Review

Introduction

Technically a student coming into a Calculus class is supposed to know both Algebra and Trigonometry. The reality is often much different however. Most students enter a Calculus class woefully unprepared for both the algebra and the trig that is in a Calculus class. This is very unfortunate since good algebra skills are absolutely vital to successfully completing any Calculus course and if your Calculus course includes trig (as this one does) good trig skills are also important in many sections.

The intent of this chapter is to do a very cursory review of some algebra and trig skills that are absolutely vital to a calculus course. This chapter is not inclusive in the algebra and trig skills that are needed to be successful in a Calculus course. It only includes those topics that most students are particularly deficient in. For instance factoring is also vital to completing a standard calculus class but is not included here. For a more in depth review you should visit my Algebra/Trig review or my full set of Algebra notes at http://tutorial.math.lamar.edu.

Note that even though these topics are very important to a Calculus class I rarely cover all of these in the actual class itself. We simply don't have the time to do that. I do cover certain portions of this chapter in class, but for the most part I leave it to the students to read this chapter on their own.

Here is a list of topics that are in this chapter. I've also denoted the sections that I typically cover during the first couple of days of a Calculus class.

Review : Functions – Here is a quick review of functions, function notation and a couple of fairly important ideas about functions.

Review : Inverse Functions – A quick review of inverse functions and the notation for inverse functions.

Review : Trig Functions – A review of trig functions, evaluation of trig functions and the unit circle. This section usually gets a quick review in my class.

Review : Solving Trig Equations – A reminder on how to solve trig equations. This section is always covered in my class.

Review : Solving Trig Equations with Calculators, Part I – The previous section worked problem whose answers were always the "standard" angles. In this section we work some problems whose answers are not "standard" and so a calculator is needed. This section is always covered in my class as most trig equations in the remainder will need a calculator.

Review : Solving Trig Equations with Calculators, Part II – Even more trig equations requiring a calculator to solve.

Review : Exponential Functions – A review of exponential functions. This section usually gets a quick review in my class.

Review : Logarithm Functions – A review of logarithm functions and logarithm properties. This section usually gets a quick review in my class.

Review : Exponential and Logarithm Equations – How to solve exponential and logarithm equations. This section is always covered in my class.

Review : Common Graphs – This section isn't much. It's mostly a collection of graphs of many of the common functions that are liable to be seen in a Calculus class.

Review : Functions

In this section we're going to make sure that you're familiar with functions and function notation. Both will appear in almost every section in a Calculus class and so you will need to be able to deal with them.

First, what exactly is a function? An equation will be a function if for any *x* in the domain of the equation (the domain is all the *x*'s that can be plugged into the equation) the equation will yield exactly one value of *y*.

This is usually easier to understand with an example.

Example 1 Determine if each of the following are functions.

(a) $y = x^2 + 1$

(b) $y^2 = x + 1$

Solution

(a) This first one is a function. Given an *x* there is only one way to square it and then add 1 to the result and so no matter what value of *x* you put into the equation there is only one possible value of *y*.

(b) The only difference between this equation and the first is that we moved the exponent off the *x* and onto the *y*. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of *x*, say $x=3$ and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of *y* that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of *y* that we get from a single *x* this equation isn't a function.

Note that this only needs to be the case for a single value of *x* to make an equation not be a function. For instance we could have used $x=-1$ and in this case we would get a single *y* ($y=0$). However, because of what happens at $x=3$ this equation will not be a function.

Next we need to take a quick look at function notation. Function notation is nothing more than a fancy way of writing the *y* in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function.

$$y = 2x^2 - 5x + 3$$

Using function notation we can write this as any of the following.

$$f(x) = 2x^2 - 5x + 3 \qquad g(x) = 2x^2 - 5x + 3$$
$$h(x) = 2x^2 - 5x + 3 \qquad R(x) = 2x^2 - 5x + 3$$
$$w(x) = 2x^2 - 5x + 3 \qquad y(x) = 2x^2 - 5x + 3$$
$$\vdots$$

Recall that this is NOT a letter times *x*, this is just a fancy way of writing *y*.

So, why is this useful? Well let's take the function above and let's get the value of the function at $x=-3$. Using function notation we represent the value of the function at $x=-3$ as $f(-3)$. Function notation gives us a nice compact way of representing function values.

Now, how do we actually evaluate the function? That's really simple. Everywhere we see an *x* on the right side we will substitute whatever is in the parenthesis on the left side. For our function this gives,

$$\begin{aligned} f(-3) &= 2(-3)^2 - 5(-3) + 3 \\ &= 2(9) + 15 + 3 \\ &= 36 \end{aligned}$$

Let's take a look at some more function evaluation.

Example 2 Given $f(x) = -x^2 + 6x - 11$ find each of the following.

(a) $f(2)$ [Solution]

(b) $f(-10)$ [Solution]

(c) $f(t)$ [Solution]

(d) $f(t-3)$ [Solution]

(e) $f(x-3)$ [Solution]

(f) $f(4x-1)$ [Solution]

Solution

(a) $f(2) = -(2)^2 + 6(2) - 11 = -3$

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(b) $f(-10) = -(-10)^2 + 6(-10) - 11 = -100 - 60 - 11 = -171$

Be careful when squaring negative numbers!

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(c) $f(t) = -t^2 + 6t - 11$

Remember that we substitute for the *x*'s WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put *t*'s in for all the *x*'s on the left.

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(d) $f\left(t-3\right)=-\left(t-3\right)^2+6\left(t-3\right)-11=-t^2+12t-38$

Often instead of evaluating functions at numbers or single letters we will have some fairly complex evaluations so make sure that you can do these kinds of evaluations.

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(e) $f\left(x-3\right)=-\left(x-3\right)^2+6\left(x-3\right)-11=-x^2+12x-38$

The only difference between this one and the previous one is that I changed the *t* to an *x*. Other than that there is absolutely no difference between the two! Don't get excited if an *x* appears inside the parenthesis on the left.

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(f) $f\left(4x-1\right)=-\left(4x-1\right)^2+6\left(4x-1\right)-11=-16x^2+32x-18$

This one is not much different from the previous part. All we did was change the equation that we were plugging into function.

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All throughout a calculus course we will be finding roots of functions. A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, *g(x)*, is equivalent to solving

$$g\left(x\right)=0$$

Example 3 Determine all the roots of $f\left(t\right)=9t^3-18t^2+6t$

Solution

So we will need to solve,

$$9t^3-18t^2+6t=0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t\left(3t^2-6t+2\right)=0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t=0 \qquad \text{OR,}$$

$$3t^2-6t+2=0$$

From the first it's clear that one of the roots must then be *t=0*. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$
\begin{aligned}
t &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \\
&= \frac{6 \pm \sqrt{12}}{6} \\
&= \frac{6 \pm \sqrt{(4)(3)}}{6} \\
&= \frac{6 \pm 2\sqrt{3}}{6} \\
&= \frac{3 \pm \sqrt{3}}{3} \\
&= 1 \pm \frac{1}{3}\sqrt{3} \\
&= 1 \pm \frac{1}{\sqrt{3}}
\end{aligned}
$$

In order to remind you how to simplify radicals we gave several forms of the answer.

To complete the problem, here is a complete list of all the roots of this function.

$$t = 0, \; t = \frac{3+\sqrt{3}}{3}, \;\; t = \frac{3-\sqrt{3}}{3}$$

Note we didn't use the final form for the roots from the quadratic. This is usually where we'll stop with the simplification for these kinds of roots. Also note that, for the sake of the practice, we broke up the compact form for the two roots of the quadratic. You will need to be able to do this so make sure that you can.

This example had a couple of points other than finding roots of functions.

The first was to remind you of the quadratic formula. This won't be the first time that you'll need it in this class.

The second was to get you used to seeing "messy" answers. In fact, the answers in the above list are not that messy. However, most students come out of an Algebra class very used to seeing only integers and the occasional "nice" fraction as answers.

So, here is fair warning. In this class I often will intentionally make the answers look "messy" just to get you out of the habit of always expecting "nice" answers. In "real life" (whatever that is) the answer is rarely a simple integer such as two. In most problems the answer will be a decimal that came about from a messy fraction and/or an answer that involved radicals.

The next topic that we need to discuss here is that of **function composition**. The composition of *f(x)* and *g(x)* is

$$(f \circ g)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will usually result in a different answer.

Example 4 Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(f \circ g)(5)$ [Solution]

(b) $(f \circ g)(x)$ [Solution]

(c) $(g \circ f)(x)$ [Solution]

(d) $(g \circ g)(x)$ [Solution]

Solution

(a) $(f \circ g)(5)$

In this case we've got a number instead of an *x* but it works in exactly the same way.

$$\begin{aligned}(f \circ g)(5) &= f(g(5)) \\ &= f(-99) = 29512\end{aligned}$$

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(b) $(f \circ g)(x)$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(1-20x) \\ &= 3(1-20x)^2 - (1-20x) + 10 \\ &= 3(1-40x+400x^2) - 1 + 20x + 10 \\ &= 1200x^2 - 100x + 12\end{aligned}$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

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(c) $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x^2 - x + 10) \\ &= 1 - 20(3x^2 - x + 10) \\ &= -60x^2 + 20x - 199\end{aligned}$$

And just to make the point. This answer is different from the previous part. Order is important in composition.

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(d) $(g \circ g)(x)$

In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(1-20x) \\ &= 1-20(1-20x) \\ &= 400x-19\end{aligned}$$

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Let's work one more example that will lead us into the next section.

Example 5 Given $f(x)=3x-2$ and $g(x)=\frac{1}{3}x+\frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

(a)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{3}x+\frac{2}{3}\right) \\ &= 3\left(\frac{1}{3}x+\frac{2}{3}\right)-2 \\ &= x+2-2=x\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x-2) \\ &= \frac{1}{3}(3x-2)+\frac{2}{3} \\ &= x-\frac{2}{3}+\frac{2}{3}=x\end{aligned}$$

In this case the two compositions where the same and in fact the answer was very simple.

$$(f \circ g)(x)=(g \circ f)(x)=x$$

This will usually not happen. However, when the two compositions are the same, or more specifically when the two compositions are both x there is a very nice relationship between the two functions. We will take a look at that relationship in the next section.

