Review : Trig Functions

The intent of this section is to remind you of some of the more important (from a Calculus standpoint…) topics from a trig class. One of the most important (but not the first) of these topics will be how to use the unit circle. We will actually leave the most important topic to the next section.

First let's start with the six trig functions and how they relate to each other.

$$\cos(x) \qquad\qquad \sin(x)$$
$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad\qquad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$
$$\sec(x) = \frac{1}{\cos(x)} \qquad\qquad \csc(x) = \frac{1}{\sin(x)}$$

Recall as well that all the trig functions can be defined in terms of a right triangle.
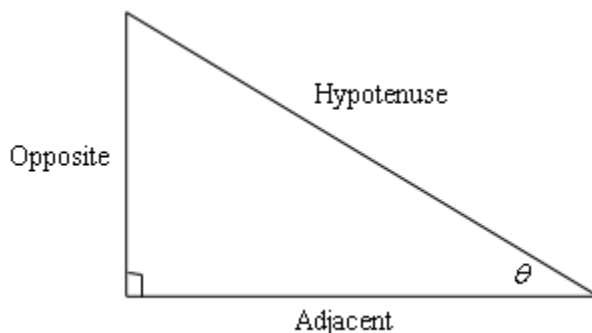
From this right triangle we get the following definitions of the six trig functions.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

Remembering both the relationship between all six of the trig functions and their right triangle definitions will be useful in this course on occasion.

Next, we need to touch on radians. In most trig classes instructors tend to concentrate on doing everything in terms of degrees (probably because it's easier to visualize degrees). The same is

true in many science classes. However, in a calculus course almost everything is done in radians. The following table gives some of the basic angles in both degrees and radians.

| Degree	0	30	45	60	90	180	270	360
Radians	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π

Know this table! We may not see these specific angles all that much when we get into the Calculus portion of these notes, but knowing these can help us to visualize each angle. Now, one more time just make sure this is clear.

Be forewarned, everything in most calculus classes will be done in radians!

Let's next take a look at one of the most overlooked ideas from a trig class. The unit circle is one of the more useful tools to come out of a trig class. Unfortunately, most people don't learn it as well as they should in their trig class.

Below is the unit circle with just the first quadrant filled in. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate is the cosine of that angle and the second coordinate is the sine of that angle. We've put some of the *basic* angles along with the coordinates of their intersections on the unit circle. So, from the unit circle below we can see that $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$ and $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$.
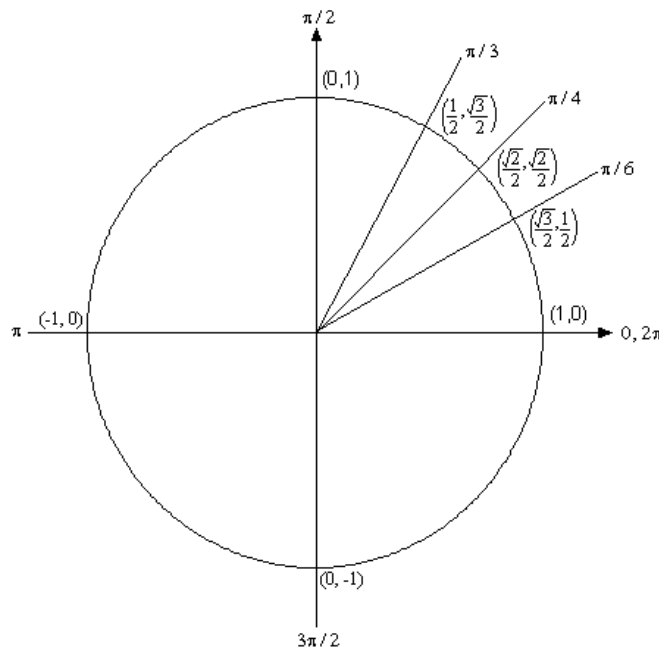
Remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is 2π, so the positive *x*-axis can correspond to either an angle of 0 or 2π (or 4π, or 6π, or -2π, or -4π, *etc*. depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate once around counter clockwise)}$$

$$\frac{\pi}{6} + 4\pi = \frac{25\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate around twice counter clockwise)}$$

$$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate once around clockwise)}$$

$$\frac{\pi}{6} - 4\pi = -\frac{23\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate around twice clockwise)}$$

etc.

In fact $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6} + 2\pi n$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ In this case *n* is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of *n* correspond to counter clockwise rotations and negative values of *n* correspond to clockwise rotations.

So, why did I only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first with a small application of geometry. You'll see how this is done in the following set of examples.

Example 1 Evaluate each of the following.

   (a) $\sin\left(\dfrac{2\pi}{3}\right)$ and $\sin\left(-\dfrac{2\pi}{3}\right)$ [Solution]

   (b) $\cos\left(\dfrac{7\pi}{6}\right)$ and $\cos\left(-\dfrac{7\pi}{6}\right)$ [Solution]

   (c) $\tan\left(-\dfrac{\pi}{4}\right)$ and $\tan\left(\dfrac{7\pi}{4}\right)$ [Solution]

   (d) $\sec\left(\dfrac{25\pi}{6}\right)$ [Solution]

Solution

(a) The first evaluation in this part uses the angle $\dfrac{2\pi}{3}$. That's not on our unit circle above,

however notice that $\dfrac{2\pi}{3} = \pi - \dfrac{\pi}{3}$. So $\dfrac{2\pi}{3}$ is found by rotating up $\dfrac{\pi}{3}$ from the negative *x*-axis.

This means that the line for $\dfrac{2\pi}{3}$ will be a mirror image of the line for $\dfrac{\pi}{3}$ only in the second

quadrant. The coordinates for $\dfrac{2\pi}{3}$ will be the coordinates for $\dfrac{\pi}{3}$ except the *x* coordinate will be

negative.

Likewise for $-\dfrac{2\pi}{3}$ we can notice that $-\dfrac{2\pi}{3} = -\pi + \dfrac{\pi}{3}$, so this angle can be found by rotating

down $\dfrac{\pi}{3}$ from the negative *x*-axis. This means that the line for $-\dfrac{2\pi}{3}$ will be a mirror image of

the line for $\dfrac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates

for $\dfrac{\pi}{3}$ except both will be negative.

Both of these angles along with their coordinates are shown on the following unit circle.
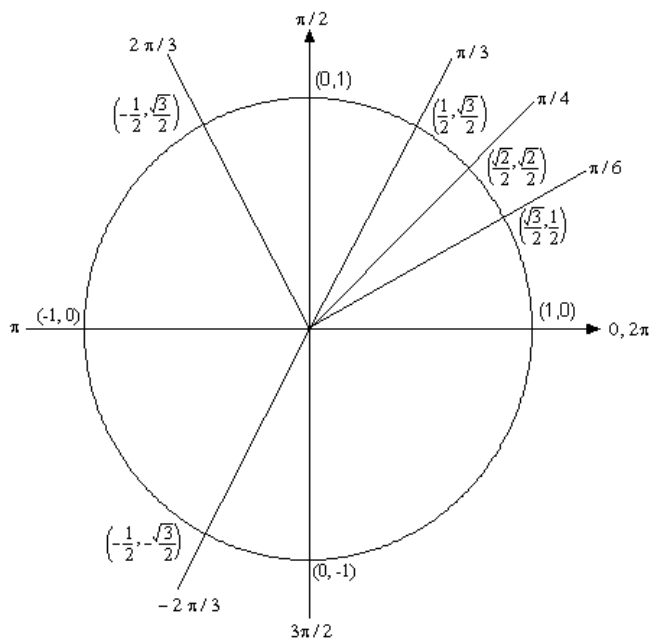
From this unit circle we can see that $\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ and $\sin\left(-\dfrac{2\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$.

This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta) = -\sin(\theta)$$

(b) For this example notice that $\dfrac{7\pi}{6} = \pi + \dfrac{\pi}{6}$ so this means we would rotate down $\dfrac{\pi}{6}$ from the negative *x*-axis to get to this angle. Also $-\dfrac{7\pi}{6} = -\pi - \dfrac{\pi}{6}$ so this means we would rotate up $\dfrac{\pi}{6}$ from the negative *x*-axis to get to this angle. So, as with the last part, both of these angles will be mirror images of $\dfrac{\pi}{6}$ in the third and second quadrants respectively and we can use this to determine the coordinates for both of these new angles.

