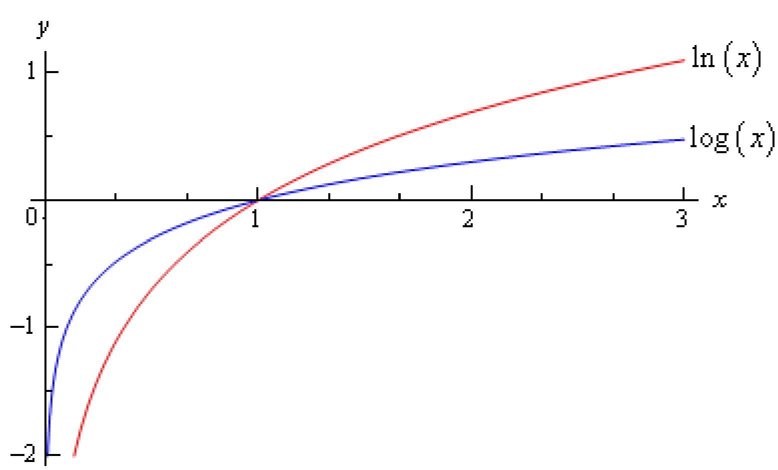
(c) $\log_5 625 = 4$	because	$5^4 = 625$
		[Return to Problems]
(d) $\log_9 \frac{1}{531441} = -6$	because	$9^{-6} = \frac{1}{9^6} = \frac{1}{531441}$
		[Return to Problems]
(e) $\log_{\frac{1}{6}} 36 = -2$	because	$\left(\frac{1}{6}\right)^{-2} = 6^2 = 36$
		[Return to Problems]
(f) $\log_{\frac{3}{2}} \frac{27}{8} = 3$	because	$\left(\frac{3}{2}\right)^3 = \frac{27}{8}$
		[Return to Problems]

There are a couple of special logarithms that arise in many places. These are,

$$\ln x = \log_{\mathbf{e}} x \qquad \text{This log is called the natural logarithm}$$
$$\log x = \log_{10} x \qquad \text{This log is called the common logarithm}$$

In the natural logarithm the base **e** is the same number as in the natural exponential logarithm that we saw in the last section. Here is a sketch of both of these logarithms.



From this graph we can get a couple of very nice properties about the natural logarithm that we will use many times in this and later Calculus courses.

$$\ln x \to \infty \quad \text{as} \quad x \to \infty$$
$$\ln x \to -\infty \quad \text{as} \quad x \to 0,\ x > 0$$

Let's take a look at a couple of more logarithm evaluations. Some of which deal with the natural or common logarithm and some of which don't.

Example 2 Without a calculator give the exact value of each of the following logarithms.

(a) $\ln \sqrt[3]{\mathbf{e}}$

(b) $\log 1000$

(c) $\log_{16} 16$

(d) $\log_{23} 1$

(e) $\log_{2} \sqrt[7]{32}$

Solution

These work exactly the same as previous example so we won't put in too many details.

(a) $\ln \sqrt[3]{\mathbf{e}} = \frac{1}{3}$ because $\mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$

(b) $\log 1000 = 3$ because $10^3 = 1000$

(c) $\log_{16} 16 = 1$ because $16^1 = 16$

(d) $\log_{23} 1 = 0$ because $23^0 = 1$

(e) $\log_{2} \sqrt[7]{32} = \frac{5}{7}$ because $\sqrt[7]{32} = 32^{\frac{1}{7}} = \left(2^5\right)^{\frac{1}{7}} = 2^{\frac{5}{7}}$

This last set of examples leads us to some of the basic properties of logarithms.

Properties

1. The domain of the logarithm function is $(0, \infty)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. $\log_b b = 1$
3. $\log_b 1 = 0$
4. $\log_b b^x = x$
5. $b^{\log_b x} = x$

The last two properties will be especially useful in the next section. Notice as well that these last two properties tell us that,

$$f(x) = b^x \qquad \text{and} \qquad g(x) = \log_b x$$

are inverses of each other.

Here are some more properties that are useful in the manipulation of logarithms.

More Properties

6. $\log_b xy = \log_b x + \log_b y$
7. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
8. $\log_b \left(x^r\right) = r \log_b x$

Note that there is no equivalent property to the first two for sums and differences. In other words,

$$\log_b (x+y) \neq \log_b x + \log_b y$$
$$\log_b (x-y) \neq \log_b x - \log_b y$$

Example 3 Write each of the following in terms of simpler logarithms.

(a) $\ln x^3 y^4 z^5$ [Solution]

(b) $\log_3 \left(\frac{9x^4}{\sqrt{y}}\right)$ [Solution]

(c) $\log \left(\frac{x^2+y^2}{(x-y)^3}\right)$ [Solution]

Solution

What the instructions really mean here is to use as many if the properties of logarithms as we can to simplify things down as much as we can.

(a) $\ln x^3 y^4 z^5$

Property 6 above can be extended to products of more than two functions. Once we've used Property 6 we can then use Property 8.

$$\begin{aligned}\ln x^3 y^4 z^5 &= \ln x^3 + \ln y^4 + \ln z^5 \\ &= 3\ln x + 4\ln y + 5\ln z\end{aligned}$$

[Return to Problems]

(b) $\log_3 \left(\frac{9x^4}{\sqrt{y}}\right)$

When using property 7 above make sure that the logarithm that you subtract is the one that contains the denominator as its argument. Also, note that that we'll be converting the root to fractional exponents in the first step.

$$\begin{aligned}\log_3 \left(\frac{9x^4}{\sqrt{y}}\right) &= \log_3 9x^4 - \log_3 y^{\frac{1}{2}} \\ &= \log_3 9 + \log_3 x^4 - \log_3 y^{\frac{1}{2}} \\ &= 2 + 4\log_3 x - \frac{1}{2}\log_3 y\end{aligned}$$

[Return to Problems]

(c) $\log\left(\frac{x^2+y^2}{(x-y)^3}\right)$

The point to this problem is mostly the correct use of property 8 above.

$$\log\left(\frac{x^2+y^2}{(x-y)^3}\right) = \log\left(x^2+y^2\right) - \log\left(x-y\right)^3$$
$$= \log\left(x^2+y^2\right) - 3\log\left(x-y\right)$$

You can use Property 8 on the second term because the WHOLE term was raised to the 3, but in the first logarithm, only the individual terms were squared and not the term as a whole so the 2's must stay where they are!