Review : Inverse Functions

In the last example from the previous section we looked at the two functions $f(x) = 3x - 2$ and $g(x) = \frac{x}{3} + \frac{2}{3}$ and saw that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

and as noted in that section this means that there is a nice relationship between these two functions. Let's see just what that relationship is. Consider the following evaluations.

$$f(-1) = 3(-1) - 2 = -5 \qquad \Rightarrow \qquad g(-5) = \frac{-5}{3} + \frac{2}{3} = \frac{-3}{3} = -1$$

$$g(2) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right) - 2 = 4 - 2 = 2$$

In the first case we plugged $x = -1$ into $f(x)$ and got a value of -5. We then turned around and plugged $x = -5$ into $g(x)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x = 2$ into $g(x)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f(x)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$(g \circ f)(-1) = g\left[f(-1)\right] = g[-5] = -1$$

and the second case is really,

$$(f \circ g)(2) = f\left[g(2)\right] = f\left[\frac{4}{3}\right] = 2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x = -1$ into $f(x)$ and then plugged the result from this function evaluation back into $g(x)$ and in some way $g(x)$ undid what $f(x)$ had done to $x = -1$ and gave us back the original x that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of x produce the same y. Mathematically this is the same as saying,

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \qquad x_1 \neq x_2$$

So, a function is one-to-one if whenever we plug different values into the function we get different function values.

Sometimes it is easier to understand this definition if we see a function that isn't one-to-one. Let's take a look at a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. In other words there are two different values of x that produce the same value of y. Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \le x < \infty$. This can sometimes be done with functions.

Showing that a function is one-to-one is often tedious and/or difficult. For the most part we are going to assume that the functions that we're going to be dealing with in this course are either one-to-one or we have restricted the domain of the function to get it to be a one-to-one function.

Now, let's formally define just what inverse functions are. Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = x \qquad \text{AND} \qquad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are **inverses** of each other. More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$$f(x) = 3x - 2 \qquad f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

$$g(x) = \frac{x}{3} + \frac{2}{3} \qquad g^{-1}(x) = 3x - 2$$

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that is sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there are a couple of steps that can on occasion be somewhat messy. Here is the process

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with since it is easy to make mistakes in those steps.

In the verification step we technically really do need to check that both $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are true. For all the functions that we are going to be looking at in this course if one is true then the other will also be true. However, there are functions (they are beyond the scope of this course however) for which it is possible for only one of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

Example 1 Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f\left(x\right)$ with *y*.

$$y = 3x - 2$$

Next, replace all *x*'s with *y* and all *y's* with *x*.

$$x = 3y - 2$$

Now, solve for *y*.

$$\begin{aligned} x + 2 &= 3y \\ \frac{1}{3}\left(x+2\right) &= y \\ \frac{x}{3} + \frac{2}{3} &= y \end{aligned}$$

Finally replace *y* with $f^{-1}\left(x\right)$.

$$f^{-1}\left(x\right) = \frac{x}{3} + \frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $\left(f \circ f^{-1}\right)\left(x\right) = x$ is true.

$$\begin{aligned} \left(f \circ f^{-1}\right)\left(x\right) &= f\left[f^{-1}\left(x\right)\right] \\ &= f\left[\frac{x}{3} + \frac{2}{3}\right] \\ &= 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

Example 2 Given $g\left(x\right) = \sqrt{x-3}$ find $g^{-1}\left(x\right)$.

Solution

The fact that we're using $g\left(x\right)$ instead of $f\left(x\right)$ doesn't change how the process works. Here are the first few steps.

$$\begin{aligned} y &= \sqrt{x-3} \\ x &= \sqrt{y-3} \end{aligned}$$

Now, to solve for *y* we will need to first square both sides and then proceed as normal.

$$\begin{aligned} x &= \sqrt{y-3} \\ x^2 &= y-3 \\ x^2+3 &= y \end{aligned}$$

This inverse is then,

$$g^{-1}(x) = x^2+3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned} \left(g^{-1}\circ g\right)(x) &= g^{-1}\left[g(x)\right] \\ &= g^{-1}\left(\sqrt{x-3}\right) \\ &= \left(\sqrt{x-3}\right)^2+3 \\ &= x-3+3 \\ &= x \end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

The next example can be a little messy so be careful with the work here.

Example 3 Given $h(x) = \dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$\begin{aligned} y &= \frac{x+4}{2x-5} \\ x &= \frac{y+4}{2y-5} \end{aligned}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$x(2y-5) = y+4$$
$$2xy - 5x = y+4$$
$$2xy - y = 4+5x$$
$$(2x-1)y = 4+5x$$
$$y = \frac{4+5x}{2x-1}$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}(x) = \frac{4+5x}{2x-1}$$

Finally we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$\begin{aligned}\left(h \circ h^{-1}\right)(x) &= h\left[h^{-1}(x)\right] \\ &= h\left[\frac{4+5x}{2x-1}\right] \\ &= \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}\end{aligned}$$

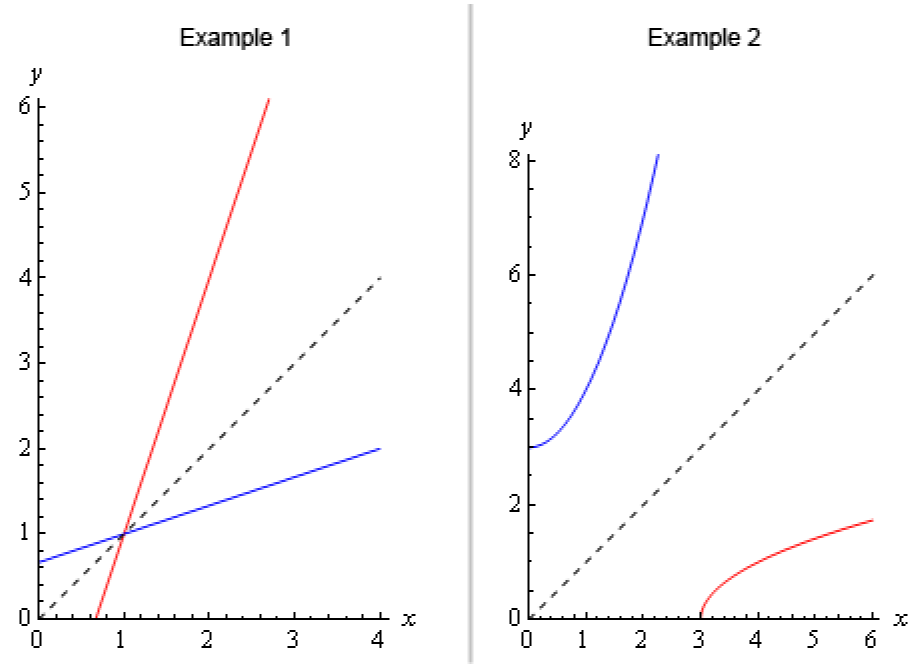
Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$\begin{aligned}\left(h \circ h^{-1}\right)(x) &= \frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5} \\ &= \frac{(2x-1)\left(\frac{4+5x}{2x-1}+4\right)}{(2x-1)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)} \\ &= \frac{4+5x+4(2x-1)}{2(4+5x)-5(2x-1)} \\ &= \frac{4+5x+8x-4}{8+10x-10x+5} \\ &= \frac{13x}{13} = x\end{aligned}$$

Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and the graph of its inverse.

Here is the graph of the function and inverse from the first two examples.



In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.

Review : Trig Functions

The intent of this section is to remind you of some of the more important (from a Calculus standpoint…) topics from a trig class. One of the most important (but not the first) of these topics will be how to use the unit circle. We will actually leave the most important topic to the next section.

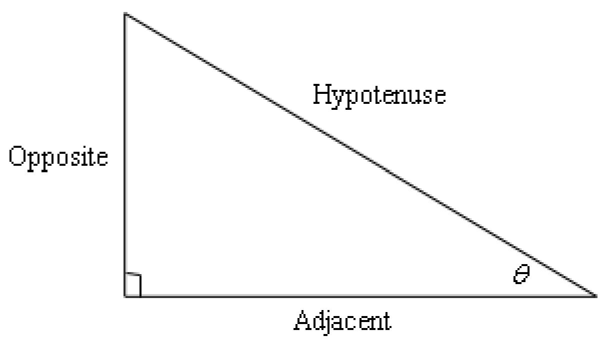
First let's start with the six trig functions and how they relate to each other.

$$\cos(x) \qquad \sin(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

$$\sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

Recall as well that all the trig functions can be defined in terms of a right triangle.



From this right triangle we get the following definitions of the six trig functions.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

Remembering both the relationship between all six of the trig functions and their right triangle definitions will be useful in this course on occasion.

Next, we need to touch on radians. In most trig classes instructors tend to concentrate on doing everything in terms of degrees (probably because it's easier to visualize degrees). The same is true in many science classes. However, in a calculus course almost everything is done in radians. The following table gives some of the basic angles in both degrees and radians.

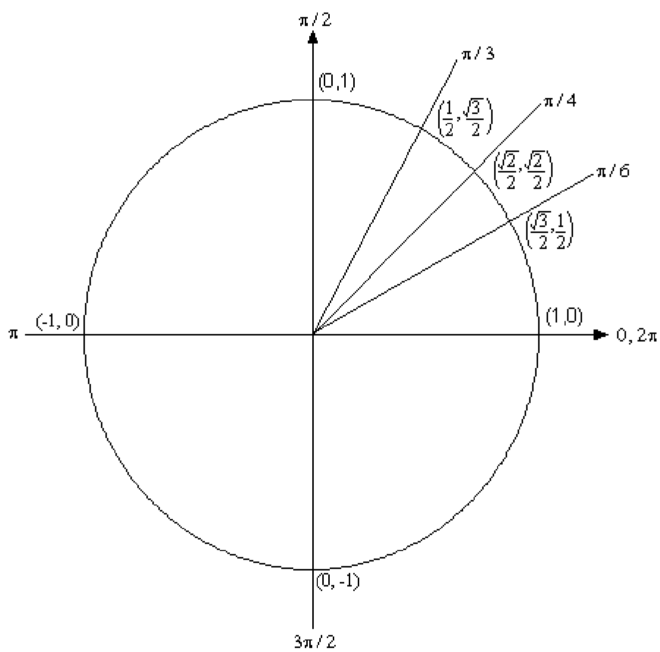
Degree	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Know this table! We may not see these specific angles all that much when we get into the Calculus portion of these notes, but knowing these can help us to visualize each angle. Now, one more time just make sure this is clear.