Both of these angles are shown on the following unit circle along with appropriate coordinates for the intersection points.
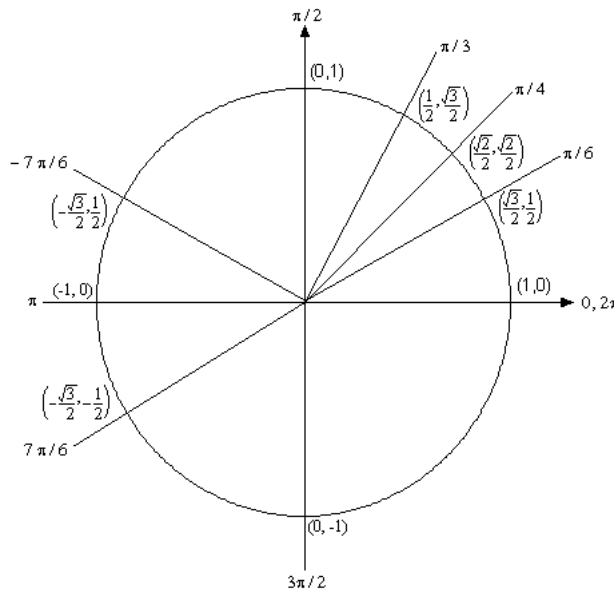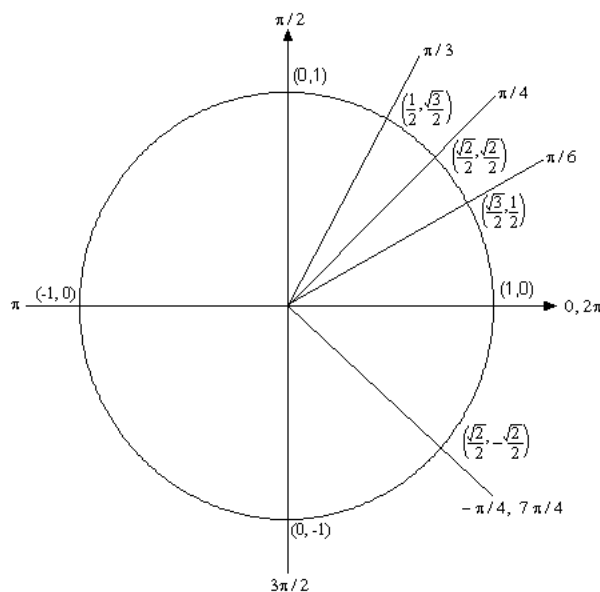

From this unit circle we can see that $\cos\left(\dfrac{7\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$ and $\cos\left(-\dfrac{7\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta) = \cos(\theta).$$

(c) Here we should note that $\dfrac{7\pi}{4} = 2\pi - \dfrac{\pi}{4}$ so $\dfrac{7\pi}{4}$ and $-\dfrac{\pi}{4}$ are in fact the same angle! Also note that this angle will be the mirror image of $\dfrac{\pi}{4}$ in the fourth quadrant. The unit circle for this angle is

Now, if we remember that $\tan\left(x\right) = \dfrac{\sin\left(x\right)}{\cos\left(x\right)}$ we can use the unit circle to find the values the tangent function. So,

$$\tan\left(\frac{7\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = \frac{\sin\left(-\pi/4\right)}{\cos\left(-\pi/4\right)} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1 \,.$$

On a side note, notice that $\tan\left(\dfrac{\pi}{4}\right) = 1$ and we can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan\left(-\theta\right) = -\tan\left(\theta\right) \,.$$

[Return to Problems]

(d) Here we need to notice that $\dfrac{25\pi}{6} = 4\pi + \dfrac{\pi}{6}$. In other words, we've started at $\dfrac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right) = \sec\left(4\pi + \frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec(x) = \frac{1}{\cos(x)}$$

and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

[Return to Problems]

So, in the last example we saw how the unit circle can be used to determine the value of the trig functions at any of the "common" angles. It's important to notice that all of these examples used the fact that if you know the first quadrant of the unit circle and can relate all the other angles to "mirror images" of one of the first quadrant angles you don't really need to know whole unit circle. If you'd like to see a complete unit circle I've got one on my Trig Cheat Sheet that is available at http://tutorial.math.lamar.edu.

Another important idea from the last example is that when it comes to evaluating trig functions all that you really need to know is how to evaluate sine and cosine. The other four trig functions are defined in terms of these two so if you know how to evaluate sine and cosine you can also evaluate the remaining four trig functions.

We've not covered many of the topics from a trig class in this section, but we did cover some of the more important ones from a calculus standpoint. There are many important trig formulas that you will use occasionally in a calculus class. Most notably are the half-angle and double-angle formulas. If you need reminded of what these are, you might want to download my Trig Cheat Sheet as most of the important facts and formulas from a trig class are listed there.

Review : Solving Trig Equations

In this section we will take a look at solving trig equations. This is something that you will be asked to do on a fairly regular basis in my class.

Let's just jump into the examples and see how to solve trig equations.

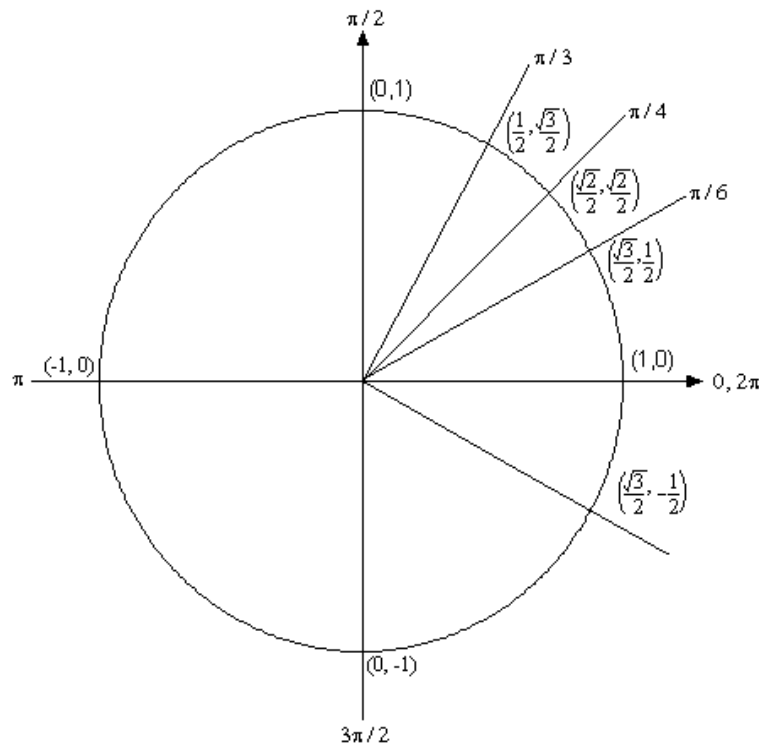
Example 1 Solve $2\cos(t) = \sqrt{3}$.

Solution
There's really not a whole lot to do in solving this kind of trig equation. All we need to do is divide both sides by 2 and the go to the unit circle.

$$2\cos(t) = \sqrt{3}$$

$$\cos(t) = \frac{\sqrt{3}}{2}$$

So, we are looking for all the values of *t* for which cosine will have the value of $\frac{\sqrt{3}}{2}$. So, let's take a look at the following unit circle.

From quick inspection we can see that $t = \frac{\pi}{6}$ is a solution. However, as I have shown on the unit

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circle there is another angle which will also be a solution. We need to determine what this angle is. When we look for these angles we typically want *positive* angles that lie between 0 and 2π. This angle will not be the only possibility of course, but by convention we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive *x*-axis, then so must the angle in the fourth quadrant. So we could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles so, we'll use

$$t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}.$$

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Recall from the previous section and you'll see there that I used

$$\frac{\pi}{6} + 2\pi n, \ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

to represent all the possible angles that can end at the same location on the unit circle, *i.e.* angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction (*n* is positive) or clockwise direction (*n* is negative) for *n* complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

$$\frac{\pi}{6} + 2\pi n, \ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\frac{11\pi}{6} + 2\pi n, \ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

Now, in a calculus class this is not a typical trig equation that we'll be asked to solve. A more typical example is the next one.

Example 2 Solve $2\cos(t) = \sqrt{3}$ on $[-2\pi, 2\pi]$.

Solution
In a calculus class we are often more interested in only the solutions to a trig equation that fall in a certain interval. The first step in this kind of problem is to first find all possible solutions. We did this in the first example.

$$\frac{\pi}{6} + 2\pi n, \ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\frac{11\pi}{6} + 2\pi n, \ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Now, to find the solutions in the interval all we need to do is start picking values of *n*, plugging them in and getting the solutions that will fall into the interval that we've been given.

n=0.

$$\frac{\pi}{6} + 2\pi(0) = \frac{\pi}{6} < 2\pi$$

$$\frac{11\pi}{6} + 2\pi(0) = \frac{11\pi}{6} < 2\pi$$

Now, notice that if we take any positive value of *n* we will be adding on positive multiples of 2π onto a positive quantity and this will take us past the upper bound of our interval and so we don't need to take any positive value of *n*.