[Return to Problems]

The last topic that we need to look at in this section is the **change of base** formula for logarithms. The change of base formula is,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

This is the most general change of base formula and will convert from base *b* to base *a*. However, the usual reason for using the change of base formula is to compute the value of a logarithm that is in a base that you can't easily deal with. Using the change of base formula means that you can write the logarithm in terms of a logarithm that you can deal with. The two most common change of base formulas are

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{and} \qquad \log_b x = \frac{\log x}{\log b}$$

In fact, often you will see one or the other listed as THE change of base formula!

In the first part of this section we computed the value of a few logarithms, but we could do these without the change of base formula because all the arguments could be written in terms of the base to a power. For instance,

$$\log_7 49 = 2 \qquad \text{because} \qquad 7^2 = 49$$

However, this only works because 49 can be written as a power of 7! We would need the change of base formula to compute $\log_7 50$.

$$\log_7 50 = \frac{\ln 50}{\ln 7} = \frac{3.91202300543}{1.94591014906} = 2.0103821378$$

OR

$$\log_7 50 = \frac{\log 50}{\log 7} = \frac{1.69897000434}{0.845098040014} = 2.0103821378$$

So, it doesn't matter which we use, we will get the same answer regardless of the logarithm that we use in the change of base formula.

Note as well that we could use the change of base formula on $\log_7 49$ if we wanted to as well.

$$\log_7 49 = \frac{\ln 49}{\ln 7} = \frac{3.89182029811}{1.94591014906} = 2$$

This is a lot of work however, and is probably not the best way to deal with this.

So, in this section we saw how logarithms work and took a look at some of the properties of logarithms. We will run into logarithms on occasion so make sure that you can deal with them when we do run into them.

Review : Exponential and Logarithm Equations

In this section we'll take a look at solving equations with exponential functions or logarithms in them.

We'll start with equations that involve exponential functions. The main property that we'll need for these equations is,

$$\log_b b^x = x$$

Example 1 Solve $7+15\mathbf{e}^{1-3z}=10$.

Solution

The first step is to get the exponential all by itself on one side of the equation with a coefficient of one.

$$\begin{aligned} 7+15\mathbf{e}^{1-3z} &= 10 \\ 15\mathbf{e}^{1-3z} &= 3 \\ \mathbf{e}^{1-3z} &= \frac{1}{5} \end{aligned}$$

Now, we need to get the z out of the exponent so we can solve for it. To do this we will use the property above. Since we have an **e** in the equation we'll use the natural logarithm. First we take the logarithm of both sides and then use the property to simplify the equation.

$$\begin{aligned} \ln\left(\mathbf{e}^{1-3z}\right) &= \ln\left(\frac{1}{5}\right) \\ 1-3z &= \ln\left(\frac{1}{5}\right) \end{aligned}$$

All we need to do now is solve this equation for z.

$$\begin{aligned} 1-3z &= \ln\left(\frac{1}{5}\right) \\ -3z &= -1+\ln\left(\frac{1}{5}\right) \\ z &= -\frac{1}{3}\left(-1+\ln\left(\frac{1}{5}\right)\right) = 0.8698126372 \end{aligned}$$

Example 2 Solve $10^{t^2-t}=100$.

Solution

Now, in this case it looks like the best logarithm to use is the common logarithm since left hand side has a base of 10. There's no initial simplification to do, so just take the log of both sides and simplify.

$$\log 10^{t^2-t} = \log 100$$
$$t^2 - t = 2$$

At this point, we've just got a quadratic that can be solved

$$t^2 - t - 2 = 0$$
$$(t-2)(t+1) = 0$$

So, it looks like the solutions in this case are $t = 2$ and $t = -1$.

Now that we've seen a couple of equations where the variable only appears in the exponent we need to see an example with variables both in the exponent and out of it.

Example 3 Solve $x - x\mathbf{e}^{5x+2} = 0$.

Solution

The first step is to factor an *x* out of both terms.

DO NOT DIVIDE AN *x* FROM BOTH TERMS!!!!

Note that it is very tempting to "simplify" the equation by dividing an *x* out of both terms. However, if you do that you'll miss a solution as we'll see.

$$x - x\mathbf{e}^{5x+2} = 0$$
$$x\left(1 - \mathbf{e}^{5x+2}\right) = 0$$

So, it's now a little easier to deal with. From this we can see that we get one of two possibilities.

$$x = 0 \qquad \text{OR}$$
$$1 - \mathbf{e}^{5x+2} = 0$$

The first possibility has nothing more to do, except notice that if we had divided both sides by an *x* we would have missed this one so be careful. In the second possibility we've got a little more to do. This is an equation similar to the first two that we did in this section.

$$\mathbf{e}^{5x+2} = 1$$
$$5x + 2 = \ln 1$$
$$5x + 2 = 0$$
$$x = -\frac{2}{5}$$

Don't forget that $\ln 1 = 0$!

So, the two solutions are $x = 0$ and $x = -\frac{2}{5}$.

The next equation is a more complicated (looking at least…) example similar to the previous one.

Example 4 Solve $5\left(x^2-4\right)=\left(x^2-4\right)\mathbf{e}^{7-x}$.

