F\, U + G = \phi \quad (1)$$

Where A, B, C, D, E, F and G are functions of x and y, and the subscript reffers to the partial derivatives:

(like $U_{xy} = \dfrac{\partial^2 U}{\partial x \partial y}$

The general formula (Eq. (1)) is classified into three catagories according to the following :

Index	classification
$B^2 - 4AC < 0$	Elliptic equation.
$B^2 - 4AC = 0$	Parabolic equation.
$B^2 - 4AC > 0$	Hyperbolic equation.

The partial differential Equations can also be classified physically into :

1- propogation problem.
2- Equilibrium problems
3- Eigenvalue problems .

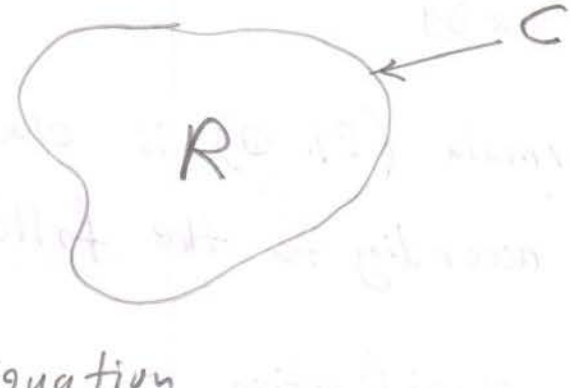
Note: The First order PDE's are allways hyparabolic.

(I) Elliplic Equation :

Poisson's Equation: $U_{xx} + U_{yy} = f(x,y)$

and Laplace Equation: $U_{xx} + U_{yy} = 0$

are both examples of Elliptic equation in two dimensions. The solution of these equation is the primitive function $u(x,y)$ which satisfys each point in the Range (R) enclosed by the boundary curve (C).

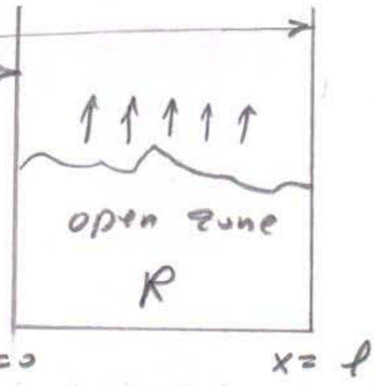


(II) Parabolic Equation :

The one-dimensional conduction equation,

$U_t = K^2 U_{xx}$, is one of the most famous examples of parabolic equations. The solution to this equation is the primitive equation $u(x,t)$, which satisfys the the spatial values of x betwen 0 and ℓ and the time t from 0 to ∞. The solution here is not bracketed in a closed domain, but it propogates in an open ended zone, and the solution needs one initial condition and two bondary conditions.

BC's
$x=0$ and $x=\ell$



(III) Hyperbolic Equations:

The one dimensional wave equation,

$U_{tt} = c^2 U_{xx}$; is one of the first examples of hyperbolic equations.

The solution to this PDE's is the displacement primitive equation $u(x,t)$ for $x = 0 \to \ell$ and $t = 0 \to \infty$ that satisfys the two initial conditions and two boundary conditions. the solution also propogate in an open domain.

Finite Difference Approximation to partial Derivatives:

$$\frac{\partial u}{\partial x} = U_x = \frac{U_{i+1,j} - U_{i-1,j}}{2h}$$

$h = \Delta x$
$K = \Delta y$

$$\frac{\partial^2 u}{\partial x^2} = U_{xx} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$$

$$\frac{\partial u}{\partial y} = U_y = \frac{U_{i,j+1} - U_{i,j-1}}{2k}$$

$$\frac{\partial^2 u}{\partial y^2} = U_{yy} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{K^2}$$

Partial Differential Equations :

$$U_{xx} + U_{yy} = 0 \Rightarrow \text{Laplace Equation}$$

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{K^2} = 0$$

If we used a square grid where $h = K \Rightarrow$

$$U_{i,j} = \frac{1}{4}\left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right)$$

where we can see that the value of $(U_{i,j})$ is the mean of the next 4-nodes ; and it is called the (Standard Five Points Equation). This equation can be written in another form :