However, just because we aren't going to take any positive value of *n* doesn't mean that we shouldn't also look at negative values of *n*.

n=-1.

$$\frac{\pi}{6} + 2\pi(-1) = -\frac{11\pi}{6} > -2\pi$$

$$\frac{11\pi}{6} + 2\pi(-1) = -\frac{\pi}{6} > -2\pi$$

These are both greater than -2π and so are solutions, but if we subtract another 2π off (*i.e* use $n = -2$) we will once again be outside of the interval so we've found all the possible solutions that lie inside the interval $[-2\pi, 2\pi]$.

So, the solutions are : $\dfrac{\pi}{6}, \dfrac{11\pi}{6}, -\dfrac{\pi}{6}, -\dfrac{11\pi}{6}$.

So, let's see if you've got all this down.

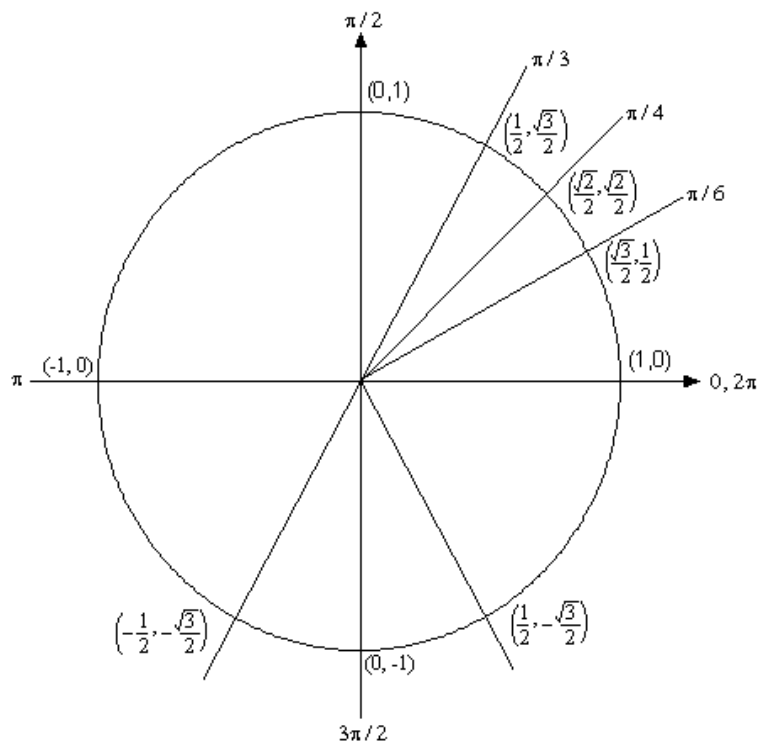***Example 3*** Solve $2\sin(5x) = -\sqrt{3}$ on $[-\pi, 2\pi]$

Solution
This problem is very similar to the other problems in this section with a very important difference. We'll start this problem in exactly the same way. We first need to find all possible solutions.

$$2\sin(5x) = -\sqrt{3}$$

$$\sin(5x) = \frac{-\sqrt{3}}{2}$$

So, we are looking for angles that will give $-\dfrac{\sqrt{3}}{2}$ out of the sine function. Let's again go to our trusty unit circle.

Now, there are no angles in the first quadrant for which sine has a value of $-\dfrac{\sqrt{3}}{2}$. However,

there are two angles in the lower half of the unit circle for which sine will have a value of $-\dfrac{\sqrt{3}}{2}$.

So, what are these angles? We'll notice $\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$, so the angle in the third quadrant will be

$\frac{\pi}{3}$ below the **negative** x-axis or $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$. Likewise, the angle in the fourth quadrant will $\frac{\pi}{3}$

below the **positive** x-axis or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$. Remember that we're typically looking for positive

angles between 0 and 2π.

Now we come to the very important difference between this problem and the previous problems in this section. The solution is **NOT**

$$x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots$$

$$x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots$$

This is not the set of solutions because we are NOT looking for values of *x* for which

$\sin(x) = -\frac{\sqrt{3}}{2}$, but instead we are looking for values of *x* for which $\sin(5x) = -\frac{\sqrt{3}}{2}$. Note the

difference in the arguments of the sine function! One is *x* and the other is $5x$. This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

$$5x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots$$

$$5x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots$$

Well, actually, that's not quite the solution. We are looking for values of *x* so divide everything by 5 to get.

$$x = \frac{4\pi}{15} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \dots$$

$$x = \frac{\pi}{3} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \dots$$

Notice that we also divided the $2\pi n$ by 5 as well! This is important! If we don't do that you **WILL** miss solutions. For instance, take $n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{11\pi}{15}\right)\right) = \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

I'll leave it to you to verify my work showing they are solutions. However it makes the point. If you didn't divided the $2\pi n$ by 5 you would have missed these solutions!

Okay, now that we've gotten all possible solutions it's time to find the solutions on the given interval. We'll do this as we did in the previous problem. Pick values of *n* and get the solutions.

$n = 0$.

$$x = \frac{4\pi}{15} + \frac{2\pi(0)}{5} = \frac{4\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(0)}{5} = \frac{\pi}{3} < 2\pi$$

$n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi(1)}{5} = \frac{2\pi}{3} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(1)}{5} = \frac{11\pi}{15} < 2\pi$$

$n = 2$.

$$x = \frac{4\pi}{15} + \frac{2\pi(2)}{5} = \frac{16\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(2)}{5} = \frac{17\pi}{15} < 2\pi$$

$n = 3$.

$$x = \frac{4\pi}{15} + \frac{2\pi(3)}{5} = \frac{22\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(3)}{5} = \frac{23\pi}{15} < 2\pi$$

$n = 4$.

$$x = \frac{4\pi}{15} + \frac{2\pi(4)}{5} = \frac{28\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(4)}{5} = \frac{29\pi}{15} < 2\pi$$

$n = 5$.

$$x = \frac{4\pi}{15} + \frac{2\pi(5)}{5} = \frac{34\pi}{15} > 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(5)}{5} = \frac{35\pi}{15} > 2\pi$$

Okay, so we finally got past the right endpoint of our interval so we don't need any more positive n. Now let's take a look at the negative n and see what we've got.

$n = -1$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-1)}{5} = -\frac{2\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-1)}{5} = -\frac{\pi}{15} > -\pi$$

$n = -2$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-2)}{5} = -\frac{8\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-2)}{5} = -\frac{7\pi}{15} > -\pi$$

$n = -3$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-3)}{5} = -\frac{14\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-3)}{5} = -\frac{13\pi}{15} > -\pi$$

$n = -4$.

$$x = \frac{4\pi}{15} + \frac{2\pi(-4)}{5} = -\frac{4\pi}{3} < -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-4)}{5} = -\frac{19\pi}{15} < -\pi$$

And we're now past the left endpoint of the interval. Sometimes, there will be many solutions as there were in this example. Putting all of this together gives the following set of solutions that lie in the given interval.

$$\frac{4\pi}{15}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{11\pi}{15}, \frac{16\pi}{15}, \frac{17\pi}{15}, \frac{22\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{29\pi}{15}$$

$$-\frac{\pi}{15}, -\frac{2\pi}{15}, -\frac{7\pi}{15}, -\frac{8\pi}{15}, -\frac{13\pi}{15}, -\frac{14\pi}{15}$$

Let's work another example.

Example 4 Solve $\sin(2x) = -\cos(2x)$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2} \right]$

Solution
This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

$$\sin(2x) = -\cos(2x)$$

$$\frac{\sin(2x)}{\cos(2x)} = -1$$

$$\tan(2x) = -1$$

So, solving $\sin(2x) = -\cos(2x)$ is the same as solving $\tan(2x) = -1$. At some level we didn't need to do this for this problem as all we're looking for is angles in which sine and cosine have the same value, but opposite signs. However, for other problems this won't be the case and we'll want to convert to tangent.

Looking at our trusty unit circle it appears that the solutions will be,

$$2x = \frac{3\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$2x = \frac{7\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by the 2 we get all possible solutions.

$$x = \frac{3\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$x = \frac{7\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now, let's determine the solutions that lie in the given interval.

$n = 0$.

$$x = \frac{3\pi}{8} + \pi(0) = \frac{3\pi}{8} < \frac{3\pi}{2}$$

$$x = \frac{7\pi}{8} + \pi(0) = \frac{7\pi}{8} < \frac{3\pi}{2}$$

$n = 1$.

$$x = \frac{3\pi}{8} + \pi(1) = \frac{11\pi}{8} < \frac{3\pi}{2}$$

$$x = \frac{7\pi}{8} + \pi(1) = \frac{15\pi}{8} > \frac{3\pi}{2}$$

Unlike the previous example only one of these will be in the interval. This will happen occasionally so don't always expect both answers from a particular *n* to work. Also, we should now check *n=2* for the first to see if it will be in or out of the interval. I'll leave it to you to check that it's out of the interval.

Now, let's check the negative *n*.