Solution

As with the previous problem do NOT divide an x^2-4 out of both sides. Doing this will lose solutions even though it "simplifies" the equation. Note however, that if you can divide a term out then you can also factor it out if the equation is written properly.

So, the first step here is to move everything to one side of the equation and then to factor out the x^2-4.

$$\begin{aligned} 5\left(x^2-4\right)-\left(x^2-4\right)\mathbf{e}^{7-x} &= 0 \\ \left(x^2-4\right)\left(5-\mathbf{e}^{7-x}\right) &= 0 \end{aligned}$$

At this point all we need to do is set each factor equal to zero and solve each.

$$\begin{aligned} x^2-4=0 \qquad & 5-\mathbf{e}^{7-x} = 0 \\ x=\pm 2 \qquad & \mathbf{e}^{7-x} = 5 \\ & 7-x = \ln\left(5\right) \\ & x = 7-\ln\left(5\right) = 5.390562088 \end{aligned}$$

The three solutions are then $x=\pm 2$ and $x=5.3906$.

As a final example let's take a look at an equation that contains two different logarithms.

Example 5 Solve $4\mathbf{e}^{1+3x}-9\mathbf{e}^{5-2x}=0$.

Solution

The first step here is to get one exponential on each side and then we'll divide both sides by one of them (which doesn't matter for the most part) so we'll have a quotient of two exponentials. The quotient can then be simplified and we'll finally get both coefficients on the other side. Doing all of this gives,

$$\begin{aligned} 4\mathbf{e}^{1+3x} &= 9\mathbf{e}^{5-2x} \\ \frac{\mathbf{e}^{1+3x}}{\mathbf{e}^{5-2x}} &= \frac{9}{4} \\ \mathbf{e}^{1+3x-(5-2x)} &= \tfrac{9}{4} \\ \mathbf{e}^{5x-4} &= \tfrac{9}{4} \end{aligned}$$

Note that while we said that it doesn't really matter which exponential we divide out by doing it the way we did here we'll avoid a negative coefficient on the *x*. Not a major issue, but those minus signs on coefficients are really easy to lose on occasion.

This is now in a form that we can deal with so here's the rest of the solution.

$$\begin{aligned} \mathbf{e}^{5x-4} &= \tfrac{9}{4} \\ 5x-4 &= \ln\left(\tfrac{9}{4}\right) \\ 5x &= 4+\ln\left(\tfrac{9}{4}\right) \\ x &= \tfrac{1}{5}\left(4+\ln\left(\tfrac{9}{4}\right)\right) = 0.9621860432 \end{aligned}$$

This equation has a single solution of $x = 0.9622$.

Now let's take a look at some equations that involve logarithms. The main property that we'll be using to solve these kinds of equations is,

$$b^{\log_b x} = x$$

Example 6 Solve $3+2\ln\left(\frac{x}{7}+3\right) = -4$.

Solution

This first step in this problem is to get the logarithm by itself on one side of the equation with a coefficient of 1.

$$2\ln\left(\frac{x}{7}+3\right) = -7$$

$$\ln\left(\frac{x}{7}+3\right) = -\frac{7}{2}$$

Now, we need to get the *x* out of the logarithm and the best way to do that is to "exponentiate" both sides using **e**. In other word,

$$\mathbf{e}^{\ln\left(\frac{x}{7}+3\right)} = \mathbf{e}^{-\frac{7}{2}}$$

So using the property above with **e**, since there is a natural logarithm in the equation, we get,

$$\frac{x}{7}+3 = \mathbf{e}^{-\frac{7}{2}}$$

Now all that we need to do is solve this for *x*.

$$\begin{aligned} \frac{x}{7}+3 &= \mathbf{e}^{-\frac{7}{2}} \\ \frac{x}{7} &= -3+\mathbf{e}^{-\frac{7}{2}} \\ x &= 7\left(-3+\mathbf{e}^{-\frac{7}{2}}\right) = -20.78861832 \end{aligned}$$

At this point we might be tempted to say that we're done and move on. However, we do need to be careful. Recall from the previous section that we can't plug a negative number into a logarithm. This, by itself, doesn't mean that our answer won't work since its negative. What we need to do is plug it into the logarithm and make sure that $\frac{x}{7}+3$ will not be negative. I'll leave it to you to verify that this is in fact positive upon plugging our solution into the logarithm and so $x=-20.78861832$ is in fact a solution to the equation.

Let's now take a look at a more complicated equation. Often there will be more than one logarithm in the equation. When this happens we will need to use on or more of the following properties to combine all the logarithms into a single logarithm. Once this has been done we can proceed as we did in the previous example.

$$\log_b xy = \log_b x + \log_b y \qquad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \qquad \log_b\left(x^r\right) = r\log_b x$$

Example 7 Solve $2\ln\left(\sqrt{x}\right)-\ln\left(1-x\right)=2$.

Solution
First get the two logarithms combined into a single logarithm.

$$\begin{aligned} 2\ln\left(\sqrt{x}\right)-\ln\left(1-x\right)&=2\\ \ln\left(\left(\sqrt{x}\right)^2\right)-\ln\left(1-x\right)&=2\\ \ln\left(x\right)-\ln\left(1-x\right)&=2\\ \ln\left(\frac{x}{1-x}\right)&=2 \end{aligned}$$

Now, exponentiate both sides and solve for *x*.

$$\begin{aligned} \frac{x}{1-x}&=\mathbf{e}^2\\ x&=\mathbf{e}^2\left(1-x\right)\\ x&=\mathbf{e}^2-\mathbf{e}^2x\\ x\left(1+\mathbf{e}^2\right)&=\mathbf{e}^2\\ x&=\frac{\mathbf{e}^2}{1+\mathbf{e}^2}=0.8807970780 \end{aligned}$$

Finally, we just need to make sure that the solution, $x=0.8807970780$, doesn't produce negative numbers in both of the original logarithms. It doesn't, so this is in fact our solution to this problem.