Be forewarned, everything in most calculus classes will be done in radians!

Let's next take a look at one of the most overlooked ideas from a trig class. The unit circle is one of the more useful tools to come out of a trig class. Unfortunately, most people don't learn it as well as they should in their trig class.

Below is the unit circle with just the first quadrant filled in. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate is the cosine of that angle and the second coordinate is the sine of that angle. We've put some of the *basic* angles along with the coordinates of their intersections on the unit circle. So, from the unit circle below we can see that $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$.



Remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is 2π, so the positive *x*-axis can correspond to either an angle of 0 or 2π (or 4π, or 6π, or -2π, or -4π, *etc*. depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$$\frac{\pi}{6}+2\pi=\frac{13\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate once around counter clockwise)}$$

$$\frac{\pi}{6}+4\pi=\frac{25\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate around twice counter clockwise)}$$

$$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate once around clockwise)}$$

$$\frac{\pi}{6}-4\pi=-\frac{23\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate around twice clockwise)}$$

etc.

In fact $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6}+2\pi n$, $n=0,\pm 1,\pm 2,\pm 3,\ldots$ In this case *n* is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of *n* correspond to counter clockwise rotations and negative values of *n* correspond to clockwise rotations.

So, why did I only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first with a small application of geometry. You'll see how this is done in the following set of examples.

Example 1 Evaluate each of the following.

(a) $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$ [Solution]

(b) $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$ [Solution]

(c) $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$ [Solution]

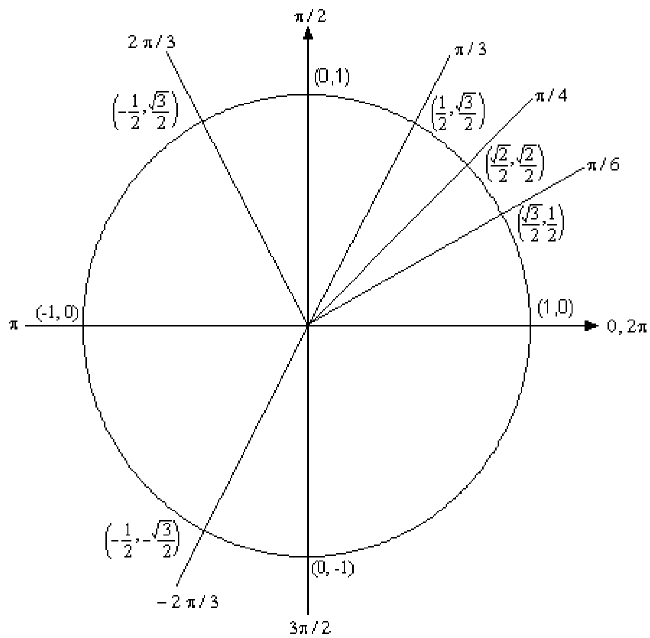
(d) $\sec\left(\frac{25\pi}{6}\right)$ [Solution]

Solution

(a) The first evaluation in this part uses the angle $\frac{2\pi}{3}$. That's not on our unit circle above, however notice that $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative *x*-axis. This means that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the *x* coordinate will be negative.

Likewise for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3} = -\pi + \frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative *x*-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Both of these angles along with their coordinates are shown on the following unit circle.



From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

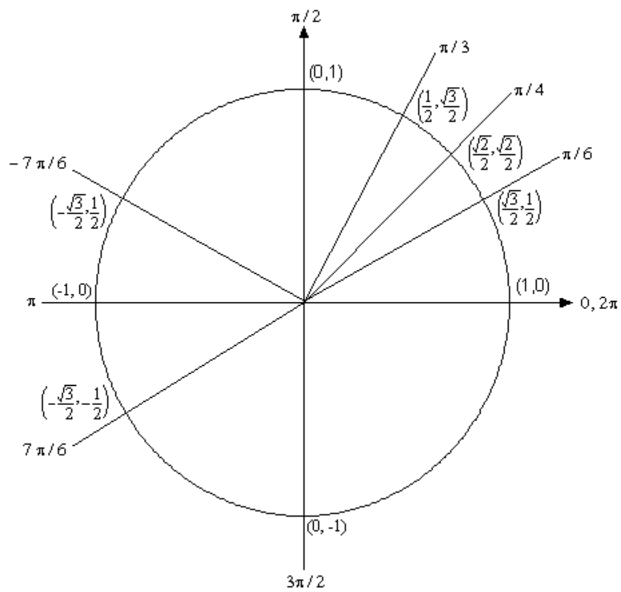
This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta) = -\sin(\theta)$$

[Return to Problems]

(b) For this example notice that $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ so this means we would rotate down $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. Also $-\frac{7\pi}{6} = -\pi - \frac{\pi}{6}$ so this means we would rotate up $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. So, as with the last part, both of these angles will be mirror images of $\frac{\pi}{6}$ in the third and second quadrants respectively and we can use this to determine the coordinates for both of these new angles.

Both of these angles are shown on the following unit circle along with appropriate coordinates for the intersection points.

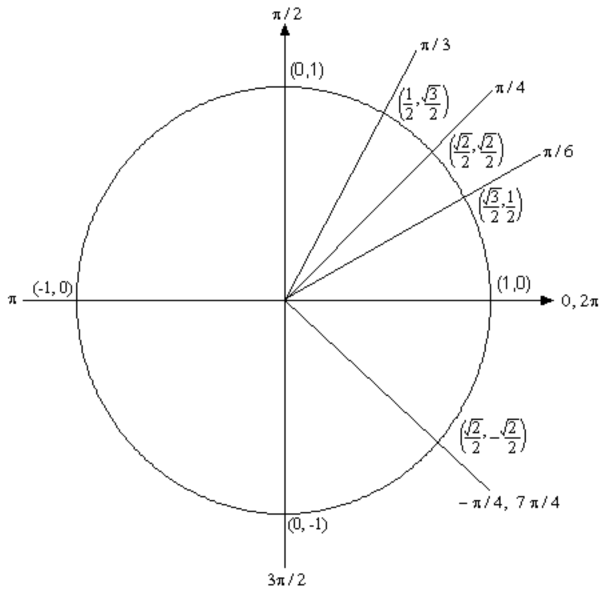


From this unit circle we can see that $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta) = \cos(\theta).$$

[Return to Problems]

(c) Here we should note that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ so $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ are in fact the same angle! Also note that this angle will be the mirror image of $\frac{\pi}{4}$ in the fourth quadrant. The unit circle for this angle is



Now, if we remember that $\tan\left(x\right) = \frac{\sin\left(x\right)}{\cos\left(x\right)}$ we can use the unit circle to find the values the tangent function. So,

$$\tan\left(\frac{7\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = \frac{\sin\left(-\pi/4\right)}{\cos\left(-\pi/4\right)} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1.$$

On a side note, notice that $\tan\left(\frac{\pi}{4}\right) = 1$ and we can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan\left(-\theta\right) = -\tan\left(\theta\right).$$

[Return to Problems]

(d) Here we need to notice that $\frac{25\pi}{6} = 4\pi + \frac{\pi}{6}$. In other words, we've started at $\frac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right) = \sec\left(4\pi + \frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec(x) = \frac{1}{\cos(x)}$$

and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

[Return to Problems]

So, in the last example we saw how the unit circle can be used to determine the value of the trig functions at any of the "common" angles. It's important to notice that all of these examples used the fact that if you know the first quadrant of the unit circle and can relate all the other angles to "mirror images" of one of the first quadrant angles you don't really need to know whole unit circle. If you'd like to see a complete unit circle I've got one on my Trig Cheat Sheet that is available at http://tutorial.math.lamar.edu.

Another important idea from the last example is that when it comes to evaluating trig functions all that you really need to know is how to evaluate sine and cosine. The other four trig functions are defined in terms of these two so if you know how to evaluate sine and cosine you can also evaluate the remaining four trig functions.

We've not covered many of the topics from a trig class in this section, but we did cover some of the more important ones from a calculus standpoint. There are many important trig formulas that you will use occasionally in a calculus class. Most notably are the half-angle and double-angle formulas. If you need reminded of what these are, you might want to download my Trig Cheat Sheet as most of the important facts and formulas from a trig class are listed there.

Review : Solving Trig Equations

In this section we will take a look at solving trig equations. This is something that you will be asked to do on a fairly regular basis in my class.

Let's just jump into the examples and see how to solve trig equations.

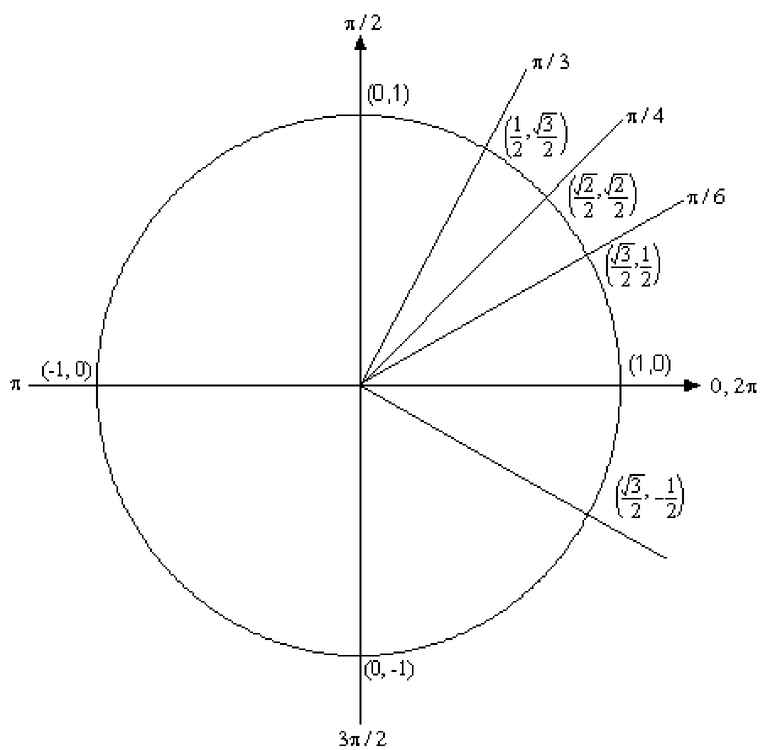
Example 1 Solve $2\cos(t) = \sqrt{3}$.