$n = -1$.

$$x = \frac{3\pi}{8} + \pi(-1) = -\frac{5\pi}{8} > -\frac{3\pi}{2}$$

$$x = \frac{7\pi}{8} + \pi(-1) = -\frac{\pi}{8} > -\frac{3\pi}{2}$$

$n = -2$.

$$x = \frac{3\pi}{8} + \pi(-2) = -\frac{13\pi}{8} < -\frac{3\pi}{2}$$

$$x = \frac{7\pi}{8} + \pi(-2) = -\frac{9\pi}{8} > -\frac{3\pi}{2}$$

Again, only one will work here. I'll leave it to you to verify that $n = -3$ will give two answers

that are both out of the interval.

The complete list of solutions is then,
$$-\frac{9\pi}{8}, -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$$

Let's work one more example so that I can make a point that needs to be understood when solving some trig equations.

Example 5 Solve $\cos(3x) = 2$.

Solution
This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \cos(\theta) \le 1$ and $-1 \le \sin(\theta) \le 1$. Therefore, since cosine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

In this section we solved some simple trig equations. There are more complicated trig equations that we can solve so don't leave this section with the feeling that there is nothing harder out there in the world to solve. In fact, we'll see at least one of the more complicated problems in the next section. Also, every one of these problems came down to solutions involving one of the "common" or "standard" angles. Most trig equations won't come down to one of those and will in fact need a calculator to solve. The next section is devoted to this kind of problem.

Review : Solving Trig Equations with Calculators, Part I

In the previous section we started solving trig equations. The only problem with the equations we solved in there is that they pretty much all had solutions that came from a handful of "standard" angles and of course there are many equations out there that simply don't. So, in this section we are going to take a look at some more trig equations, the majority of which will require the use of a calculator to solve (a couple won't need a calculator).

The fact that we are using calculators in this section does not however mean that the problems in the previous section aren't important. It is going to be assumed in this section that the basic ideas of solving trig equations are known and that we don't need to go back over them here. In particular, it is assumed that you can use a unit circle to help you find all answers to the equation (although the process here is a little different as we'll see) and it is assumed that you can find answers in a given interval. If you are unfamiliar with these ideas you should first go to the previous section and go over those problems.

Before proceeding with the problems we need to go over how our calculators work so that we can get the correct answers. Calculators are great tools but if you don't know how they work and how to interpret their answers you can get in serious trouble.

First, as already pointed out in previous sections, everything we are going to be doing here will be in radians so make sure that your calculator is set to radians before attempting the problems in this section. Also, we are going to use 4 decimal places of accuracy in the work here. You can use more if you want, but in this class we'll always use at least 4 decimal places of accuracy.

Next, and somewhat more importantly, we need to understand how calculators give answers to inverse trig functions. We didn't cover inverse trig functions in this review, but they are just inverse functions and we have talked a little bit about inverse functions in a review section. The only real difference is that we are now using trig functions. We'll only be looking at three of them and they are:

$$\text{Inverse Cosine} \; : \; \cos^{-1}(x) = \arccos(x)$$
$$\text{Inverse Sine} \quad : \; \sin^{-1}(x) = \arcsin(x)$$
$$\text{Inverse Tangent} : \tan^{-1}(x) = \arctan(x)$$

As shown there are two different notations that are commonly used. In these notes we'll be using the first form since it is a little more compact. Most calculators these days will have buttons on them for these three so make sure that yours does as well.

We now need to deal with how calculators give answers to these. Let's suppose, for example, that we wanted our calculator to compute $\cos^{-1}\left(\frac{3}{4}\right)$. First, remember that what the calculator is actually computing is the angle, let's say x, that we would plug into cosine to get a value of $\frac{3}{4}$, or

$$x = \cos^{-1}\left(\frac{3}{4}\right) \qquad \Rightarrow \qquad \cos(x) = \frac{3}{4}$$

So, in other words, when we are using our calculator to compute an inverse trig function we are really solving a simple trig equation.

Having our calculator compute $\cos^{-1}\left(\frac{3}{4}\right)$ and hence solve $\cos(x) = \frac{3}{4}$ gives,

$$x = \cos^{-1}\left(\frac{3}{4}\right) = 0.7227$$

From the previous section we know that there should in fact be an infinite number of answers to this including a second angle that is in the interval $[0, 2\pi]$. However, our calculator only gave us a single answer. How to determine what the other angles are will be covered in the following examples so we won't go into detail here about that. We did need to point out however, that the calculators will only give a single answer and that we're going to have more work to do than just plugging a number into a calculator.

Since we know that there are supposed to be an infinite number of solutions to $\cos(x) = \frac{3}{4}$ the next question we should ask then is just how did the calculator decide to return the answer that it did? Why this one and not one of the others? Will it give the same answer every time?

There are rules that determine just what answer the calculator gives. All calculators will give answers in the following ranges.

$$0 \le \cos^{-1}(x) \le \pi \qquad\qquad -\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2} \qquad\qquad -\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$$

If you think back to the unit circle and recall that we think of cosine as the horizontal axis the we can see that we'll cover all possible values of cosine in the upper half of the circle and this is exactly the range give above for the inverse cosine. Likewise, since we think of sine as the vertical axis in the unit circle we can see that we'll cover all possible values of sine in the right half of the unit circle and that is the range given above.

For the tangent range look back to the graph of the tangent function itself and we'll see that one branch of the tangent is covered in the range given above and so that is the range we'll use for inverse tangent. Note as well that we don't include the endpoints in the range for inverse tangent since tangent does not exist there.

So, if we can remember these rules we will be able to determine the remaining angle in $[0, 2\pi]$ that also works for each solution.

As a final quick topic let's note that it will, on occasion, be useful to remember the decimal representations of some basic angles. So here they are,

$$\frac{\pi}{2} = 1.5708 \qquad\qquad \pi = 3.1416 \qquad\qquad \frac{3\pi}{2} = 4.7124 \qquad\qquad 2\pi = 6.2832$$

Using these we can quickly see that $\cos^{-1}\left(\frac{3}{4}\right)$ must be in the first quadrant since 0.7227 is between 0 and 1.5708. This will be of great help when we go to determine the remaining angles

So, once again, we can't stress enough that calculators are great tools that can be of tremendous help to us, but it you don't understand how they work you will often get the answers to problems wrong.

So, with all that out of the way let's take a look at our first problem.

***Example 1*** Solve $4\cos(t) = 3$ on[-8,10].

Solution
Okay, the first step here is identical to the problems in the previous section. We first need to isolate the cosine on one side by itself and then use our calculator to get the first answer.

$$\cos(t) = \frac{3}{4} \qquad\qquad \Rightarrow \qquad\qquad t = \cos^{-1}\left(\frac{3}{4}\right) = 0.7227$$

So, this is the one we were using above in the opening discussion of this section. At the time we mentioned that there were infinite number of answers and that we'd be seeing how to find them later. Well that time is now.

First, let's take a quick look at a unit circle for this example.
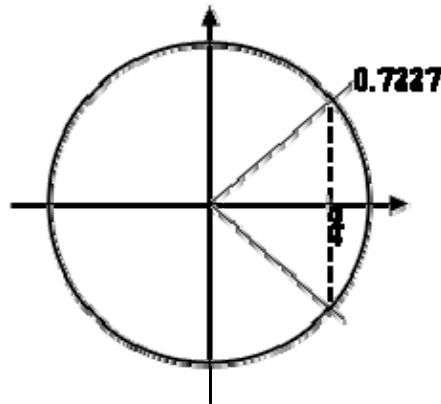
The angle that we've found is shown on the circle as well as the other angle that we know should also be an answer. Finding this angle here is just as easy as in the previous section. Since the

---

line segment in the first quadrant forms an angle of 0.7227 radians with the positive *x*-axis then so does the line segment in the fourth quadrant. This means that we can use either -0.7227 as the second angle or $2\pi - 0.7227 = 5.5605$. Which you use depends on which you prefer. We'll pretty much always use the positive angle to avoid the possibility that we'll lose the minus sign.

So, all possible solutions, ignoring the interval for a second, are then,

$$t = 0.7227 + 2\pi n$$
$$t = 5.5605 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now, all we need to do is plug in values of *n* to determine the angle that are actually in the interval. Here's the work for that.

$n = -2$:	$t = \cancel{-11.8437}$	and	-7.0059
$n = -1$:	$t = -5.5605$	and	-0.7227
$n = 0$:	$t = 0.7227$	and	5.5605
$n = 1$:	$t = 7.0059$	and	$\cancel{11.8437}$

So, the solutions to this equation, in the given interval, are,

$$t = -7.0059, \, -5.5605, \, -0.7227, \, 0.7227, \, 5.5605, \, 7.0059$$

Note that we had a choice of angles to use for the second angle in the previous example. The choice of angles there will also affect the value(s) of *n* that we'll need to use to get all the solutions. In the end, regardless of the angle chosen, we'll get the same list of solutions, but the value(s) of *n* that give the solutions will be different depending on our choice.

Also, in the above example we put in a little more explanation than we'll show in the remaining examples in this section to remind you how these work.