Let's take a look at one more example.

Example 8 Solve $\log x + \log(x-3) = 1$.

Solution
As with the last example, first combine the logarithms into a single logarithm.

$$\log x + \log(x-3) = 1$$
$$\log(x(x-3)) = 1$$

Now exponentiate, using 10 this time instead of **e** because we've got common logs in the equation, both sides.

$$10^{\log(x^2-3x)} = 10^1$$
$$x^2 - 3x = 10$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$

So, potential solutions are $x = 5$ and $x = -2$. Note, however that if we plug $x = -2$ into either of the two original logarithms we would get negative numbers so this can't be a solution. We can however, use $x = 5$.

Therefore, the solution to this equation is $x = 5$.

When solving equations with logarithms it is important to check your potential solutions to make sure that they don't generate logarithms of negative numbers or zero. It is also important to make sure that you do the checks in the **original** equation. If you check them in the second logarithm above (after we've combined the two logs) both solutions will appear to work! This is because in combining the two logarithms we've actually changed the problem. In fact, it is this change that introduces the extra solution that we couldn't use!

Also be careful in solving equations containing logarithms to not get locked into the idea that you will get two potential solutions and only one of these will work. It is possible to have problems where both are solutions and where neither are solutions.

Review : Common Graphs

The purpose of this section is to make sure that you're familiar with the graphs of many of the basic functions that you're liable to run across in a calculus class.

Example 1 Graph $y = -\frac{2}{5}x + 3$.

Solution

This is a line in the slope intercept form

$$y = mx + b$$

In this case the line has a *y* intercept of (0,*b*) and a slope of *m*. Recall that slope can be thought of as

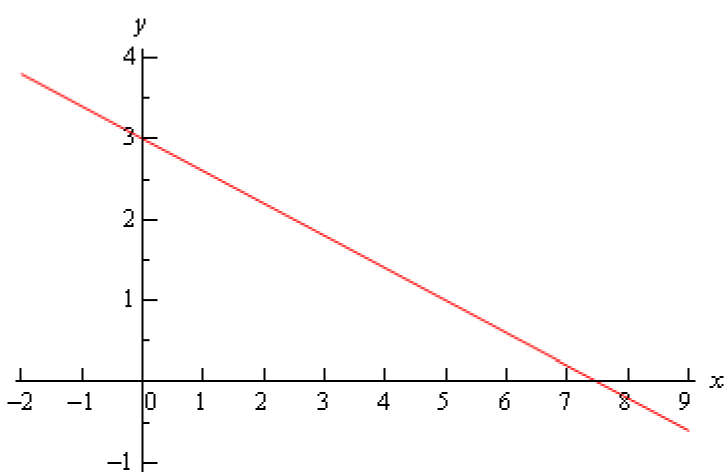
$$m = \frac{\text{rise}}{\text{run}}$$

Note that if the slope is negative we tend to think of the rise as a fall.

The slope allows us to get a second point on the line. Once we have any point on the line and the slope we move right by *run* and up/down by *rise* depending on the sign. This will be a second point on the line.

In this case we know (0,3) is a point on the line and the slope is $-\frac{2}{5}$. So starting at (0,3) we'll move 5 to the right (*i.e.* $0 \rightarrow 5$) and down 2 (*i.e.* $3 \rightarrow 1$) to get (5,1) as a second point on the line. Once we've got two points on a line all we need to do is plot the two points and connect them with a line.

Here's the sketch for this line.



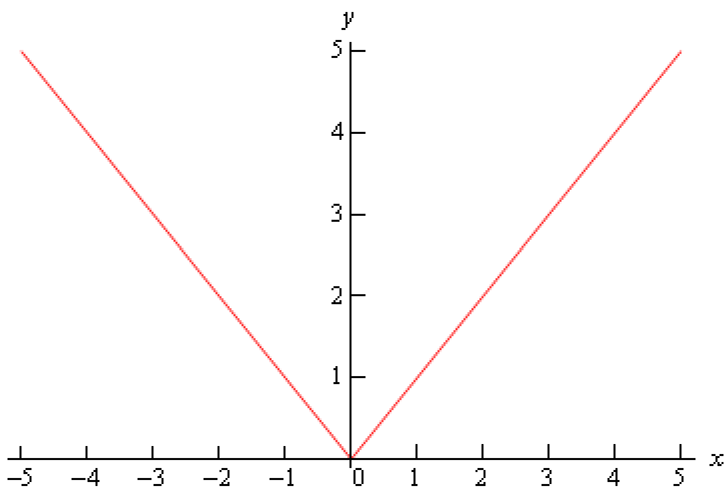
Example 2 Graph $f(x) = |x|$

Solution

There really isn't much to this problem outside of reminding ourselves of what absolute value is. Recall that the absolute value function is defined as,

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph is then,



Example 3 Graph $f(x) = -x^2 + 2x + 3$.

Solution

This is a parabola in the general form.

$$f(x) = ax^2 + bx + c$$

In this form, the *x*-coordinate of the vertex (the highest or lowest point on the parabola) is $x = -\frac{b}{2a}$ and we get the *y*-coordinate is $y = f\left(-\frac{b}{2a}\right)$. So, for our parabola the coordinates of the vertex will be.

$$x = -\frac{2}{2(-1)} = 1$$

$$y = f(1) = -(1)^2 + 2(1) + 3 = 4$$

So, the vertex for this parabola is (1,4).