Solution

There's really not a whole lot to do in solving this kind of trig equation. All we need to do is divide both sides by 2 and the go to the unit circle.

$$2\cos(t) = \sqrt{3}$$
$$\cos(t) = \frac{\sqrt{3}}{2}$$

So, we are looking for all the values of *t* for which cosine will have the value of $\frac{\sqrt{3}}{2}$. So, let's take a look at the following unit circle.



From quick inspection we can see that $t = \frac{\pi}{6}$ is a solution. However, as I have shown on the unit circle there is another angle which will also be a solution. We need to determine what this angle is. When we look for these angles we typically want *positive* angles that lie between 0 and 2π. This angle will not be the only possibility of course, but by convention we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive *x*-axis, then so must the angle in the fourth quadrant. So we could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles so, we'll use $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Recall from the previous section and you'll see there that I used

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

to represent all the possible angles that can end at the same location on the unit circle, *i.e.* angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction (*n* is positive) or clockwise direction (*n* is negative) for *n* complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

Now, in a calculus class this is not a typical trig equation that we'll be asked to solve. A more typical example is the next one.

Example 2 Solve $2\cos(t) = \sqrt{3}$ on $[-2\pi, 2\pi]$.

Solution

In a calculus class we are often more interested in only the solutions to a trig equation that fall in a certain interval. The first step in this kind of problem is to first find all possible solutions. We did this in the first example.

$$\frac{\pi}{6} + 2\pi\, n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\frac{11\pi}{6} + 2\pi\, n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Now, to find the solutions in the interval all we need to do is start picking values of *n*, plugging them in and getting the solutions that will fall into the interval that we've been given.

n=0.

$$\frac{\pi}{6} + 2\pi(0) = \frac{\pi}{6} < 2\pi$$

$$\frac{11\pi}{6} + 2\pi(0) = \frac{11\pi}{6} < 2\pi$$

Now, notice that if we take any positive value of *n* we will be adding on positive multiples of 2π onto a positive quantity and this will take us past the upper bound of our interval and so we don't need to take any positive value of *n*.

However, just because we aren't going to take any positive value of *n* doesn't mean that we shouldn't also look at negative values of *n*.

n=-1.

$$\frac{\pi}{6} + 2\pi(-1) = -\frac{11\pi}{6} > -2\pi$$

$$\frac{11\pi}{6} + 2\pi(-1) = -\frac{\pi}{6} > -2\pi$$

These are both greater than -2π and so are solutions, but if we subtract another 2π off (*i.e* use $n = -2$) we will once again be outside of the interval so we've found all the possible solutions that lie inside the interval $[-2\pi, 2\pi]$.

So, the solutions are : $\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{11\pi}{6}$.

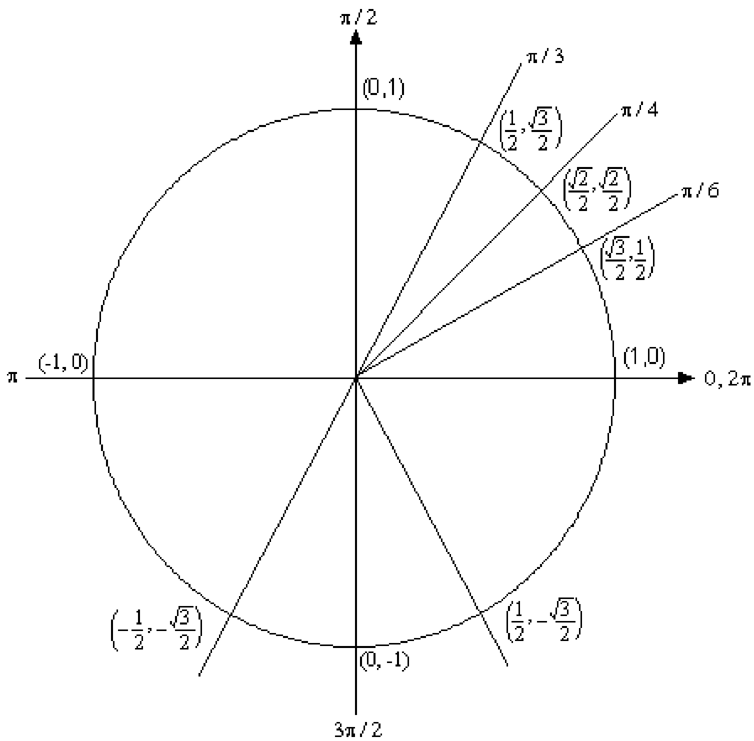
So, let's see if you've got all this down.

Example 3 Solve $2\sin(5x) = -\sqrt{3}$ on $[-\pi, 2\pi]$

Solution
This problem is very similar to the other problems in this section with a very important difference. We'll start this problem in exactly the same way. We first need to find all possible solutions.

$$2\sin(5x) = -\sqrt{3}$$
$$\sin(5x) = \frac{-\sqrt{3}}{2}$$

So, we are looking for angles that will give $-\frac{\sqrt{3}}{2}$ out of the sine function. Let's again go to our trusty unit circle.



Now, there are no angles in the first quadrant for which sine has a value of $-\frac{\sqrt{3}}{2}$. However, there are two angles in the lower half of the unit circle for which sine will have a value of $-\frac{\sqrt{3}}{2}$.

So, what are these angles? We'll notice $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, so the angle in the third quadrant will be

$\frac{\pi}{3}$ below the ***negative*** *x*-axis or $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$. Likewise, the angle in the fourth quadrant will $\frac{\pi}{3}$ below the ***positive*** *x*-axis or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$. Remember that we're typically looking for positive angles between 0 and 2π.

Now we come to the very important difference between this problem and the previous problems in this section. The solution is **NOT**

$$x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

This is not the set of solutions because we are NOT looking for values of *x* for which $\sin\left(x\right) = -\frac{\sqrt{3}}{2}$, but instead we are looking for values of *x* for which $\sin\left(5x\right) = -\frac{\sqrt{3}}{2}$. Note the difference in the arguments of the sine function! One is *x* and the other is $5x$. This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

$$5x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$5x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Well, actually, that's not quite the solution. We are looking for values of *x* so divide everything by 5 to get.

$$x = \frac{4\pi}{15} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{\pi}{3} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Notice that we also divided the $2\pi n$ by 5 as well! This is important! If we don't do that you **WILL** miss solutions. For instance, take $n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{11\pi}{15}\right)\right) = \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

I'll leave it to you to verify my work showing they are solutions. However it makes the point. If you didn't divided the $2\pi n$ by 5 you would have missed these solutions!

Okay, now that we've gotten all possible solutions it's time to find the solutions on the given interval. We'll do this as we did in the previous problem. Pick values of *n* and get the solutions.

$n = 0$.

$$x = \frac{4\pi}{15} + \frac{2\pi(0)}{5} = \frac{4\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(0)}{5} = \frac{\pi}{3} < 2\pi$$

$n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi(1)}{5} = \frac{2\pi}{3} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(1)}{5} = \frac{11\pi}{15} < 2\pi$$

$n = 2$.

$$x = \frac{4\pi}{15} + \frac{2\pi(2)}{5} = \frac{16\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(2)}{5} = \frac{17\pi}{15} < 2\pi$$

$n = 3$.

$$x = \frac{4\pi}{15} + \frac{2\pi(3)}{5} = \frac{22\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(3)}{5} = \frac{23\pi}{15} < 2\pi$$

$n = 4$.

$$x = \frac{4\pi}{15} + \frac{2\pi(4)}{5} = \frac{28\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(4)}{5} = \frac{29\pi}{15} < 2\pi$$

$n = 5$.

$$x = \frac{4\pi}{15} + \frac{2\pi(5)}{5} = \frac{34\pi}{15} > 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(5)}{5} = \frac{35\pi}{15} > 2\pi$$

Okay, so we finally got past the right endpoint of our interval so we don't need any more positive *n*. Now let's take a look at the negative *n* and see what we've got.

$n = -1$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-1)}{5} = -\frac{2\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-1)}{5} = -\frac{\pi}{15} > -\pi$$

$n = -2$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-2)}{5} = -\frac{8\pi}{15} > -\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(-2)}{5} = -\frac{7\pi}{15} > -\pi$$

$n = -3$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-3)}{5} = -\frac{14\pi}{15} > -\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(-3)}{5} = -\frac{13\pi}{15} > -\pi$$

$n = -4$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-4)}{5} = -\frac{4\pi}{3} < -\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(-4)}{5} = -\frac{19\pi}{15} < -\pi$$

And we're now past the left endpoint of the interval. Sometimes, there will be many solutions as there were in this example. Putting all of this together gives the following set of solutions that lie in the given interval.

$$\frac{4\pi}{15}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{11\pi}{15}, \frac{16\pi}{15}, \frac{17\pi}{15}, \frac{22\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{29\pi}{15}$$
$$-\frac{\pi}{15}, -\frac{2\pi}{15}, -\frac{7\pi}{15}, -\frac{8\pi}{15}, -\frac{13\pi}{15}, -\frac{14\pi}{15}$$

Let's work another example.

Example 4 Solve $\sin(2x) = -\cos(2x)$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

Solution

This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

$$\sin(2x) = -\cos(2x)$$
$$\frac{\sin(2x)}{\cos(2x)} = -1$$
$$\tan(2x) = -1$$

So, solving $\sin(2x) = -\cos(2x)$ is the same as solving $\tan(2x) = -1$. At some level we didn't need to do this for this problem as all we're looking for is angles in which sine and cosine have the same value, but opposite signs. However, for other problems this won't be the case and we'll want to convert to tangent.