$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = 0$$

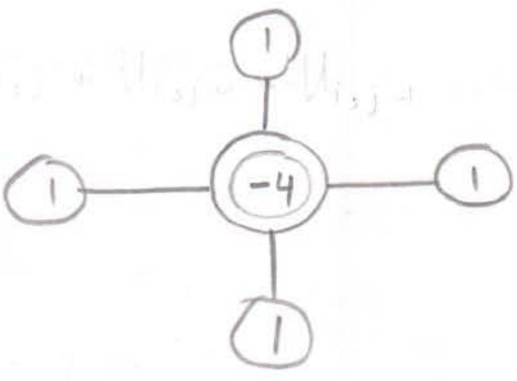
→ (I) Elliptic Equations: (i) Laplace Equation:-

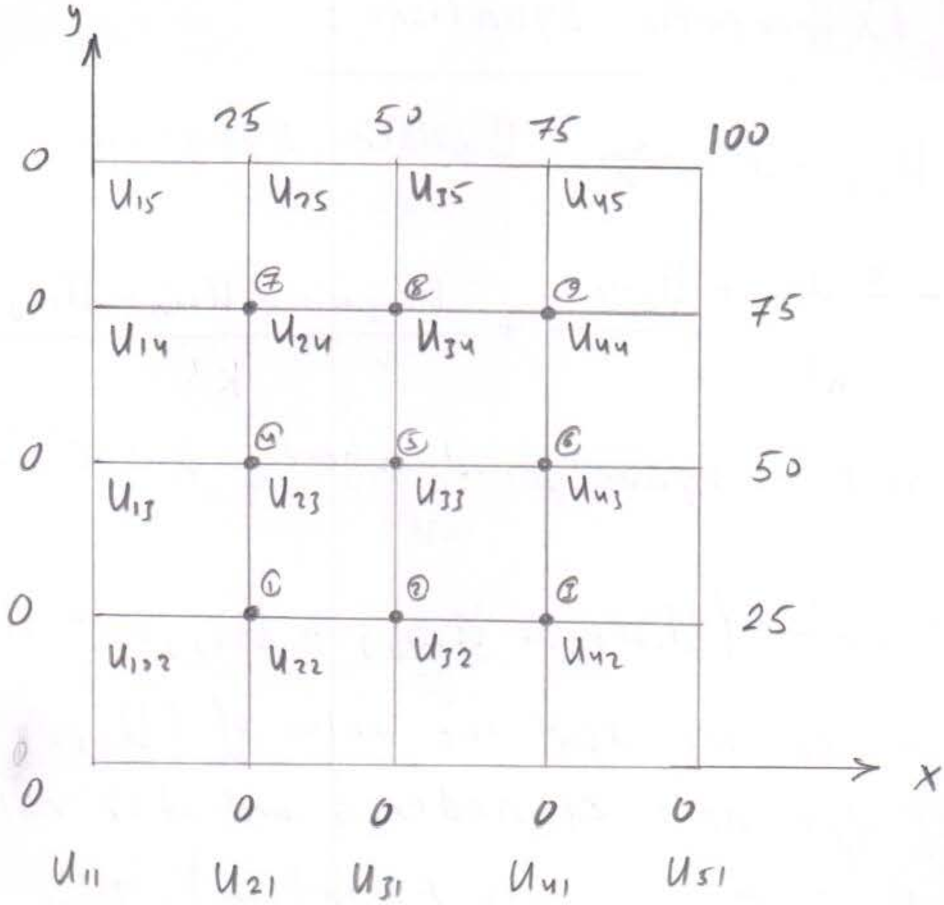
Ex: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with the boundary conditions

$U(0,y) = 0$, $U(x,0) = 0$, $U(x,l) = 100x$, $U(l,y) = 100y$

taking $h = 0.25\,l$?

Sol: Using Stencil form of FDE :

$\nabla^2 =$  $= 0$



$$\partial_{2,2} \quad \overset{0}{\cancel{U_{21}}} + \overset{0}{\cancel{U_{12}}} + U_{23} + U_{32} - 4U_{22} = 0$$

$$\partial_{3,2} \quad \overset{0}{\cancel{U_{31}}} + U_{22} + U_{33} + U_{42} - 4U_{32} = 0$$

$$\partial_{4,2} \quad \overset{0}{\cancel{U_{41}}} + U_{32} + U_{43} + \overset{25}{\cancel{U_{52}}} - 4U_{42} = 0$$

$$\partial_{2,3} \quad U_{22} + \overset{0}{\cancel{U_{13}}} + U_{24} + U_{33} - 4U_{23} = 0$$