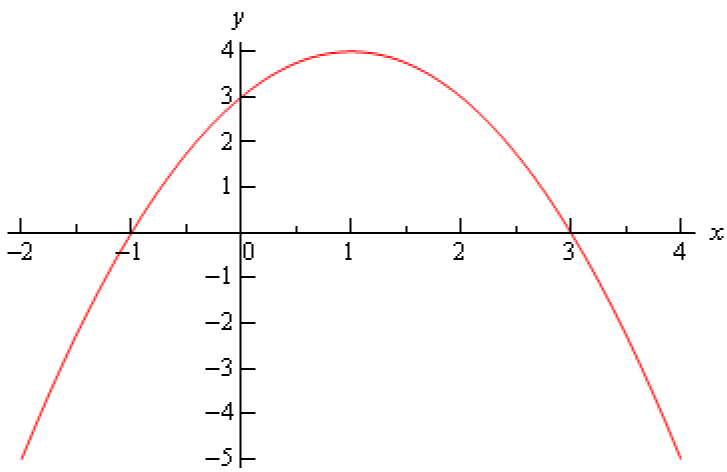
We can also determine which direction the parabola opens from the sign of *a*. If *a* is positive the parabola opens up and if *a* is negative the parabola opens down. In our case the parabola opens down.

Now, because the vertex is above the *x*-axis and the parabola opens down we know that we'll have *x*-intercepts (*i.e.* values of *x* for which we'll have $f(x)=0$) on this graph. So, we'll solve the following.

$$-x^2+2x+3=0$$
$$x^2-2x-3=0$$
$$(x-3)(x+1)=0$$

So, we will have *x*-intercepts at $x=-1$ and $x=3$. Notice that to make our life easier in the solution process we multiplied everything by -1 to get the coefficient of the x^2 positive. This made the factoring easier.

Here's a sketch of this parabola.



Example 4 Graph $f(y)=y^2-6y+5$

Solution

Most people come out of an Algebra class capable of dealing with functions in the form $y=f(x)$. However, many functions that you will have to deal with in a Calculus class are in the form $x=f(y)$ and can only be easily worked with in that form. So, you need to get used to working with functions in this form.

The nice thing about these kinds of function is that if you can deal with functions in the form $y=f(x)$ then you can deal with functions in the form $x=f(y)$ even if you aren't that familiar with them.

Let's first consider the equation.

$$y=x^2-6x+5$$

This is a parabola that opens up and has a vertex of (3,-4), as we know from our work in the previous example.

For our function we have essentially the same equation except the x and y's are switched around. In other words, we have a parabola in the form,

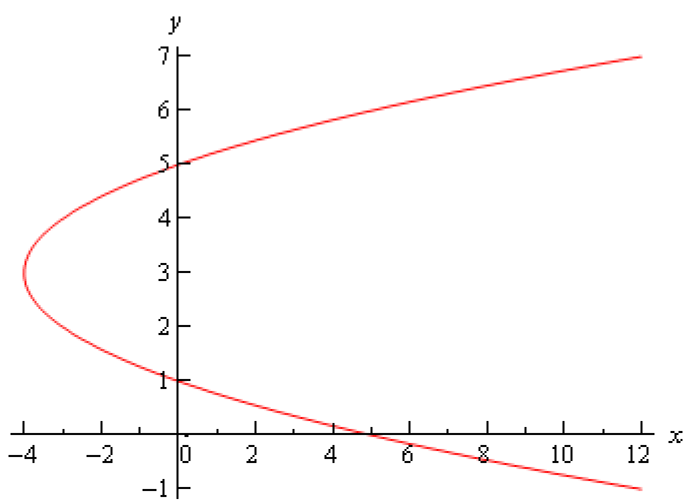
$$x = ay^2 + by + c$$

This is the general form of this kind of parabola and this will be a parabola that opens left or right depending on the sign of a. The y-coordinate of the vertex is given by $y = -\frac{b}{2a}$ and we find the x-coordinate by plugging this into the equation. So, you can see that this is very similar to the type of parabola that you're already used to dealing with.

Now, let's get back to the example. Our function is a parabola that opens to the right (a is positive) and has a vertex at (-4,3). The vertex is to the left of the y-axis and opens to the right so we'll need the y-intercepts (*i.e.* values of y for which we'll have $f(y) = 0$)). We find these just like we found x-intercepts in the previous problem.

$$y^2 - 6y + 5 = 0$$
$$(y-5)(y-1) = 0$$

So, our parabola will have y-intercepts at $y = 1$ and $y = 5$. Here's a sketch of the graph.



Example 5 Graph $x^2 + 2x + y^2 - 8y + 8 = 0$.

Solution

To determine just what kind of graph we've got here we need complete the square on both the x and the y.

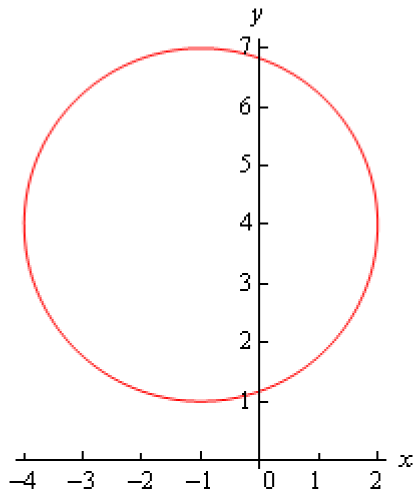
$$x^2 + 2x + y^2 - 8y + 8 = 0$$
$$x^2 + 2x + 1 - 1 + y^2 - 8y + 16 - 16 + 8 = 0$$
$$(x+1)^2 + (y-4)^2 = 9$$

Recall that to complete the square we take the half of the coefficient of the x (or the y), square this and then add and subtract it to the equation.

Upon doing this we see that we have a circle and it's now written in standard form.

$$(x-h)^2+(y-k)^2=r^2$$

When circles are in this form we can easily identify the center : (*h*, *k*) and radius : *r*. Once we have these we can graph the circle simply by starting at the center and moving right, left, up and down by *r* to get the rightmost, leftmost, top most and bottom most points respectively.
Our circle has a center at (-1, 4) and a radius of 3. Here's a sketch of this circle.