***Example 2*** Solve $-10\cos(3t) = 7$ on [-2,5].

Solution
Okay, let's first get the inverse cosine portion of this problem taken care of.

$$\cos(3t) = -\frac{7}{10} \qquad \Rightarrow \qquad 3t = \cos^{-1}\left(-\frac{7}{10}\right) = 2.3462$$

Don't forget that we still need the "3"!

Now, let's look at a quick unit circle for this problem. As we can see the angle 2.3462 radians is in the second quadrant and the other angle that we need is in the third quadrant. We can find this second angle in exactly the same way we did in the previous example. We can use either -2.3462

or we can use $2\pi - 2.3462 = 3.9370$. As with the previous example we'll use the positive choice, but that is purely a matter of preference. You could use the negative if you wanted to.
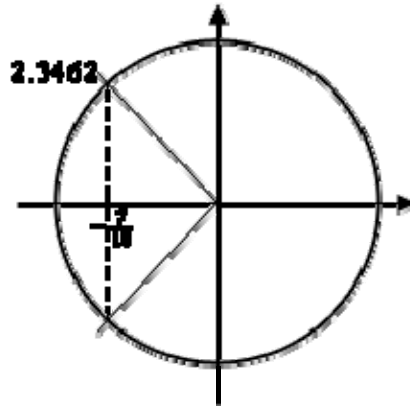
So, let's now finish out the problem. First, let's acknowledge that the values of $3t$ that we need are,

$$3t = 2.3462 + 2\pi n$$
$$3t = 3.9370 + 2\pi n$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Now, we need to properly deal with the 3, so divide that out to get all the solutions to the trig equation.

$$t = 0.7821 + \frac{2\pi n}{3}$$
$$t = 1.3123 + \frac{2\pi n}{3}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Finally, we need to get the values in the given interval.

$$n = -2: \qquad t = \cancel{-3.4067} \quad \text{and} \quad \cancel{-2.8765}$$
$$n = -1: \qquad t = -1.3123 \quad \text{and} \quad -0.7821$$
$$n = 0: \qquad t = 0.7821 \quad \text{and} \quad 1.3123$$
$$n = 1: \qquad t = 2.8765 \quad \text{and} \quad 3.4067$$
$$n = 2: \qquad t = 4.9709 \quad \text{and} \quad \cancel{5.5011}$$

The solutions to this equation, in the given interval are then,

$$t = -1.3123, \, -0.7821, \, 0.7821, \, 1.3123, \, 2.8765, \, 3.4067, \, 4.9709$$

We've done a couple of basic problems with cosines, now let's take a look at how solving equations with sines work.
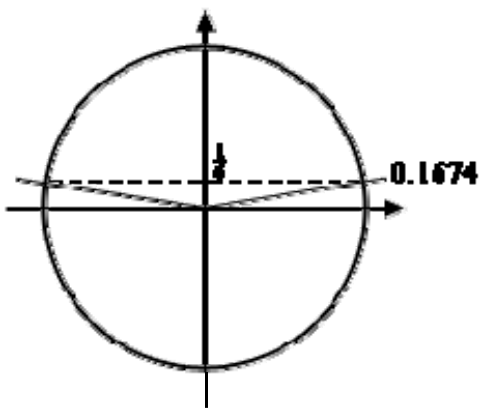
Example 3 Solve $6\sin\left(\dfrac{x}{2}\right) = 1$ on [-20,30]

Solution

Let's first get the calculator work out of the way since that isn't where the difference comes into play.

$$\sin\left(\frac{x}{2}\right) = \frac{1}{6} \qquad \Rightarrow \qquad \frac{x}{2} = \sin^{-1}\left(\frac{1}{6}\right) = 0.1674$$

Here's a unit circle for this example.

To find the second angle in this case we can notice that the line in the first quadrant makes an angle of 0.1674 with the positive *x*-axis and so the angle in the second quadrant will then make an angle of 0.1674 with the negative *x*-axis and so the angle that we're after is then, $\pi - 0.1674 = 2.9742$.

Here's the rest of the solution for this example. We're going to assume from this point on that you can do this work without much explanation.

$$\frac{x}{2} = 0.1674 + 2\pi n$$
$$\frac{x}{2} = 2.9742 + 2\pi n \qquad \Rightarrow \qquad \begin{array}{l} x = 0.3348 + 4\pi n \\ x = 5.9484 + 4\pi n \end{array} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\begin{array}{lll}
n = -1: & x = \cancel{-24.7980} & \text{and} \quad -19.1844 \\
n = 0: & x = 0.3348 & \text{and} \quad 5.9484 \\
n = 1: & x = 25.4676 & \text{and} \quad \cancel{31.0812}
\end{array}$$
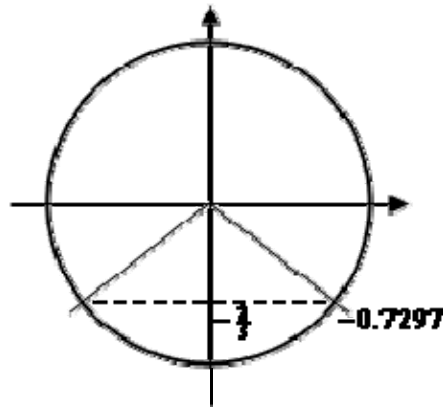
The solutions to this equation are then,

$$x = -19.1844, \ 0.3348, \ 5.9484, \ 25.4676$$

Example 4 Solve $3\sin(5z) = -2$ on [0,1].

Solution
You should be getting pretty good at these by now, so we won't be putting much explanation in for this one. Here we go.

$$\sin(5z) = -\frac{2}{3} \qquad \Rightarrow \qquad 5z = \sin^{-1}\left(-\frac{2}{3}\right) = -0.7297$$

Okay, with this one we're going to do a little more work than with the others. For the first angle we could use the answer our calculator gave us. However, it's easy to lose minus signs so we'll instead use $2\pi - 0.7297 = 5.5535$. Again, there is no reason to this other than a worry about losing the minus sign in the calculator answer. If you'd like to use the calculator answer you are more than welcome to. For the second angle we'll note that the lines in the third and fourth quadrant make an angle of 0.7297 with the *x*-axis and so the second angle is $\pi + 0.7297 = 3.8713$.

Here's the rest of the work for this example.

$$\begin{aligned}5z &= 5.5535 + 2\pi n \\ 5z &= 3.8713 + 2\pi n\end{aligned} \quad \Rightarrow \quad \begin{aligned}z &= 1.1107 + \frac{2\pi n}{5} \\ z &= 0.7743 + \frac{2\pi n}{5}\end{aligned} \qquad n = 0, \pm1, \pm2, \ldots$$

$$\begin{aligned}n = -1: & \qquad x = \cancel{-0.1460} \quad \text{and} \quad \cancel{-0.4823} \\ n = 0: & \qquad \cancel{x = 1.1107} \quad \text{and} \quad 0.7743\end{aligned}$$

So, in this case we get a single solution of 0.7743.

Note that in the previous example we only got a single solution. This happens on occasion so don't get worried about it. Also, note that it was the second angle that gave this solution and so if

we'd just relied on our calculator without worrying about other angles we would not have gotten this solution. Again, it can't be stressed enough that while calculators are a great tool if we don't understand how to correctly interpret/use the result we can (and often will) get the solution wrong.

To this point we've only worked examples involving sine and cosine. Let's no work a couple of examples that involve other trig functions to see how they work.

Example 5 Solve $9\sin(2x) = -5\cos(2x)$ on[-10,0].

Solution
At first glance this problem seems to be at odds with the sentence preceding the example. However, it really isn't.

First, when we have more than one trig function in an equation we need a way to get equations that only involve one trig function. There are many ways of doing this that depend on the type of equation we're starting with. In this case we can simply divide both sides by a cosine and we'll get a single tangent in the equation. We can now see that this really is an equation that doesn't involve a sine or a cosine.

So, let's get started on this example.
$$\frac{\sin(2x)}{\cos(2x)} = \tan(2x) = -\frac{5}{9} \qquad \Rightarrow \qquad 2x = \tan^{-1}\left(-\frac{5}{9}\right) = -0.5071$$

Now, the unit circle doesn't involve tangents, however we can use it to illustrate the second angle in the range $[0, 2\pi]$.
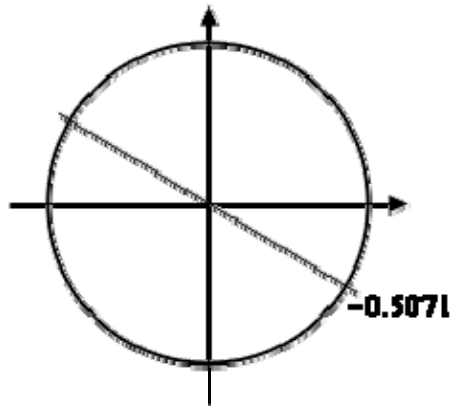
The angles that we're looking for here are those whose quotient of $\dfrac{\text{sine}}{\text{cosine}}$ is the same. The second angle were we will get the same value of tangent will be exactly opposite of the given point. For this angle the values of sine and cosine are the same except they will have opposite signs. In the quotient however, the difference in signs will cancel out and we'll get the same

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value of tangent. So, the second angle will always be the first angle plus π.