$$\partial_{3,3} \quad U_{32} + U_{23} + U_{34} + U_{43} - 4U_{33} = 0$$

$$\partial_{4,3} \quad U_{42} + U_{33} + U_{44} + \overset{\cancel{50}\ 25}{\cancel{U_{52}}} - 4U_{43} = 0$$

$$\partial_{2,4} \quad U_{23} + \overset{0}{\cancel{U_{14}}} + \overset{75}{\cancel{U_{25}}} + U_{34} - 4U_{24} = 0$$

$$\partial_{3,4} \quad U_{33} + U_{24} + \overset{50}{\cancel{U_{35}}} + U_{44} - 4U_{34} = 0$$

$$\partial_{4,4} \quad U_{43} + U_{34} + \overset{75}{\cancel{U_{45}}} + \overset{75}{\cancel{U_{54}}} - 4U_{44} = 0$$

H.W

Ex Solve the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -36\left(x^3 + y^3 + 5\right)$$

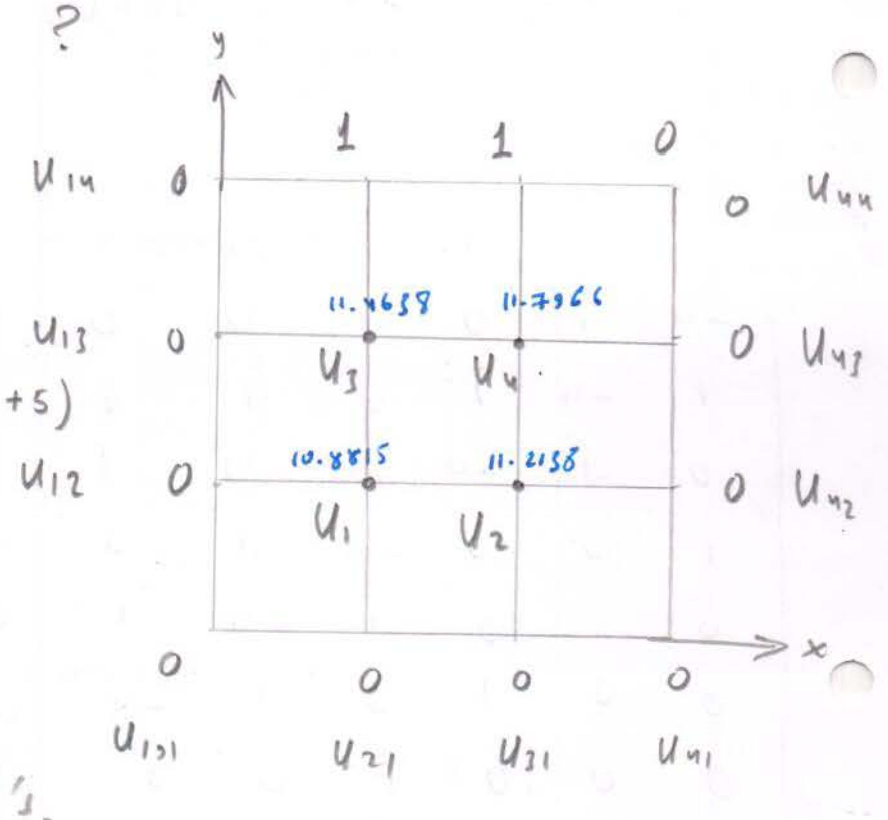
subject to the conditions :

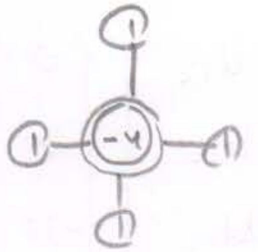
$U = 0$ at $x = 0$ and $x = \ell$

$U = 0$ at $y = 0$

$U = 1$ at $y = \ell$ in $0 < x < \ell$

use $\left(h = \frac{1}{3}\ell\right)$?





$$= -36\left(\tfrac{1}{3}\ell\right)^2\left(x_i^3 + y_j^3 + 5\right)$$

sub. to get 4-eq's.

and solve. ?