Example 6 Graph $\dfrac{(x-2)^2}{9}+4(y+2)^2=1$

Solution
This is an ellipse. The standard form of the ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

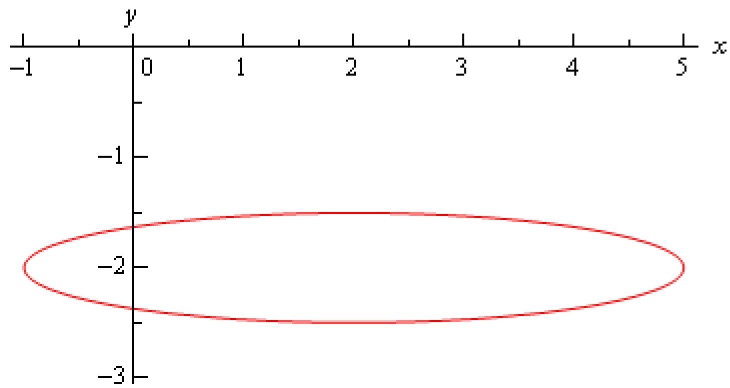
This is an ellipse with center (*h*, *k*) and the right most and left most points are a distance of *a* away from the center and the top most and bottom most points are a distance of *b* away from the center.

The ellipse for this problem has center (2, -2) and has $a=3$ and $b=\frac{1}{2}$. Note that to get the *b* we're really rewriting the equation as,

$$\frac{(x-2)^2}{9}+\frac{(y+2)^2}{1/4}=1$$

to get it into standard from.

Here's a sketch of the ellipse.



Example 7 Graph $\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{4} = 1$

Solution

This is a hyperbola. There are actually two standard forms for a hyperbola. Here are the basics for each form.

Form	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$
Center	(h, k)	(h, k)
Opens	Opens right and left	Opens up and down
Vertices	*a* units right and left of center.	*b* units up and down from center.
Slope of Asymptotes	$\pm \dfrac{b}{a}$	$\pm \dfrac{b}{a}$

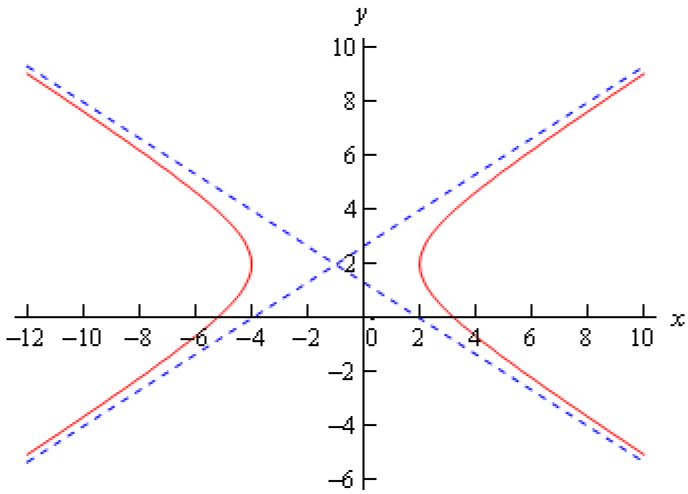
So, what does all this mean? First, notice that one of the terms is positive and the other is negative. This will determine which direction the two parts of the hyperbola open. If the *x* term is positive the hyperbola opens left and right. Likewise, if the *y* term is positive the parabola opens up and down.

Both have the same "center". Note that hyperbolas don't really have a center in the sense that circles and ellipses have centers. The center is the starting point in graphing a hyperbola. It tells up how to get to the vertices and how to get the asymptotes set up.

The asymptotes of a hyperbola are two lines that intersect at the center and have the slopes listed above. As you move farther out from the center the graph will get closer and closer to the asymptotes.

For the equation listed here the hyperbola will open left and right. Its center is (-1, 2). The two vertices are (-4, 2) and (2, 2). The asymptotes will have slopes $\pm\frac{2}{3}$.

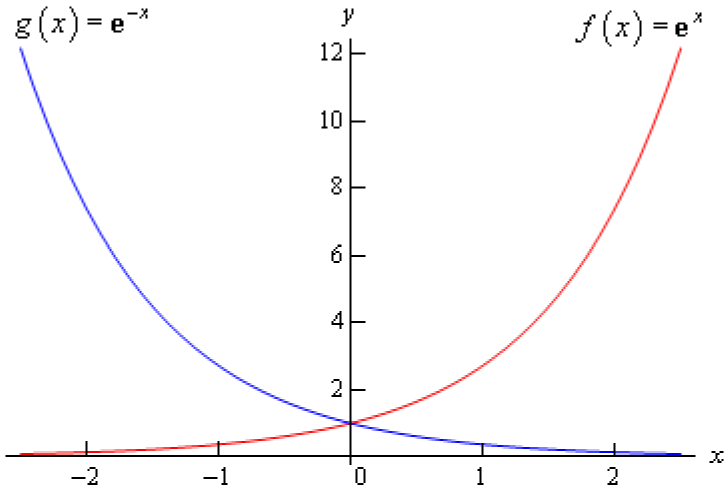
Here is a sketch of this hyperbola. Note that the asymptotes are denoted by the two dashed lines.



Example 8 Graph $f(x) = \mathbf{e}^{x}$ and $g(x) = \mathbf{e}^{-x}$.

Solution
There really isn't a lot to this problem other than making sure that both of these exponentials are graphed somewhere.