Before getting the second angle let's also note that, like the previous example, we'll use the $2\pi - 0.5071 = 5.7761$ for the first angle. Again, this is only because of a concern about losing track of the minus sign in our calculator answer. We could just as easily do the work with the original angle our calculator gave us.

Now, this is where is seems like we're just randomly making changes and doing things for no reason. The second angle that we're going to use is,

$$\pi + (-0.5071) = \pi - 0.5071 = 2.6345$$

The fact that we used the calculator answer here seems to contradict the fact that we used a different angle for the first above. The reason for doing this here is to give a second angle that is in the range $[0, 2\pi]$. Had we used 5.7761 to find the second angle we'd get $\pi + 5.7761 = 8.9177$. This is a perfectly acceptable answer, however it is larger than 2π (6.2832) and the general rule of thumb is to keep the initial angles as small as possible.

Here are all the solutions to the equation.

$$
\begin{array}{lll}
2x = 5.7761 + 2\pi n & & x = 2.8881 + \pi n \\
& \Rightarrow & \qquad\qquad\qquad\qquad n = 0, \pm 1, \pm 2, \ldots \\
2x = 2.6345 + 2\pi n & & x = 1.3173 + \pi n
\end{array}
$$

$$
\begin{array}{llll}
n = -2: & x = -9.6783 & \text{and} & \cancel{-11.2491} \\
n = -1: & x = -3.3951 & \text{and} & -4.9659 \\
n = 0: & \cancel{x = 2.8881} & \text{and} & \cancel{1.3173}
\end{array}
$$

The three solutions to this equation are then,

$$-3.3951, \ -4.9659, \ -9.6783$$

Note as well that we didn't need to do the $n = 0$ and computation since we could see from the given interval that we only wanted negative answers and these would clearly give positive answers.

Most calculators today can only do inverse sine, inverse cosine, and inverse tangent. So, let's see an example that uses one of the other trig functions.

Example 6 Solve $7\sec(3t) = -10$.

Solution
We'll start this one in exactly the same way we've done all the others.

$$\sec(3t) = -\frac{10}{7} \qquad \Rightarrow \qquad 3t = \sec^{-1}\left(-\frac{10}{7}\right)$$

Now we reach the problem. As noted above, most calculators can't handle inverse secant so we're going to need a different solution method for this one. To finish the solution here we'll simply recall the definition of secant in terms of cosine and convert this into an equation involving cosine instead and we already know how to solve those kinds of trig equations.

$$\frac{1}{\cos(3t)} = \sec(3t) = -\frac{10}{7} \qquad \Rightarrow \qquad \cos(3t) = -\frac{7}{10}$$

Now, we solved this equation in the second example above so we won't redo our work here. The solution is,

$$t = 0.7821 + \frac{2\pi n}{3}$$
$$t = 1.3123 + \frac{2\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \dots$$

We weren't given an interval in this problem so here is nothing else to do here.

For the remainder of the examples in this section we're not going to be finding solutions in an interval to save some space. If you followed the work from the first few examples in which we were given intervals you should be able to do any of the remaining examples if given an interval.

Also, we will no longer be including sketches of unit circles in the remaining solutions. We are going to assume that you can use the above sketches as guides for sketching unit circles to verify our claims in the following examples.

The next three examples don't require a calculator but are important enough or cause enough problems for students to include in this section in case you run across them and haven't seen them anywhere else.

Example 7 Solve $\cos(4\theta) = -1$.

Solution
There really isn't too much to do with this problem. It is, however, different from all the others

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done to this point. All the others done to this point have had two angles in the interval $[0, 2\pi]$ that were solutions to the equation. This only has one. Here is the solution to this equation.

$$4\theta = \pi + 2\pi n \qquad \Rightarrow \qquad \theta = \frac{\pi}{4} + \frac{\pi n}{2} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Example 8 Solve $\sin\left(\dfrac{\alpha}{7}\right) = 0$.

Solution
Again, not much to this problem. Using a unit circle it isn't too hard to see that the solutions to this equation are,

$$\begin{aligned} \frac{\alpha}{7} &= 0 + 2\pi n \\[2mm] \frac{\alpha}{7} &= \pi + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \alpha &= 14\pi n \\[2mm] \alpha &= 7\pi + 14\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

This next example has an important point that needs to be understood when solving some trig equations.

Example 9 Solve $\sin(3t) = 2$.

Solution
This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \sin(\theta) \le 1$ and $-1 \le \cos(\theta) \le 1$. Therefore, since sine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

Because this document is also being prepared for viewing on the web we're going to split this section in two in order to keep the page size (and hence load time in a browser) to a minimum. In the next section we're going to take a look at some slightly more "complicated" equations. Although, as you'll see, they aren't as complicated as they may at first seem.

Review : Solving Trig Equations with Calculators, Part II

Because this document is also being prepared for viewing on the web we split this section into two parts to keep the size of the pages to a minimum.

Also, as with the last few examples in the previous part of this section we are not going to be looking for solutions in an interval in order to save space. The important part of these examples is to find the solutions to the equation. If we'd been given an interval it would be easy enough to find the solutions that actually fall in the interval.

In all the examples in the previous section all the arguments, the $3t$, $\dfrac{\alpha}{7}$, *etc*., were fairly simple.

Let's take a look at an example that has a slightly more complicated argument.

Example 1 Solve $5\cos(2x-1)=-3$.

Solution
Note that the argument here is not really all that complicated but the addition of the "-1" often seems to confuse people so we need to a quick example with this kind of argument. The solution process is identical to all the problems we've done to this point so we won't be putting in much explanation. Here is the solution.

$$\cos(2x-1)=-\frac{3}{5} \qquad \Rightarrow \qquad 2x-1=\cos^{-1}\left(-\frac{3}{5}\right)=2.2143$$

This angle is in the second quadrant and so we can use either -2.2143 or $2\pi-2.2143=4.0689$ for the second angle. As usual for these notes we'll use the positive one. Therefore the two angles are,

$$\begin{aligned}2x-1&=2.2143+2\pi n\\ 2x-1&=4.0689+2\pi n\end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Now, we still need to find the actual values of *x* that are the solutions. These are found in the same manner as all the problems above. We'll first add 1 to both sides and then divide by 2. Doing this gives,

$$\begin{aligned}x&=1.6072+\pi n\\ x&=2.5345+\pi n\end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

So, in this example we saw an argument that was a little different from those seen previously, but not all that different when it comes to working the problems so don't get too excited about it.

We now need to move into a different type of trig equation. All of the trig equations solved to this point (the previous example as well as the previous section) were, in some way, more or less the "standard" trig equation that is usually solved in a trig class. There are other types of equations involving trig functions however that we need to take a quick look at. The remaining examples show some of these different kinds of trig equations.

Example 2 Solve $2\cos(6y) + 11\cos(6y)\sin(3y) = 0$.

Solution
So, this definitely doesn't look like any of the equations we've solved to this point and initially the process is different as well. First, notice that there is a $\cos(6y)$ in each term, so let's factor that out and see what we have.

$$\cos(6y)(2 + 11\sin(3y)) = 0$$

We now have a product of two terms that is zero and so we know that we must have,

$$\cos(6y) = 0 \qquad \text{OR} \qquad 2 + 11\sin(3y) = 0$$

Now, at this point we have two trig equations to solve and each is identical to the type of equation we were solving earlier. Because of this we won't put in much detail about solving these two equations.

First, solving $\cos(6y) = 0$ gives,

$$
\begin{aligned}
6y &= \frac{\pi}{2} + 2\pi n \\
6y &= \frac{3\pi}{2} + 2\pi n
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
y &= \frac{\pi}{12} + \frac{\pi n}{3} \\
y &= \frac{\pi}{4} + \frac{\pi n}{3}
\end{aligned}
\qquad n = 0, \pm 1, \pm 2, \ldots
$$

Next, solving $2 + 11\sin(3y) = 0$ gives,

$$
\begin{aligned}
3y &= 6.1004 + 2\pi n \\
3y &= 3.3244 + 2\pi n
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
y &= 2.0335 + \frac{2\pi n}{3} \\
y &= 1.1081 + \frac{2\pi n}{3}
\end{aligned}
\qquad n = 0, \pm 1, \pm 2, \ldots
$$

Remember that in these notes we tend to take positive angles and so the first solution here is in fact $2\pi - 0.1828$ where our calculator gave us -0.1828 as the answer when using the inverse sine function.