Looking at our trusty unit circle it appears that the solutions will be,

$$2x = \frac{3\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$2x = \frac{7\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by the 2 we get all possible solutions.

$$x = \frac{3\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{7\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now, let's determine the solutions that lie in the given interval.

n = 0.

$$x = \frac{3\pi}{8} + \pi(0) = \frac{3\pi}{8} < \frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(0) = \frac{7\pi}{8} < \frac{3\pi}{2}$$

n = 1.

$$x = \frac{3\pi}{8} + \pi(1) = \frac{11\pi}{8} < \frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(1) = \frac{15\pi}{8} > \frac{3\pi}{2}$$

Unlike the previous example only one of these will be in the interval. This will happen occasionally so don't always expect both answers from a particular *n* to work. Also, we should now check *n=2* for the first to see if it will be in or out of the interval. I'll leave it to you to check that it's out of the interval.

Now, let's check the negative *n*.

n = –1.

$$x = \frac{3\pi}{8} + \pi(-1) = -\frac{5\pi}{8} > -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(-1) = -\frac{\pi}{8} > -\frac{3\pi}{2}$$

n = –2.

$$x = \frac{3\pi}{8} + \pi(-2) = -\frac{13\pi}{8} < -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(-2) = -\frac{9\pi}{8} > -\frac{3\pi}{2}$$

Again, only one will work here. I'll leave it to you to verify that *n* = –3 will give two answers that are both out of the interval.

The complete list of solutions is then,

$$-\frac{9\pi}{8}, -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$$

Let’s work one more example so that I can make a point that needs to be understood when solving some trig equations.

Example 5 Solve $\cos(3x) = 2$.

Solution

This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$. Therefore, since cosine will never be greater that 1 it definitely can’t be 2. So **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

In this section we solved some simple trig equations. There are more complicated trig equations that we can solve so don’t leave this section with the feeling that there is nothing harder out there in the world to solve. In fact, we’ll see at least one of the more complicated problems in the next section. Also, every one of these problems came down to solutions involving one of the “common” or “standard” angles. Most trig equations won’t come down to one of those and will in fact need a calculator to solve. The next section is devoted to this kind of problem.

Review : Solving Trig Equations with Calculators, Part I

In the previous section we started solving trig equations. The only problem with the equations we solved in there is that they pretty much all had solutions that came from a handful of "standard" angles and of course there are many equations out there that simply don't. So, in this section we are going to take a look at some more trig equations, the majority of which will require the use of a calculator to solve (a couple won't need a calculator).

The fact that we are using calculators in this section does not however mean that the problems in the previous section aren't important. It is going to be assumed in this section that the basic ideas of solving trig equations are known and that we don't need to go back over them here. In particular, it is assumed that you can use a unit circle to help you find all answers to the equation (although the process here is a little different as we'll see) and it is assumed that you can find answers in a given interval. If you are unfamiliar with these ideas you should first go to the previous section and go over those problems.

Before proceeding with the problems we need to go over how our calculators work so that we can get the correct answers. Calculators are great tools but if you don't know how they work and how to interpret their answers you can get in serious trouble.

First, as already pointed out in previous sections, everything we are going to be doing here will be in radians so make sure that your calculator is set to radians before attempting the problems in this section. Also, we are going to use 4 decimal places of accuracy in the work here. You can use more if you want, but in this class we'll always use at least 4 decimal places of accuracy.

Next, and somewhat more importantly, we need to understand how calculators give answers to inverse trig functions. We didn't cover inverse trig functions in this review, but they are just inverse functions and we have talked a little bit about inverse functions in a review section. The only real difference is that we are now using trig functions. We'll only be looking at three of them and they are:

$$\begin{aligned} &\text{Inverse Cosine} \;:\; \cos^{-1}(x) = \arccos(x) \\ &\text{Inverse Sine} \;\;\;\;:\; \sin^{-1}(x) = \arcsin(x) \\ &\text{Inverse Tangent} : \tan^{-1}(x) = \arctan(x) \end{aligned}$$

As shown there are two different notations that are commonly used. In these notes we'll be using the first form since it is a little more compact. Most calculators these days will have buttons on them for these three so make sure that yours does as well.

We now need to deal with how calculators give answers to these. Let's suppose, for example, that we wanted our calculator to compute $\cos^{-1}\left(\frac{3}{4}\right)$. First, remember that what the calculator is actually computing is the angle, let's say *x*, that we would plug into cosine to get a value of $\frac{3}{4}$, or

$$x = \cos^{-1}\left(\frac{3}{4}\right) \qquad \Rightarrow \qquad \cos\left(x\right) = \frac{3}{4}$$

So, in other words, when we are using our calculator to compute an inverse trig function we are really solving a simple trig equation.

Having our calculator compute $\cos^{-1}\left(\frac{3}{4}\right)$ and hence solve $\cos\left(x\right) = \frac{3}{4}$ gives,

$$x = \cos^{-1}\left(\frac{3}{4}\right) = 0.7227$$

From the previous section we know that there should in fact be an infinite number of answers to this including a second angle that is in the interval $\left[0, 2\pi\right]$. However, our calculator only gave us a single answer. How to determine what the other angles are will be covered in the following examples so we won't go into detail here about that. We did need to point out however, that the calculators will only give a single answer and that we're going to have more work to do than just plugging a number into a calculator.

Since we know that there are supposed to be an infinite number of solutions to $\cos\left(x\right) = \frac{3}{4}$ the next question we should ask then is just how did the calculator decide to return the answer that it did? Why this one and not one of the others? Will it give the same answer every time?

There are rules that determine just what answer the calculator gives. All calculators will give answers in the following ranges.

$$0 \le \cos^{-1}\left(x\right) \le \pi \qquad -\frac{\pi}{2} \le \sin^{-1}\left(x\right) \le \frac{\pi}{2} \qquad -\frac{\pi}{2} < \tan^{-1}\left(x\right) < \frac{\pi}{2}$$

If you think back to the unit circle and recall that we think of cosine as the horizontal axis the we can see that we'll cover all possible values of cosine in the upper half of the circle and this is exactly the range give above for the inverse cosine. Likewise, since we think of sine as the vertical axis in the unit circle we can see that we'll cover all possible values of sine in the right half of the unit circle and that is the range given above.

For the tangent range look back to the graph of the tangent function itself and we'll see that one branch of the tangent is covered in the range given above and so that is the range we'll use for inverse tangent. Note as well that we don't include the endpoints in the range for inverse tangent since tangent does not exist there.

So, if we can remember these rules we will be able to determine the remaining angle in $\left[0, 2\pi\right]$ that also works for each solution.

As a final quick topic let’s note that it will, on occasion, be useful to remember the decimal representations of some basic angles. So here they are,

$$\frac{\pi}{2} = 1.5708 \qquad \pi = 3.1416 \qquad \frac{3\pi}{2} = 4.7124 \qquad 2\pi = 6.2832$$

Using these we can quickly see that $\cos^{-1}\left(\frac{3}{4}\right)$ must be in the first quadrant since 0.7227 is between 0 and 1.5708. This will be of great help when we go to determine the remaining angles

So, once again, we can’t stress enough that calculators are great tools that can be of tremendous help to us, but it you don’t understand how they work you will often get the answers to problems wrong.

So, with all that out of the way let’s take a look at our first problem.

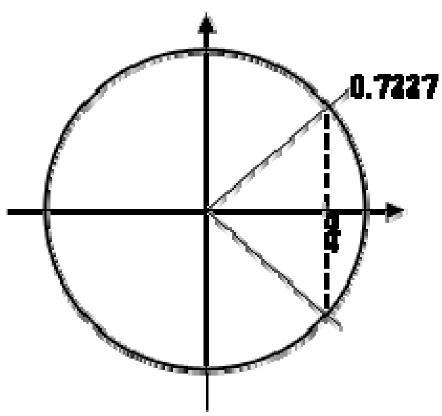
Example 1 Solve $4\cos\left(t\right) = 3$ on[-8,10].

Solution
Okay, the first step here is identical to the problems in the previous section. We first need to isolate the cosine on one side by itself and then use our calculator to get the first answer.

$$\cos\left(t\right) = \frac{3}{4} \qquad \Rightarrow \qquad t = \cos^{-1}\left(\frac{3}{4}\right) = 0.7227$$

So, this is the one we were using above in the opening discussion of this section. At the time we mentioned that there were infinite number of answers and that we’d be seeing how to find them later. Well that time is now.

First, let’s take a quick look at a unit circle for this example.



The angle that we’ve found is shown on the circle as well as the other angle that we know should also be an answer. Finding this angle here is just as easy as in the previous section. Since the line segment in the first quadrant forms an angle of 0.7227 radians with the positive *x*-axis then so does the line segment in the fourth quadrant. This means that we can use either -0.7227 as the second angle or $2\pi - 0.7227 = 5.5605$. Which you use depends on which you prefer. We'll pretty much always use the positive angle to avoid the possibility that we'll lose the minus sign.

So, all possible solutions, ignoring the interval for a second, are then,

$$\begin{aligned} t &= 0.7227 + 2\pi n \\ t &= 5.5605 + 2\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now, all we need to do is plug in values of *n* to determine the angle that are actually in the interval. Here's the work for that.

$$\begin{array}{llll} n = -2: & t = \cancel{-11.8437} & \text{and} & -7.0059 \\ n = -1: & t = -5.5605 & \text{and} & -0.7227 \\ n = 0: & t = 0.7227 & \text{and} & 5.5605 \\ n = 1: & t = 7.0059 & \text{and} & \cancel{11.8437} \end{array}$$

So, the solutions to this equation, in the given interval, are,

$$t = -7.0059,\ -5.5605,\ -0.7227,\ 0.7227,\ 5.5605,\ 7.0059$$

Note that we had a choice of angles to use for the second angle in the previous example. The choice of angles there will also affect the value(s) of *n* that we'll need to use to get all the solutions. In the end, regardless of the angle chosen, we'll get the same list of solutions, but the value(s) of *n* that give the solutions will be different depending on our choice.