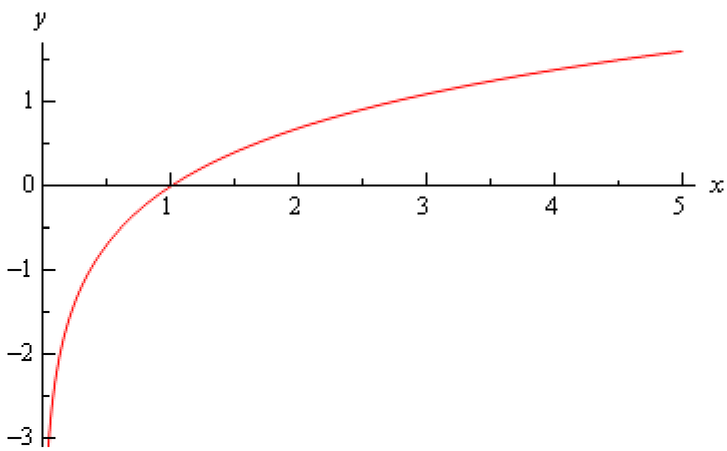
These will both show up with some regularity in later sections and their behavior as *x* goes to both plus and minus infinity will be needed and from this graph we can clearly see this behavior.



Example 9 Graph $f(x) = \ln(x)$.

Solution

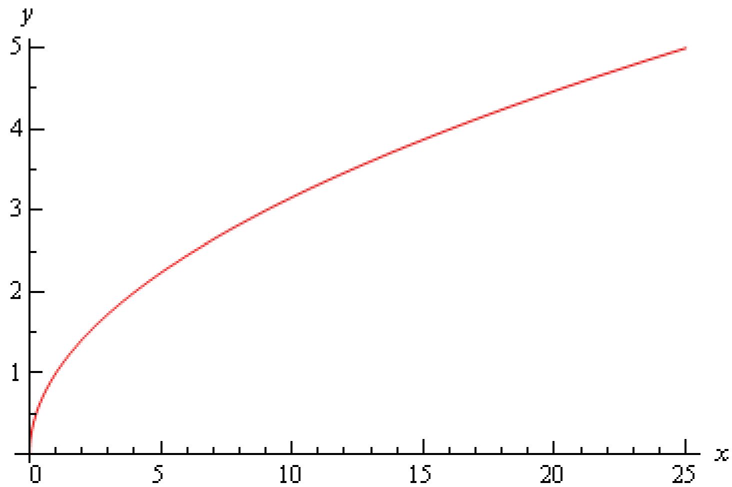
This has already been graphed once in this review, but this puts it here with all the other "important" graphs.



Example 10 Graph $y = \sqrt{x}$.

Solution

This one is fairly simple, we just need to make sure that we can graph it when need be.

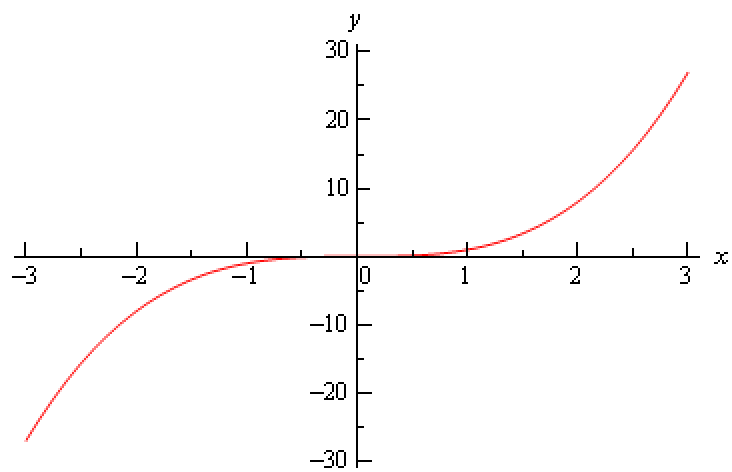


Remember that the domain of the square root function is $x \geq 0$.

Example 11 Graph $y = x^3$

Solution

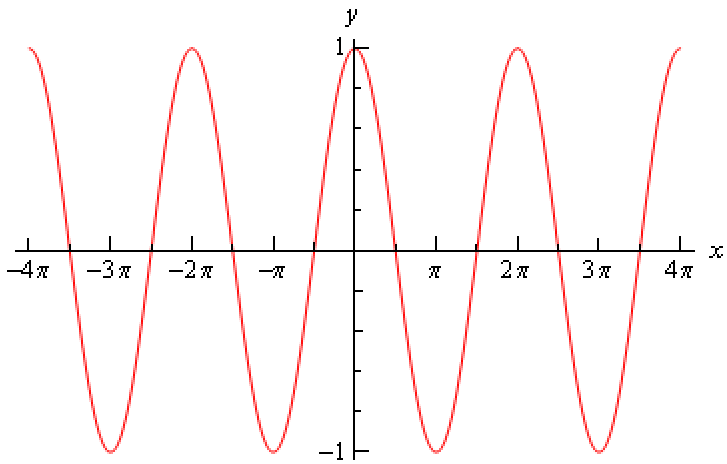
Again, there really isn't much to this other than to make sure it's been graphed somewhere so we can say we've done it.



Example 12 Graph $y = \cos(x)$

Solution

There really isn't a whole lot to this one. Here's the graph for $-4\pi \le x \le 4\pi$.