The solutions to this equation are then,

$$y = \frac{\pi}{12} + \frac{\pi n}{3}$$

$$y = \frac{\pi}{4} + \frac{\pi n}{3}$$

$$y = 2.0335 + \frac{2\pi n}{3}$$

$$y = 1.1081 + \frac{2\pi n}{3}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

This next example also involves "factoring" trig equations but in a slightly different manner than the previous example.

Example 3 Solve $4\sin^2\left(\frac{t}{3}\right) - 3\sin\left(\frac{t}{3}\right) = 1$.

Solution

Before solving this equation let's solve an apparently unrelated equation.

$$4x^2 - 3x = 1 \qquad \Rightarrow \qquad 4x^2 - 3x - 1 = (4x+1)(x-1) = 0 \qquad \Rightarrow \qquad x = -\frac{1}{4}, 1$$

This is an easy (or at least I hope it's easy as this point) equation to solve. The obvious question then is, why did we do this? We'll, if you compare the two equations you'll see that the only real difference is that the one we just solved has an *x* everywhere the equation we want to solve has a sine. What this tells us is that we can work the two equations in exactly the same way.

We, will first "factor" the equation as follows,

$$4\sin^2\left(\frac{t}{3}\right) - 3\sin\left(\frac{t}{3}\right) - 1 = \left(4\sin\left(\frac{t}{3}\right) + 1\right)\left(\sin\left(\frac{t}{3}\right) - 1\right) = 0$$

Now, set each of the two factors equal to zero and solve for the sine,

$$\sin\left(\frac{t}{3}\right) = -\frac{1}{4} \qquad\qquad \sin\left(\frac{t}{3}\right) = 1$$

We now have two trig equations that we can easily (hopefully…) solve at this point. We'll leave the details to you to verify that the solutions to each of these and hence the solutions to the original equation are,

$$t = 18.0915 + 6\pi n$$

$$t = 10.1829 + 6\pi n \qquad\qquad n = 0, \pm 1, \pm 2, \ldots$$

$$t = \frac{3\pi}{2} + 6\pi n$$

The first two solutions are from the first equation and the third solution is from the second equation.

Let's work one more trig equation that involves solving a quadratic equation. However, this time, unlike the previous example this one won't factor and so we'll need to use the quadratic formula.

Example 4 Solve $8\cos^2(1-x)+13\cos(1-x)-5=0$.

Solution
Now, as mentioned prior to starting the example this quadratic does not factor. However, that doesn't mean all is lost. We can solve the following equation with the quadratic formula (you do remember this and how to use it right?),

$$8t^2+13t-5=0 \qquad \Rightarrow \qquad t=\frac{-13\pm\sqrt{329}}{16}=0.3211, \ -1.9461$$

So, if we can use the quadratic formula on this then we can also use it on the equation we're asked to solve. Doing this gives us,

$$\cos(1-x)=0.3211 \qquad\qquad \text{OR} \qquad\qquad \cos(1-x)=-1.9461$$

Now, recall Example 9 from the previous section. In that example we noted that $-1\le\cos(\theta)\le1$ and so the second equation will have no solutions. Therefore, the solutions to the first equation will yield the only solutions to our original equation. Solving this gives the following set of solutions,

$$\begin{aligned} x&=-0.2439-2\pi n \\ x&=-4.0393-2\pi n \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Note that we did get some negative numbers here and that does seem to violate the general form that we've been using in most of these examples. However, in this case the "-" are coming about when we solved for x after computing the inverse cosine in our calculator.

There is one more example in this section that we need to work that illustrates another way in which factoring can arise in solving trig equations. This equation is also the only one where the variable appears both inside and outside of the trig equation. Not all equations in this form can be easily solved, however some can so we want to do a quick example of one.

Example 5 Solve $5x \tan(8x) = 3x$.

Solution
First, before we even start solving we need to make one thing clear. **DO NOT CANCEL AN x FROM BOTH SIDES!!!** While this may seem like a natural thing to do it **WILL** cause us to lose a solution here.

So, to solve this equation we'll first get all the terms on one side of the equation and then factor an x out of the equation. If we can cancel an x from all terms then it can be factored out. Doing this gives,

$$5x \tan(8x) - 3x = x\left(5 \tan(8x) - 3\right) = 0$$

Upon factoring we can see that we must have either,

$$x = 0 \qquad\qquad \text{OR} \qquad\qquad \tan(8x) = \frac{3}{5}$$

Note that if we'd canceled the x we would have missed the first solution. Now, we solved an equation with a tangent in it in Example 5 of the previous section so we'll not go into the details of this solution here. Here is the solution to the trig equation.

$$x = 0.0676 + \frac{\pi n}{4}$$
$$x = 0.4603 + \frac{\pi n}{4}$$
$$n = 0, \pm 1, \pm 2, \ldots$$

The complete set of solutions then to the original equation are,

$$x = 0$$
$$x = 0.0676 + \frac{\pi n}{4}$$
$$x = 0.4603 + \frac{\pi n}{4}$$
$$n = 0, \pm 1, \pm 2, \ldots$$

Review : Exponential Functions

In this section we're going to review one of the more common functions in both calculus and the sciences. However, before getting to this function let's take a much more general approach to things.

Let's start with $b > 0$, $b \neq 1$. An exponential function is then a function in the form,

$$f(x) = b^x$$

Note that we avoid $b = 1$ because that would give the constant function, $f(x) = 1$. We avoid $b = 0$ since this would also give a constant function and we avoid negative values of b for the following reason. Let's, for a second, suppose that we did allow b to be negative and look at the following function.

$$g(x) = (-4)^x$$

Let's do some evaluation.

$$g(2) = (-4)^2 = 16 \qquad\qquad g\left(\frac{1}{2}\right) = -(-4)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

So, for some values of x we will get real numbers and for other values of x we well get complex numbers. We want to avoid this and so if we require $b > 0$ this will not be a problem.

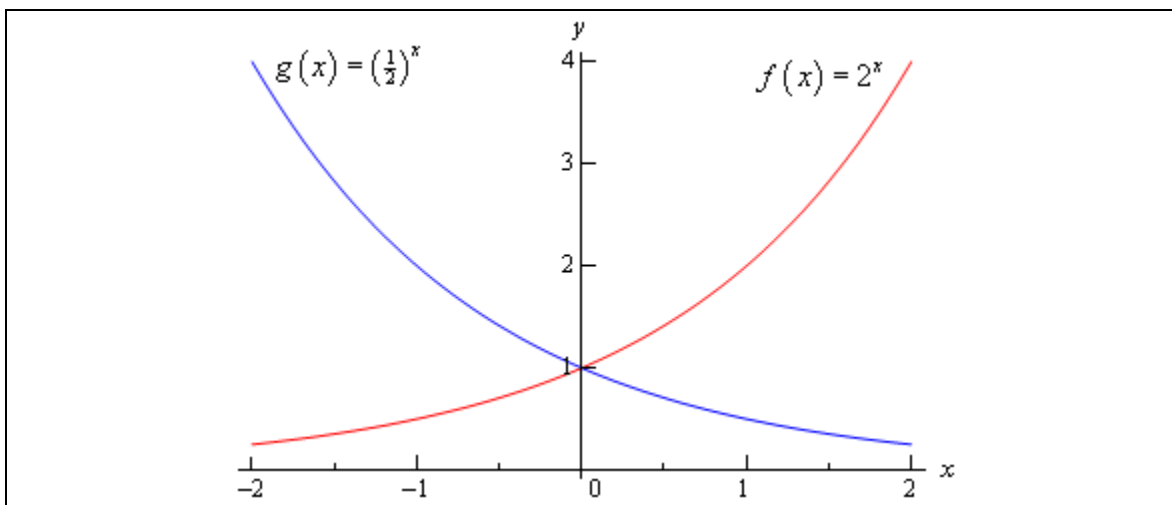Let's take a look at a couple of exponential functions.

Example 1 Sketch the graph of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$

Solution
Let's first get a table of values for these two functions.

x	$f(x)$	$g(x)$
-2	$f(-2) = 2^{-2} = \dfrac{1}{4}$	$g(-2) = \left(\dfrac{1}{2}\right)^{-2} = 4$
-1	$f(-1) = 2^{-1} = \dfrac{1}{2}$	$g(-1) = \left(\dfrac{1}{2}\right)^{-1} = 2$
0	$f(0) = 2^0 = 1$	$g(0) = \left(\dfrac{1}{2}\right)^0 = 1$
1	$f(1) = 2$	$g(1) = \dfrac{1}{2}$
2	$f(2) = 4$	$g(2) = \dfrac{1}{4}$

Here's the sketch of both of these functions.

This graph illustrates some very nice properties about exponential functions in general.

Properties of $f(x) = b^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$.

2. $f(x) \neq 0$. An exponential function will never be zero.

3. $f(x) > 0$. An exponential function is always positive.

4. The previous two properties can be summarized by saying that the range of an exponential function is $(0, \infty)$.

5. The domain of an exponential function is $(-\infty, \infty)$. In other words, you can plug every x into an exponential function.