Also, in the above example we put in a little more explanation than we'll show in the remaining examples in this section to remind you how these work.

Example 2 Solve $-10\cos(3t) = 7$ on [-2,5].

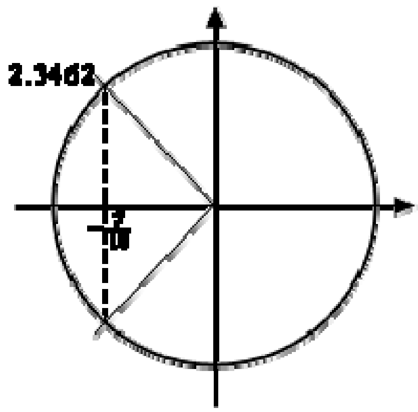
Solution

Okay, let's first get the inverse cosine portion of this problem taken care of.

$$\cos(3t) = -\frac{7}{10} \qquad \Rightarrow \qquad 3t = \cos^{-1}\left(-\frac{7}{10}\right) = 2.3462$$

Don't forget that we still need the "3"!

Now, let's look at a quick unit circle for this problem. As we can see the angle 2.3462 radians is in the second quadrant and the other angle that we need is in the third quadrant. We can find this second angle in exactly the same way we did in the previous example. We can use either -2.3462 or we can use $2\pi - 2.3462 = 3.9370$. As with the previous example we'll use the positive choice, but that is purely a matter of preference. You could use the negative if you wanted to.



So, let's now finish out the problem. First, let's acknowledge that the values of 3*t* that we need are,

$$\begin{aligned} 3t &= 2.3462 + 2\pi n \\ 3t &= 3.9370 + 2\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now, we need to properly deal with the 3, so divide that out to get all the solutions to the trig equation.

$$\begin{aligned} t &= 0.7821 + \frac{2\pi n}{3} \\ t &= 1.3123 + \frac{2\pi n}{3} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, we need to get the values in the given interval.

$n = -2$:	$t =$ ~~-3.4067~~	and	~~-2.8765~~
$n = -1$:	$t = -1.3123$	and	-0.7821
$n = 0$:	$t = 0.7821$	and	1.3123
$n = 1$:	$t = 2.8765$	and	3.4067
$n = 2$:	$t = 4.9709$	and	~~5.5011~~

The solutions to this equation, in the given interval are then,

$$t = -1.3123,\ -0.7821,\ 0.7821,\ 1.3123,\ 2.8765,\ 3.4067,\ 4.9709$$

We've done a couple of basic problems with cosines, now let's take a look at how solving equations with sines work.

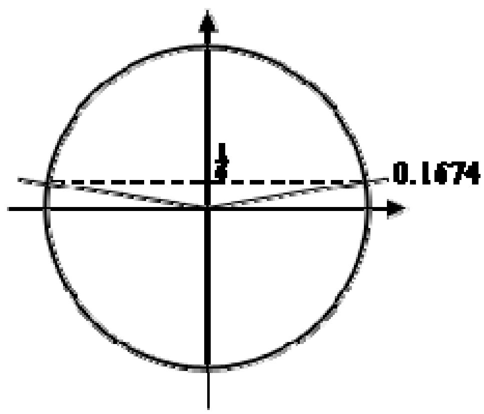
Example 3 Solve $6\sin\left(\frac{x}{2}\right)=1$ on [-20,30]

Solution

Let's first get the calculator work out of the way since that isn't where the difference comes into play.

$$\sin\left(\frac{x}{2}\right)=\frac{1}{6} \qquad \Rightarrow \qquad \frac{x}{2}=\sin^{-1}\left(\frac{1}{6}\right)=0.1674$$

Here's a unit circle for this example.



To find the second angle in this case we can notice that the line in the first quadrant makes an angle of 0.1674 with the positive *x*-axis and so the angle in the second quadrant will then make an angle of 0.1674 with the negative *x*-axis and so the angle that we're after is then, $\pi-0.1674=2.9742$.

Here's the rest of the solution for this example. We're going to assume from this point on that you can do this work without much explanation.

$$\begin{aligned}\frac{x}{2}&=0.1674+2\pi n\\ \frac{x}{2}&=2.9742+2\pi n\end{aligned} \qquad \Rightarrow \qquad \begin{aligned}x&=0.3348+4\pi n\\ x&=5.9484+4\pi n\end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

$$\begin{array}{lll} n=-1 : & x=\cancel{-24.7980} & \text{and} \quad -19.1844\\ n=0 : & x=0.3348 & \text{and} \quad 5.9484\\ n=1 : & x=25.4676 & \text{and} \quad \cancel{31.0812}\end{array}$$

The solutions to this equation are then,

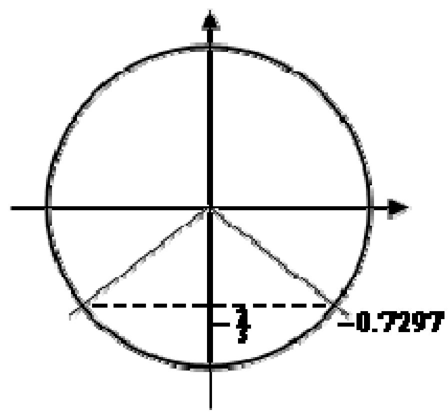
$$x=-19.1844,\ 0.3348,\ 5.9484,\ 25.4676$$

Example 4 Solve $3\sin\left(5z\right) = -2$ on [0,1].

Solution

You should be getting pretty good at these by now, so we won't be putting much explanation in for this one. Here we go.

$$\sin\left(5z\right) = -\frac{2}{3} \qquad \Rightarrow \qquad 5z = \sin^{-1}\left(-\frac{2}{3}\right) = -0.7297$$



Okay, with this one we're going to do a little more work than with the others. For the first angle we could use the answer our calculator gave us. However, it's easy to lose minus signs so we'll instead use $2\pi - 0.7297 = 5.5535$. Again, there is no reason to this other than a worry about losing the minus sign in the calculator answer. If you'd like to use the calculator answer you are more than welcome to. For the second angle we'll note that the lines in the third and fourth quadrant make an angle of 0.7297 with the *x*-axis and so the second angle is $\pi + 0.7297 = 3.8713$.

Here's the rest of the work for this example.

$$\begin{aligned} 5z &= 5.5535 + 2\pi n \\ 5z &= 3.8713 + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} z &= 1.1107 + \frac{2\pi n}{5} \\ z &= 0.7743 + \frac{2\pi n}{5} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$n = -1$: $\quad x =$ ~~-0.1460~~ and ~~-0.4823~~

$n = 0$: $\quad$ ~~$x = 1.1107$~~ and 0.7743

So, in this case we get a single solution of 0.7743.

Note that in the previous example we only got a single solution. This happens on occasion so don't get worried about it. Also, note that it was the second angle that gave this solution and so if we'd just relied on our calculator without worrying about other angles we would not have gotten this solution. Again, it can't be stressed enough that while calculators are a great tool if we don't understand how to correctly interpret/use the result we can (and often will) get the solution wrong.

To this point we've only worked examples involving sine and cosine. Let's no work a couple of examples that involve other trig functions to see how they work.

Example 5 Solve $9\sin(2x) = -5\cos(2x)$ on[-10,0].

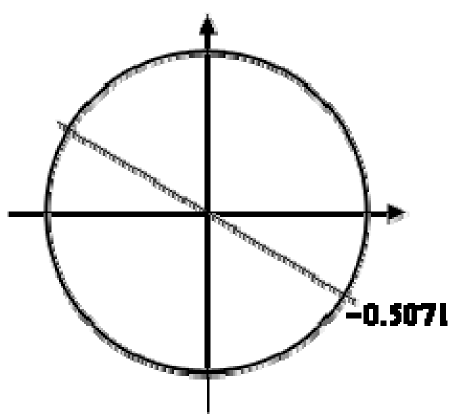
Solution
At first glance this problem seems to be at odds with the sentence preceding the example. However, it really isn't.

First, when we have more than one trig function in an equation we need a way to get equations that only involve one trig function. There are many ways of doing this that depend on the type of equation we're starting with. In this case we can simply divide both sides by a cosine and we'll get a single tangent in the equation. We can now see that this really is an equation that doesn't involve a sine or a cosine.

So, let's get started on this example.

$$\frac{\sin(2x)}{\cos(2x)} = \tan(2x) = -\frac{5}{9} \qquad \Rightarrow \qquad 2x = \tan^{-1}\left(-\frac{5}{9}\right) = -0.5071$$

Now, the unit circle doesn't involve tangents, however we can use it to illustrate the second angle in the range $[0, 2\pi]$.



The angles that we're looking for here are those whose quotient of $\frac{\text{sine}}{\text{cosine}}$ is the same. The second angle were we will get the same value of tangent will be exactly opposite of the given point. For this angle the values of sine and cosine are the same except they will have opposite signs. In the quotient however, the difference in signs will cancel out and we'll get the same value of tangent. So, the second angle will always be the first angle plus π.

Before getting the second angle let's also note that, like the previous example, we'll use the $2\pi - 0.5071 = 5.7761$ for the first angle. Again, this is only because of a concern about losing track of the minus sign in our calculator answer. We could just as easily do the work with the original angle our calculator gave us.

Now, this is where is seems like we're just randomly making changes and doing things for no reason. The second angle that we're going to use is,

$$\pi + \left(-0.5071\right) = \pi - 0.5071 = 2.6345$$

The fact that we used the calculator answer here seems to contradict the fact that we used a different angle for the first above. The reason for doing this here is to give a second angle that is in the range $\left[0, 2\pi\right]$. Had we used 5.7761 to find the second angle we'd get $\pi + 5.7761 = 8.9177$. This is a perfectly acceptable answer, however it is larger than 2π (6.2832) and the general rule of thumb is to keep the initial angles as small as possible.

