(II) Parabolic Equations:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

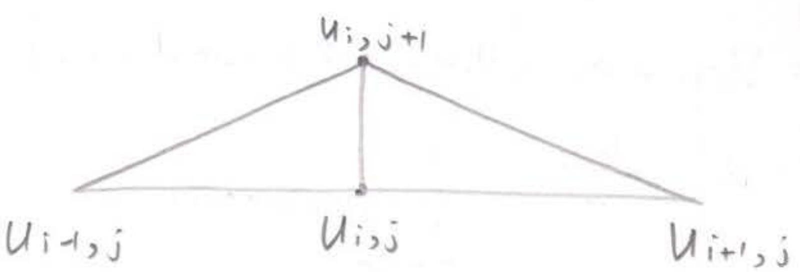
Assuming a rectangular grid in the $(x-t)$ plane ; such as

$$x = i\,h \qquad (i = 0, 1, 2, 3, \dots)$$
$$t = j\,K \qquad (j = 0, 1, 2, 3, \dots) \qquad \Rightarrow$$

$$\frac{1}{K}\underbrace{\left(u_{i,j+1} - u_{i,j}\right)}_{\text{Forward Diff.}} = \frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$\boxed{\therefore\ u_{i,j+1} = \alpha\left(u_{i+1,j} + u_{i-1,j}\right) + (1-2\alpha)\,u_{i,j} \qquad \text{where} \quad \alpha = \frac{K}{h^2}}$$

and this is called the Explicit scheme.



The solution of the Explicit Scheme is stable when $\alpha < \frac{1}{2}$; and when $\alpha = \frac{1}{2}$ the equation becomes :

$$u_{i,j+1} = \frac{1}{2}\left(u_{i+1,j} + u_{i-1,j}\right)$$

Ex] Obtain the Numerical Solution of the differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \; t \geq 0$$

under the condition that:

$$U(0,t) = U(1,t) = 0.0 \text{ and}$$

$$U(x,0) = 2x \quad \text{for} \quad 0 \leq x \leq \tfrac{1}{2}$$

$$= 2(1-x) \quad \text{for} \quad \tfrac{1}{2} \leq x \leq 1$$

solve using the explicit scheme?

Sol.

$$\frac{1}{K}\left(U_{i,j+1} - U_{i,j}\right) = \frac{1}{h^2}\left(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}\right)$$

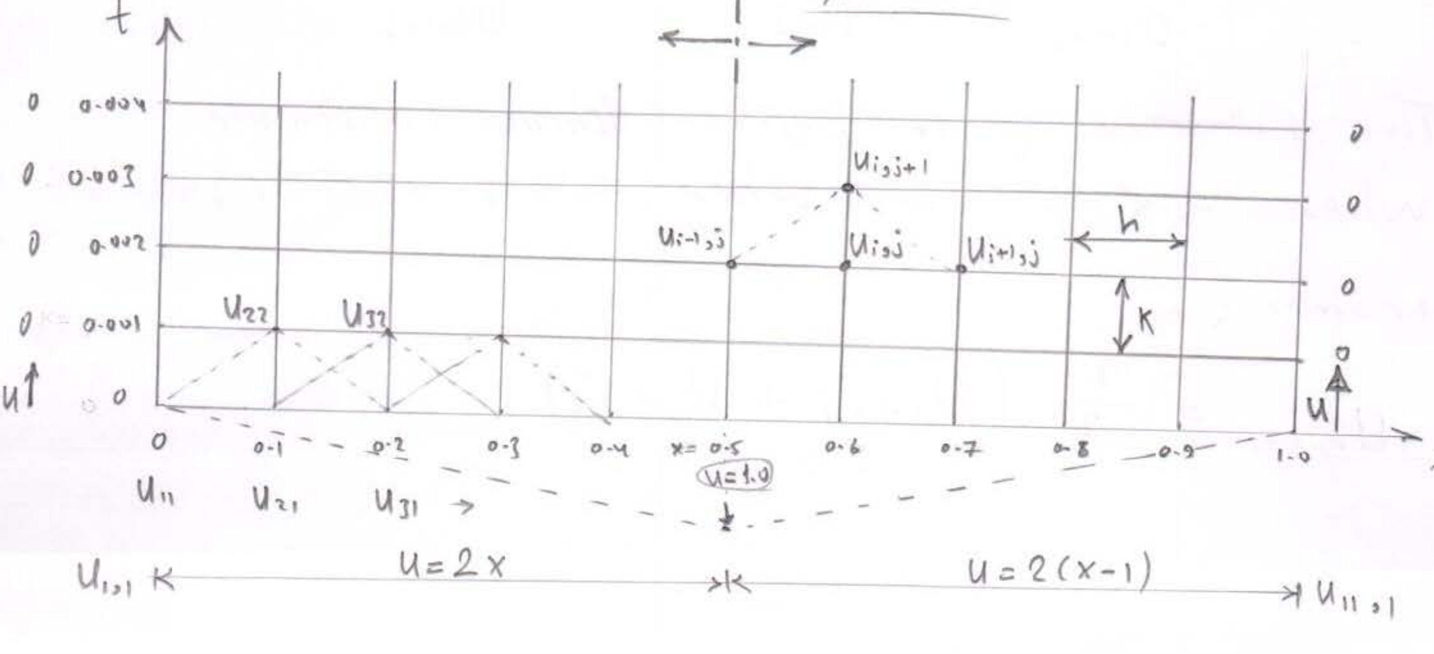
where: $x = i.h \quad (i = 0, 1, 2 \ldots)$