Let's also note here that we can put all values of *x* into cosine (which won't be the case for most of the trig functions) and so the domain is all real numbers. Also note that

$$-1 \le \cos(x) \le 1$$

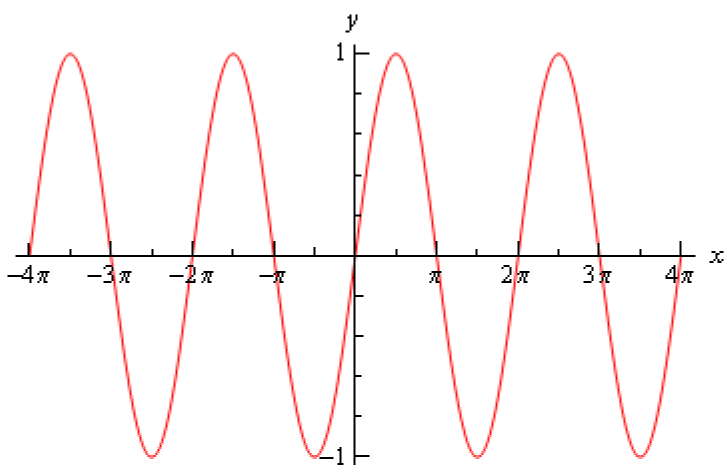
It is important to notice that cosine will never be larger than 1 or smaller than -1. This will be useful on occasion in a calculus class. In general we can say that

$$-R \le R\cos(\omega x) \le R$$

Example 13 Graph $y = \sin(x)$

Solution

As with the first problem in this section there really isn't a lot to do other than graph it. Here is the graph.



From this graph we can see that sine has the same range that cosine does. In general

$$-R \le R\sin(\omega x) \le R$$

As with cosine, sine itself will never be larger than 1 and never smaller than -1. Also the domain of sine is all real numbers.

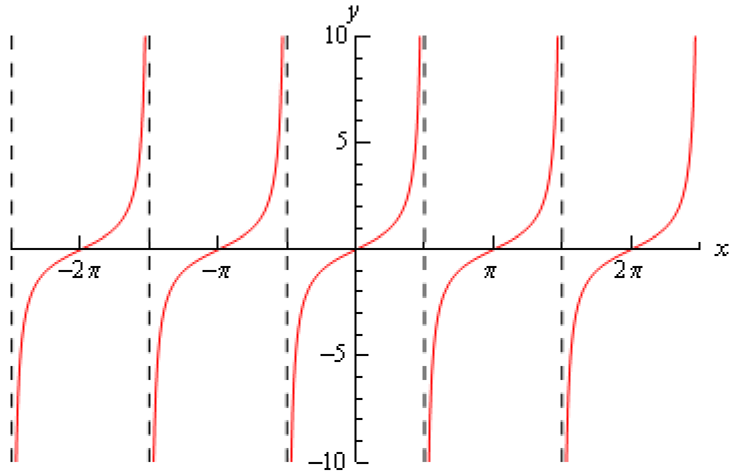
Example 14 Graph $y = \tan(x)$.

Solution

In the case of tangent we have to be careful when plugging *x*'s in since tangent doesn't exist wherever cosine is zero (remember that $\tan x = \frac{\sin x}{\cos x}$). Tangent will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of tangent on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



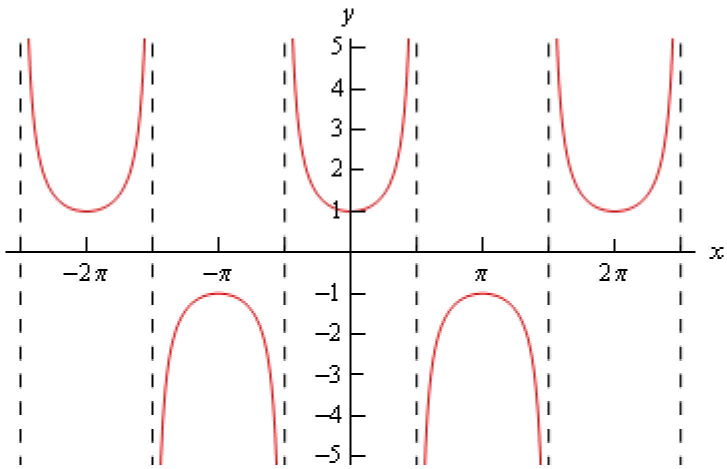
Example 15 Graph $y = \sec(x)$

Solution

As with tangent we will have to avoid *x*'s for which cosine is zero (remember that $\sec x = \frac{1}{\cos x}$). Secant will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of secant on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



Notice that the graph is always greater than 1 and less than -1. This should not be terribly surprising. Recall that $-1 \le \cos(x) \le 1$. So, one divided by something less than one will be greater than 1. Also, $1/\pm 1 = \pm 1$ and so we get the following ranges for secant.

$$\sec(\omega x) \ge 1 \qquad \text{and} \qquad \sec(\omega x) \le -1$$

Limits

Introduction

The topic that we will be examining in this chapter is that of Limits. This is the first of three major topics that we will be covering in this course. While we will be spending the least amount of time on limits in comparison to the other two topics limits are very important in the study of Calculus. We will be seeing limits in a variety of places once we move out of this chapter. In particular we will see that limits are part of the formal definition of the other two major topics.

Here is a quick listing of the material that will be covered in this chapter.

Tangent Lines and Rates of Change – In this section we will take a look at two problems that we will see time and again in this course. These problems will be used to introduce the topic of limits.

The Limit – Here we will take a conceptual look at limits and try to get a grasp on just what they are and what they can tell us.

One-Sided Limits – A brief introduction to one-sided limits.

Limit Properties – Properties of limits that we'll need to use in computing limits. We will also compute some basic limits in this section

Computing Limits – Many of the limits we'll be asked to compute will not be "simple" limits. In other words, we won't be able to just apply the properties and be done. In this section we will look at several types of limits that require some work before we can use the limit properties to compute them.

Infinite Limits – Here we will take a look at limits that have a value of infinity or negative infinity. We'll also take a brief look at vertical asymptotes.

Limits At Infinity, Part I – In this section we'll look at limits at infinity. In other words, limits in which the variable gets very large in either the positive or negative sense. We'll also take a brief look at horizontal asymptotes in this section. We'll be concentrating on polynomials and rational expression involving polynomials in this section.

Limits At Infinity, Part II – We'll continue to look at limits at infinity in this section, but this time we'll be looking at exponential, logarithms and inverse tangents.