Here are all the solutions to the equation.

$$\begin{aligned} 2x &= 5.7761 + 2\pi n \\ 2x &= 2.6345 + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= 2.8881 + \pi n \\ x &= 1.3173 + \pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\begin{array}{lllll} n = -2 : & & x = -9.6783 & \text{and} & \cancel{-11.2491} \\ n = -1 : & & x = -3.3951 & \text{and} & -4.9659 \\ n = 0 \ \ : & & \cancel{x = 2.8881} & \text{and} & \cancel{1.3173} \end{array}$$

The three solutions to this equation are then,

$$-3.3951, \ -4.9659, \ -9.6783$$

Note as well that we didn't need to do the $n = 0$ and computation since we could see from the given interval that we only wanted negative answers and these would clearly give positive answers.

Most calculators today can only do inverse sine, inverse cosine, and inverse tangent. So, let's see an example that uses one of the other trig functions.

Example 6 Solve $7\sec(3t) = -10$.

Solution
We'll start this one in exactly the same way we've done all the others.

$$\sec(3t) = -\frac{10}{7} \qquad \Rightarrow \qquad 3t = \sec^{-1}\left(-\frac{10}{7}\right)$$

Now we reach the problem. As noted above, most calculators can't handle inverse secant so we're going to need a different solution method for this one. To finish the solution here we'll simply recall the definition of secant in terms of cosine and convert this into an equation involving cosine instead and we already know how to solve those kinds of trig equations.

$$\frac{1}{\cos(3t)} = \sec(3t) = -\frac{10}{7} \qquad \Rightarrow \qquad \cos(3t) = -\frac{7}{10}$$

Now, we solved this equation in the second example above so we won't redo our work here. The solution is,

$$\begin{aligned} t &= 0.7821 + \frac{2\pi n}{3} \\ t &= 1.3123 + \frac{2\pi n}{3} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

We weren't given an interval in this problem so here is nothing else to do here.

For the remainder of the examples in this section we're not going to be finding solutions in an interval to save some space. If you followed the work from the first few examples in which we were given intervals you should be able to do any of the remaining examples if given an interval.

Also, we will no longer be including sketches of unit circles in the remaining solutions. We are going to assume that you can use the above sketches as guides for sketching unit circles to verify our claims in the following examples.

The next three examples don't require a calculator but are important enough or cause enough problems for students to include in this section in case you run across them and haven't seen them anywhere else.

Example 7 Solve $\cos(4\theta) = -1$.

Solution
There really isn't too much to do with this problem. It is, however, different from all the others

done to this point. All the others done to this point have had two angles in the interval $\left[0, 2\pi\right]$ that were solutions to the equation. This only has one. Here is the solution to this equation.

$$4\theta = \pi + 2\pi n \qquad \Rightarrow \qquad \theta = \frac{\pi}{4} + \frac{\pi n}{2} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Example 8 Solve $\sin\left(\frac{\alpha}{7}\right) = 0$.

Solution
Again, not much to this problem. Using a unit circle it isn't too hard to see that the solutions to this equation are,

$$\begin{aligned} \frac{\alpha}{7} &= 0 + 2\pi n \\ \frac{\alpha}{7} &= \pi + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \alpha &= 14\pi n \\ \alpha &= 7\pi + 14\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

This next example has an important point that needs to be understood when solving some trig equations.

Example 9 Solve $\sin\left(3t\right) = 2$.

Solution
This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \sin\left(\theta\right) \le 1$ and $-1 \le \cos\left(\theta\right) \le 1$. Therefore, since sine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

Because this document is also being prepared for viewing on the web we're going to split this section in two in order to keep the page size (and hence load time in a browser) to a minimum. In the next section we're going to take a look at some slightly more "complicated" equations. Although, as you'll see, they aren't as complicated as they may at first seem.

Review : Solving Trig Equations with Calculators, Part II

Because this document is also being prepared for viewing on the web we split this section into two parts to keep the size of the pages to a minimum.

Also, as with the last few examples in the previous part of this section we are not going to be looking for solutions in an interval in order to save space. The important part of these examples is to find the solutions to the equation. If we'd been given an interval it would be easy enough to find the solutions that actually fall in the interval.

In all the examples in the previous section all the arguments, the $3t$, $\frac{\alpha}{7}$, *etc*., were fairly simple. Let's take a look at an example that has a slightly more complicated argument.

Example 1 Solve $5\cos\left(2x-1\right)=-3$.

Solution
Note that the argument here is not really all that complicated but the addition of the "-1" often seems to confuse people so we need to a quick example with this kind of argument. The solution process is identical to all the problems we've done to this point so we won't be putting in much explanation. Here is the solution.

$$\cos\left(2x-1\right)=-\frac{3}{5} \qquad \Rightarrow \qquad 2x-1=\cos^{-1}\left(-\frac{3}{5}\right)=2.2143$$

This angle is in the second quadrant and so we can use either -2.2143 or $2\pi-2.2143=4.0689$ for the second angle. As usual for these notes we'll use the positive one. Therefore the two angles are,

$$\begin{aligned} 2x-1&=2.2143+2\pi n \\ 2x-1&=4.0689+2\pi n \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Now, we still need to find the actual values of *x* that are the solutions. These are found in the same manner as all the problems above. We'll first add 1 to both sides and then divide by 2. Doing this gives,

$$\begin{aligned} x&=1.6072+\pi n \\ x&=2.5345+\pi n \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

So, in this example we saw an argument that was a little different from those seen previously, but not all that different when it comes to working the problems so don't get too excited about it.

We now need to move into a different type of trig equation. All of the trig equations solved to this point (the previous example as well as the previous section) were, in some way, more or less the "standard" trig equation that is usually solved in a trig class. There are other types of equations involving trig functions however that we need to take a quick look at. The remaining examples show some of these different kinds of trig equations.

Example 2 Solve $2\cos(6y)+11\cos(6y)\sin(3y)=0$.

Solution

So, this definitely doesn't look like any of the equations we've solved to this point and initially the process is different as well. First, notice that there is a $\cos(6y)$ in each term, so let's factor that out and see what we have.

$$\cos(6y)\left(2+11\sin(3y)\right)=0$$

We now have a product of two terms that is zero and so we know that we must have,

$$\cos(6y)=0 \qquad \text{OR} \qquad 2+11\sin(3y)=0$$

Now, at this point we have two trig equations to solve and each is identical to the type of equation we were solving earlier. Because of this we won't put in much detail about solving these two equations.

First, solving $\cos(6y)=0$ gives,

$$\begin{aligned} 6y &= \frac{\pi}{2}+2\pi n \\ 6y &= \frac{3\pi}{2}+2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} y &= \frac{\pi}{12}+\frac{\pi n}{3} \\ y &= \frac{\pi}{4}+\frac{\pi n}{3} \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Next, solving $2+11\sin(3y)=0$ gives,

$$\begin{aligned} 3y &= 6.1004+2\pi n \\ 3y &= 3.3244+2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} y &= 2.0335+\frac{2\pi n}{3} \\ y &= 1.1081+\frac{2\pi n}{3} \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Remember that in these notes we tend to take positive angles and so the first solution here is in fact $2\pi-0.1828$ where our calculator gave us -0.1828 as the answer when using the inverse sine function.

The solutions to this equation are then,

$$y = \frac{\pi}{12} + \frac{\pi n}{3}$$
$$y = \frac{\pi}{4} + \frac{\pi n}{3} \qquad\qquad n = 0, \pm 1, \pm 2, \ldots$$
$$y = 2.0335 + \frac{2\pi n}{3}$$
$$y = 1.1081 + \frac{2\pi n}{3}$$

This next example also involves "factoring" trig equations but in a slightly different manner than the previous example.

Example 3 Solve $4\sin^2\left(\frac{t}{3}\right) - 3\sin\left(\frac{t}{3}\right) = 1$.

Solution
Before solving this equation let's solve an apparently unrelated equation.

$$4x^2 - 3x = 1 \qquad \Rightarrow \qquad 4x^2 - 3x - 1 = (4x+1)(x-1) = 0 \qquad \Rightarrow \qquad x = -\frac{1}{4}, 1$$

This is an easy (or at least I hope it's easy as this point) equation to solve. The obvious question then is, why did we do this? We'll, if you compare the two equations you'll see that the only real difference is that the one we just solved has an *x* everywhere the equation we want to solve has a sine. What this tells us is that we can work the two equations in exactly the same way.

We, will first "factor" the equation as follows,

$$4\sin^2\left(\frac{t}{3}\right) - 3\sin\left(\frac{t}{3}\right) - 1 = \left(4\sin\left(\frac{t}{3}\right) + 1\right)\left(\sin\left(\frac{t}{3}\right) - 1\right) = 0$$

Now, set each of the two factors equal to zero and solve for the sine,

$$\sin\left(\frac{t}{3}\right) = -\frac{1}{4} \qquad\qquad \sin\left(\frac{t}{3}\right) = 1$$

We now have two trig equations that we can easily (hopefully…) solve at this point. We'll leave the details to you to verify that the solutions to each of these and hence the solutions to the original equation are,

$$t = 18.0915 + 6\pi n$$
$$t = 10.1829 + 6\pi n \qquad\qquad n = 0, \pm 1, \pm 2, \ldots$$
$$t = \frac{3\pi}{2} + 6\pi n$$

The first two solutions are from the first equation and the third solution is from the second equation.

Let's work one more trig equation that involves solving a quadratic equation. However, this time, unlike the previous example this one won't factor and so we'll need to use the quadratic formula.

Example 4 Solve $8\cos^2(1-x)+13\cos(1-x)-5=0$.

Solution
Now, as mentioned prior to starting the example this quadratic does not factor. However, that doesn't mean all is lost. We can solve the following equation with the quadratic formula (you do remember this and how to use it right?),

$$8t^2+13t-5=0 \qquad \Rightarrow \qquad t=\frac{-13\pm\sqrt{329}}{16}=0.3211,\ -1.9461$$

So, if we can use the quadratic formula on this then we can also use it on the equation we're asked to solve. Doing this gives us,

$$\cos(1-x)=0.3211 \qquad \text{OR} \qquad \cos(1-x)=-1.9461$$

Now, recall Example 9 from the previous section. In that example we noted that $-1\le\cos(\theta)\le1$ and so the second equation will have no solutions. Therefore, the solutions to the first equation will yield the only solutions to our original equation. Solving this gives the following set of solutions,

$$\begin{aligned} x&=-0.2439-2\pi n \\ x&=-4.0393-2\pi n \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Note that we did get some negative numbers here and that does seem to violate the general form that we've been using in most of these examples. However, in this case the "-" are coming about when we solved for *x* after computing the inverse cosine in our calculator.