$t = j.K \quad (j = 0, 1, 2, \ldots)$ and

$$U_{i,j+1} = U_{i,j} + \frac{K}{h^2}\left(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}\right) \quad \text{or}$$

$$U_{i,j+1} = U_{i,j} + \alpha\left(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}\right)$$

where: $\alpha = \frac{K}{h^2}$



Sol:

$\alpha = \frac{k^2}{h^2} = 1.0 \Rightarrow$

$$\boxed{u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}} \quad \text{———} \ (*)$$

B.C's and I.C's:

$$u_{0,j} = u_{1,j} = 0 \quad \text{———} \ (1)$$

Initial conditions:

$$u_{i,0} = \frac{1}{2} x_i (1 - x_i) \quad \text{———} \ (2)$$

$$\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \quad \text{at } t = 0$$

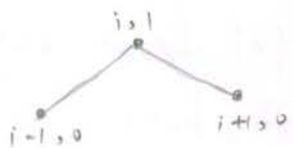
$$\Rightarrow u_{i,1} - u_{i,-1} = 0 \Rightarrow$$

$$u_{i,1} = u_{i,-1} \quad \text{———} \ (3)$$

Because of the symetry we will solve only for $\boxed{i = 1 \rightarrow i = 5}$. $\Leftrightarrow x = 0.1 \rightarrow x = 0.5$

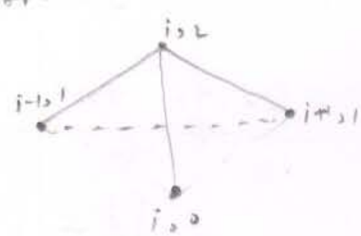
For $j = 0 \Rightarrow u_{i,1} = u_{i,-1}$ ——— (3) sub. in ——— (*) $\Rightarrow$

$$u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right)$$



For $j = 1, 2, 3, 4, 5$ and From ——— (*)

$j = 1 \quad u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$ → from ——— (2)



$j = 2 \quad u_{i,3} = u_{i-1,2} + u_{i+1,2} - u_{i,1}$

$j = 3 \quad u_{i,4} = u_{i-1,3} + u_{i+1,3} - u_{i,2}$

$j = 4 \quad u_{i,5} = u_{i-1,4} + u_{i+1,4} - u_{i,3}$

$i = 1 \rightarrow 5$

Let $h = 0.1$; $K = 0.001$ $\Rightarrow \alpha = \frac{K}{h^2} = 0.1$

$$U_{i,j+1} = U_{i,j} + \alpha \left(U_{i-1,j} - 2U_{i,j} + U_{i+1,j} \right) \Rightarrow$$

$$U_{i,j+1} = U_{i,j} + \frac{1}{10} \left(U_{i-1,j} - 2U_{i,j} + U_{i+1,j} \right) \Rightarrow$$

$$U_{i,j+1} = \frac{1}{10} \left(U_{i-1,j} + 8U_{i,j} + U_{i+1,j} \right) \quad \text{———} (*)$$

substitute in each Unknown points :

For $j=1 \Rightarrow$ to find $j+1$

$i=2 \Rightarrow U_{22} = \frac{1}{10} \left(U_{11} + 8U_{21} + U_{31} \right) = \frac{1}{10} \left(0 + 8(0.2) + 0.4 \right) = 0.2$

$i=3 \Rightarrow U_{32} = \frac{1}{10} \left(U_{21} + 8U_{31} + U_{41} \right) = \frac{1}{10} \left(0.2 + 8(0.4) + 0.6 \right) = 0.4$

$i=4 \Rightarrow U_{42} = \frac{1}{10} \left(U_{31} + 8U_{41} + U_{51} \right) = \frac{1}{10} \left(0.4 + 8(0.6) + 0.8 \right) = 0.6$

$i=5 \Rightarrow U_{52} = \frac{1}{10} \left(U_{41} + 8U_{51} + U_{61} \right) = \frac{1}{10} \left(0.6 + 8(0.8) + 1.0 \right) = 0.8$

$i=6 \Rightarrow U_{62} = \frac{1}{10} \left(U_{5,1} + 8U_{61} + U_{71} \right) = \frac{1}{10} \left(0.8 + 8(1.0) + 0.8 \right) = 0.96$

and From Symmetry :-

$U_{7,2} = U_{5,2} = 0.8$

$U_{8,2} = U_{4,2} = 0.6$

$U_{9,2} = U_{3,2} = 0.4$

$U_{10,2} = U_{2,2} = 0.2$

$U_{11,2} = U_{1,2} = 0.0$

and by same method using $j = 2$ & $i = 2 \to 11$

$j = 3$ & $i = 2 \to 11$

$\vdots$

$j = n$ & $i = 2 \to 11$

	i=0	1	2	3	4	5	6
	X=0	0.1	0.2	0.3	0.4	0.5	0.6
j=0 , t=0.000	0.0	0.200	0.40	0.60	0.80	1.00	0.8
j=1 , t=0.001	0.0	0.2	0.40	0.60	0.80	0.960	
j=2 , t=0.002	0.0	0.2	0.40	0.60	0.7960	0.928	
j=3 , t=0.003	0.0	0.2	0.40	0.5996	0.7896	0.9016	Symm.
j=4 , t=0.004	0.0	0.2	0.40	0.5986	0.7818	0.8792	
j=5 , t=0.005	0.0	0.2	0.3999	0.5971	0.7732	0.8597	
j=10 , t=0.01	0.0	0.1996	0.3968	0.5822	0.7281	0.7867	
j=20 , t=0.02	0.0	0.1938	0.3781	0.5373	0.6486	0.6891	

H.W. Consolidation Example