[*] يمكن تبسيط قيم x_i و y_i عن طريق استخدام i و j حيث للنقطة

U_1	U_2	U_3	U_4
$i=1$	$i=2$	$i=1$	$i=2$
$j=1$	$j=1$	$j=2$	$j=2$

وهنا يكون $h = \frac{1}{3}$ فقط بدون ℓ

$$-4U_{22} + U_{32} + U_{23} = 0$$

$$U_{22} - 4U_{32} + U_{42} + U_{33} = 0$$

$$U_{32} - 4U_{42} + U_{43} = -25$$

$$U_{22} - 4U_{23} + U_{33} + U_{24} = 0$$

$$U_{32} + U_{23} - 4U_{33} + U_{43} + U_{34} = 0$$

$$U_{42} + U_{33} - 4U_{43} + U_{44} = -50$$

$$+U_{23} - 4U_{24} + U_{34} = -25$$

$$+U_{33} + U_{24} - 4U_{34} + U_{44} = -50$$

$$U_{43} + U_{34} - 4U_{44} = -150$$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} U_{22} \\ U_{32} \\ U_{42} \\ U_{24} \\ U_{33} \\ U_{43} \\ U_{24} \\ U_{34} \\ U_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -25 \\ 0 \\ 0 \\ -50 \\ -25 \\ -50 \\ -150 \end{bmatrix}$$

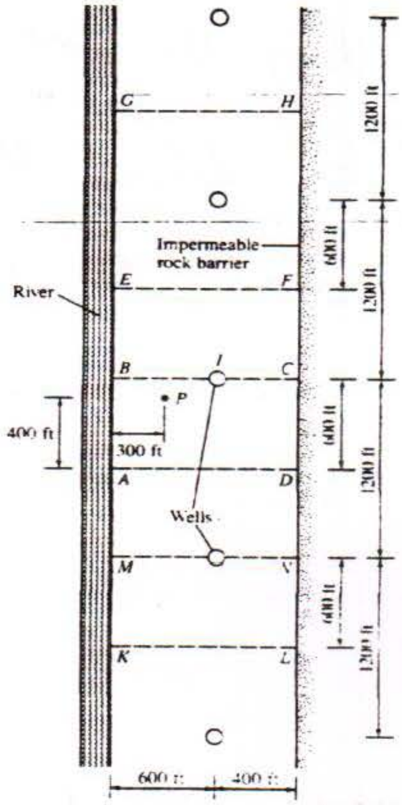
$$\Rightarrow$$

$U_{22} = 6.25$ $\quad U_{32} = 12.5$ $\quad U_{42} = 18.75$

$U_{24} = 12.5$ $\quad U_{33} = 25$ $\quad U_{43} = 37.5$

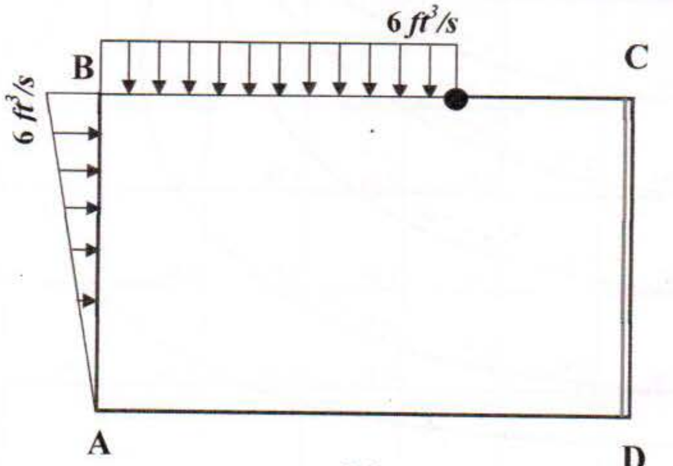
$U_{24} = 18.75$ $\quad U_{34} = 37.5$ $\quad U_{44} = 56.25$

Example: A line of wells to be drilled in an alluvial aquifer as shown in the next Figure . The wells are equally spaced along a line 600 ft from the river. An impermeable barrier is located parallel to the river and 1000 ft from the river. Each well is pumped at a rate of 12 ft^3/s. Determine the streamline and constant piezometric head pattern, and the velocity at point P. Assume that the aquifer is confined with a depth of 100 ft.



Solution:

Considering the geometry of the flow field will show us that the flow pattern in AEFD will have a flow pattern exactly like that within EGHF. Thus, we need to solve for the flow pattern within AEFD only, since that solution will apply to EGHF and other similar areas such as ADKL. Further consideration of the geometry of AEFD also shows that we can expect the flow pattern within ABCD to be mirror image of that within BEFC or ADMN. Therefore we need to solve for the flow pattern in ABCD only, since that solution will apply to all other similar areas in the flow field.



(a)

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

Using the same Technique and $h = 100$ ft to define 45 interior node and 32 Boundary node. We get

$$\nabla^2 = \begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix} = 0$$