There is one more example in this section that we need to work that illustrates another way in which factoring can arise in solving trig equations. This equation is also the only one where the variable appears both inside and outside of the trig equation. Not all equations in this form can be easily solved, however some can so we want to do a quick example of one.

Example 5 Solve $5x\tan(8x) = 3x$.

Solution

First, before we even start solving we need to make one thing clear. **DO NOT CANCEL AN *x* FROM BOTH SIDES!!!** While this may seem like a natural thing to do it **WILL** cause us to lose a solution here.

So, to solve this equation we'll first get all the terms on one side of the equation and then factor an x out of the equation. If we can cancel an x from all terms then it can be factored out. Doing this gives,

$$5x\tan(8x) - 3x = x\left(5\tan(8x) - 3\right) = 0$$

Upon factoring we can see that we must have either,

$$x = 0 \qquad \text{OR} \qquad \tan(8x) = \frac{3}{5}$$

Note that if we'd canceled the x we would have missed the first solution. Now, we solved an equation with a tangent in it in Example 5 of the previous section so we'll not go into the details of this solution here. Here is the solution to the trig equation.

$$\begin{aligned} x &= 0.0676 + \frac{\pi n}{4} \\ x &= 0.4603 + \frac{\pi n}{4} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

The complete set of solutions then to the original equation are,

$$\begin{aligned} x &= 0 \\ x &= 0.0676 + \frac{\pi n}{4} \\ x &= 0.4603 + \frac{\pi n}{4} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Review : Exponential Functions

In this section we're going to review one of the more common functions in both calculus and the sciences. However, before getting to this function let's take a much more general approach to things.

Let's start with $b > 0$, $b \neq 1$. An exponential function is then a function in the form,

$$f(x) = b^x$$

Note that we avoid $b = 1$ because that would give the constant function, $f(x) = 1$. We avoid $b = 0$ since this would also give a constant function and we avoid negative values of *b* for the following reason. Let's, for a second, suppose that we did allow *b* to be negative and look at the following function.

$$g(x) = (-4)^x$$

Let's do some evaluation.

$$g(2) = (-4)^2 = 16 \qquad\qquad g\left(\frac{1}{2}\right) = -(-4)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

So, for some values of *x* we will get real numbers and for other values of *x* we well get complex numbers. We want to avoid this and so if we require $b > 0$ this will not be a problem.

Let's take a look at a couple of exponential functions.

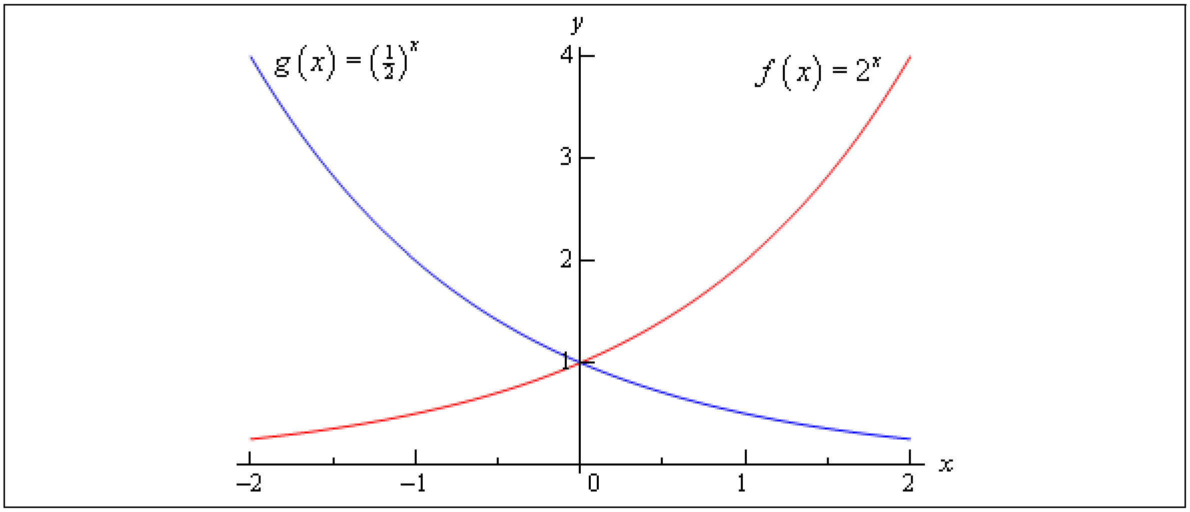
Example 1 Sketch the graph of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$

Solution

Let's first get a table of values for these two functions.

x	*f(x)*	*g(x)*
-2	$f(-2) = 2^{-2} = \frac{1}{4}$	$g(-2) = \left(\frac{1}{2}\right)^{-2} = 4$
-1	$f(-1) = 2^{-1} = \frac{1}{2}$	$g(-1) = \left(\frac{1}{2}\right)^{-1} = 2$
0	$f(0) = 2^0 = 1$	$g(0) = \left(\frac{1}{2}\right)^0 = 1$
1	$f(1) = 2$	$g(1) = \frac{1}{2}$
2	$f(2) = 4$	$g(2) = \frac{1}{4}$

Here's the sketch of both of these functions.



This graph illustrates some very nice properties about exponential functions in general.

Properties of $f(x) = b^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$.
2. $f(x) \neq 0$. An exponential function will never be zero.
3. $f(x) > 0$. An exponential function is always positive.
4. The previous two properties can be summarized by saying that the range of an exponential function is $(0, \infty)$.
5. The domain of an exponential function is $(-\infty, \infty)$. In other words, you can plug every x into an exponential function.
6. If $0 < b < 1$ then,
   a. $f(x) \to 0$ as $x \to \infty$
   b. $f(x) \to \infty$ as $x \to -\infty$
7. If $b > 1$ then,
   a. $f(x) \to \infty$ as $x \to \infty$
   b. $f(x) \to 0$ as $x \to -\infty$

These will all be very useful properties to recall at times as we move throughout this course (and later Calculus courses for that matter…).

There is a very important exponential function that arises naturally in many places. This function is called the **natural exponential function**. However, for must people this is simply the exponential function.

Definition : The **natural exponential function** is $f(x) = \mathbf{e}^x$ where,

$\mathbf{e} = 2.71828182845905\ldots$.

So, since $\mathbf{e} > 1$ we also know that $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

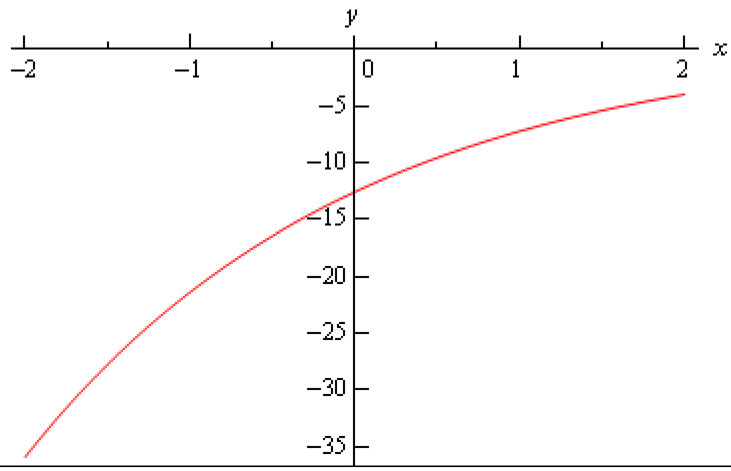
Let's take a quick look at an example.

Example 2 Sketch the graph of $h(t) = 1 - 5\mathbf{e}^{1-\frac{t}{2}}$

Solution
Let's first get a table of values for this function.

t	-2	-1	0	1	2	3
$h(t)$	-35.9453	-21.4084	-12.5914	-7.2436	-4	-2.0327

Here is the sketch.



The main point behind this problem is to make sure you can do this type of evaluation so make sure that you can get the values that we graphed in this example. You will be asked to do this kind of evaluation on occasion in this class.

You will be seeing exponential functions in pretty much every chapter in this class so make sure that you are comfortable with them.

Review : Logarithm Functions

In this section we'll take a look at a function that is related to the exponential functions we looked at in the last section. We will look logarithms in this section. Logarithms are one of the functions that students fear the most. The main reason for this seems to be that they simply have never really had to work with them. Once they start working with them, students come to realize that they aren't as bad as they first thought.

We'll start with $b > 0$, $b \neq 1$ just as we did in the last section. Then we have

$$y = \log_b x \qquad \text{is equivalent to} \qquad x = b^y$$

The first is called logarithmic form and the second is called the exponential form. Remembering this equivalence is the key to evaluating logarithms. The number, *b*, is called the base.

Example 1 Without a calculator give the exact value of each of the following logarithms.

(a) $\log_2 16$ [Solution]

(b) $\log_4 16$ [Solution]

(c) $\log_5 625$ [Solution]

(d) $\log_9 \dfrac{1}{531441}$ [Solution]

(e) $\log_{\frac{1}{6}} 36$ [Solution]

(f) $\log_{\frac{3}{2}} \dfrac{27}{8}$ [Solution]

Solution

To quickly evaluate logarithms the easiest thing to do is to convert the logarithm to exponential form. So, let's take a look at the first one.

(a) $\log_2 16$

First, let's convert to exponential form.

$$\log_2 16 = ? \qquad \text{is equivalent to} \qquad 2^? = 16$$

So, we're really asking 2 raised to what gives 16. Since 2 raised to 4 is 16 we get,

$$\log_2 16 = 4 \qquad \text{because} \qquad 2^4 = 16$$

We'll not do the remaining parts in quite this detail, but they were all worked in this way.

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(b) $\log_4 16$

$$\log_4 16 = 2 \qquad \text{because} \qquad 4^2 = 16$$

Note the difference the first and second logarithm! The base is important! It can completely change the answer.

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