6. If $0 < b < 1$ then,
   a. $f(x) \to 0$ as $x \to \infty$
   b. $f(x) \to \infty$ as $x \to -\infty$

7. If $b > 1$ then,
   a. $f(x) \to \infty$ as $x \to \infty$
   b. $f(x) \to 0$ as $x \to -\infty$

These will all be very useful properties to recall at times as we move throughout this course (and later Calculus courses for that matter…).

There is a very important exponential function that arises naturally in many places. This function is called the **natural exponential function**. However, for must people this is simply the exponential function.

Definition : The **natural exponential function** is $f(x) = \mathbf{e}^x$ where,

$\mathbf{e} = 2.71828182845905\ldots$.

So, since $\mathbf{e} > 1$ we also know that $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

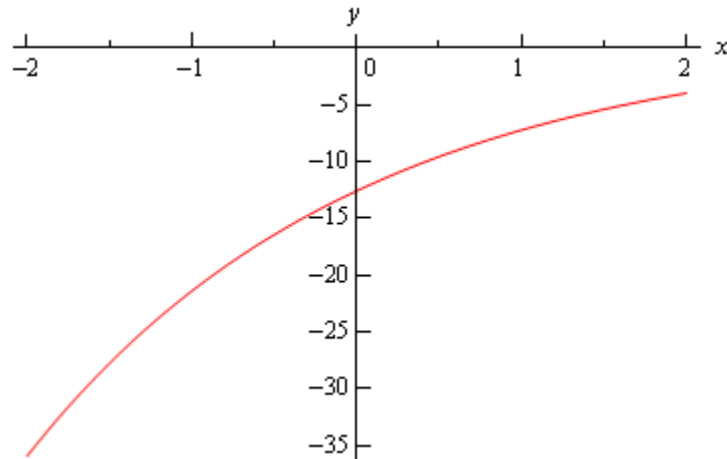Let's take a quick look at an example.

Example 2 Sketch the graph of $h(t) = 1 - 5\mathbf{e}^{1-\frac{t}{2}}$

Solution
Let's first get a table of values for this function.

t	-2	-1	0	1	2	3
$h(t)$	-35.9453	-21.4084	-12.5914	-7.2436	-4	-2.0327

Here is the sketch.

The main point behind this problem is to make sure you can do this type of evaluation so make sure that you can get the values that we graphed in this example. You will be asked to do this kind of evaluation on occasion in this class.

You will be seeing exponential functions in pretty much every chapter in this class so make sure that you are comfortable with them.

Review : Logarithm Functions

In this section we'll take a look at a function that is related to the exponential functions we looked at in the last section. We will look logarithms in this section. Logarithms are one of the functions that students fear the most. The main reason for this seems to be that they simply have never really had to work with them. Once they start working with them, students come to realize that they aren't as bad as they first thought.

We'll start with $b > 0$, $b \neq 1$ just as we did in the last section. Then we have

$$y = \log_b x \qquad\qquad \text{is equivalent to} \qquad\qquad x = b^y$$

The first is called logarithmic form and the second is called the exponential form. Remembering this equivalence is the key to evaluating logarithms. The number, b, is called the base.

Example 1 Without a calculator give the exact value of each of the following logarithms.

        (a) $\log_2 16$ [Solution]

        (b) $\log_4 16$ [Solution]

        (c) $\log_5 625$ [Solution]

        (d) $\log_9 \dfrac{1}{531441}$ [Solution]

        (e) $\log_{\frac{1}{6}} 36$ [Solution]

        (f) $\log_{\frac{3}{2}} \dfrac{27}{8}$ [Solution]

Solution

To quickly evaluate logarithms the easiest thing to do is to convert the logarithm to exponential form. So, let's take a look at the first one.

(a) $\log_2 16$

First, let's convert to exponential form.

$$\log_2 16 = ? \qquad\qquad \text{is equivalent to} \qquad\qquad 2^? = 16$$

So, we're really asking 2 raised to what gives 16. Since 2 raised to 4 is 16 we get,

$$\log_2 16 = 4 \qquad\qquad \text{because} \qquad\qquad 2^4 = 16$$

We'll not do the remaining parts in quite this detail, but they were all worked in this way.

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(b) $\log_4 16$

$$\log_4 16 = 2 \qquad\qquad \text{because} \qquad\qquad 4^2 = 16$$

Note the difference the first and second logarithm! The base is important! It can completely change the answer.

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(c) $\log_5 625 = 4$ because $5^4 = 625$

(d) $\log_9 \dfrac{1}{531441} = -6$ because $9^{-6} = \dfrac{1}{9^6} = \dfrac{1}{531441}$

(e) $\log_{\frac{1}{6}} 36 = -2$ because $\left(\dfrac{1}{6}\right)^{-2} = 6^2 = 36$

(f) $\log_{\frac{3}{2}} \dfrac{27}{8} = 3$ because $\left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$

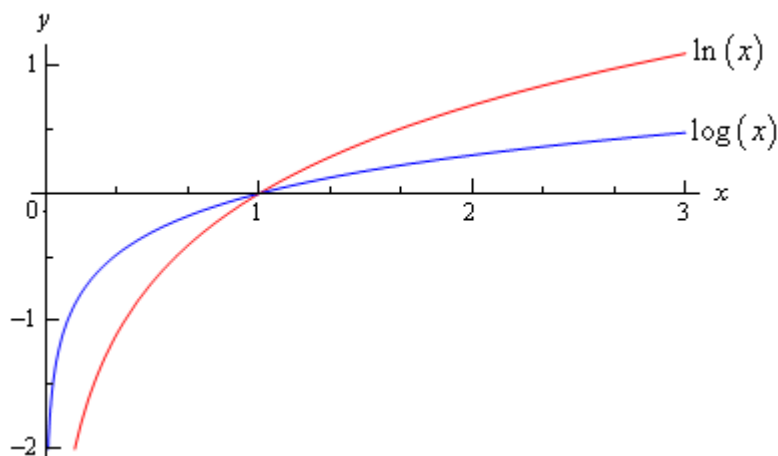There are a couple of special logarithms that arise in many places. These are,

$$\ln x = \log_e x \qquad \text{This log is called the natural logarithm}$$
$$\log x = \log_{10} x \qquad \text{This log is called the common logarithm}$$

In the natural logarithm the base **e** is the same number as in the natural exponential logarithm that we saw in the last section. Here is a sketch of both of these logarithms.

From this graph we can get a couple of very nice properties about the natural logarithm that we will use many times in this and later Calculus courses.

$$\ln x \to \infty \quad \text{as} \quad x \to \infty$$
$$\ln x \to -\infty \quad \text{as} \quad x \to 0, \; x > 0$$

Let's take a look at a couple of more logarithm evaluations. Some of which deal with the natural or common logarithm and some of which don't.

Example 2 Without a calculator give the exact value of each of the following logarithms.

   (a) $\ln \sqrt[3]{\mathbf{e}}$

   (b) $\log 1000$

   (c) $\log_{16} 16$

   (d) $\log_{23} 1$

   (e) $\log_2 \sqrt[7]{32}$

Solution

These work exactly the same as previous example so we won't put in too many details.

(a) $\ln \sqrt[3]{\mathbf{e}} = \dfrac{1}{3}$ because $\mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$

(b) $\log 1000 = 3$ because $10^3 = 1000$

(c) $\log_{16} 16 = 1$ because $16^1 = 16$

(d) $\log_{23} 1 = 0$ because $23^0 = 1$

(e) $\log_2 \sqrt[7]{32} = \dfrac{5}{7}$ because $\sqrt[7]{32} = 32^{\frac{1}{7}} = \left(2^5\right)^{\frac{1}{7}} = 2^{\frac{5}{7}}$

This last set of examples leads us to some of the basic properties of logarithms.

Properties

1. The domain of the logarithm function is $(0, \infty)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. $\log_b b = 1$
3. $\log_b 1 = 0$
4. $\log_b b^x = x$
5. $b^{\log_b x} = x$

The last two properties will be especially useful in the next section. Notice as well that these last two properties tell us that,

$$f(x) = b^x \qquad \text{and} \qquad g(x) = \log_b x$$

are inverses of each other.

Here are some more properties that are useful in the manipulation of logarithms.

More Properties

6. $\log_b xy = \log_b x + \log_b y$

7. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

8. $\log_b\left(x^r\right) = r\log_b x$

Note that there is no equivalent property to the first two for sums and differences. In other words,

$$\log_b(x+y) \neq \log_b x + \log_b y$$
$$\log_b(x-y) \neq \log_b x - \log_b y$$

Example 3 Write each of the following in terms of simpler logarithms.

    (a) $\ln x^3 y^4 z^5$ [Solution]

    (b) $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$ [Solution]

    (c) $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$ [Solution]

Solution

What the instructions really mean here is to use as many if the properties of logarithms as we can to simplify things down as much as we can.

(a) $\ln x^3 y^4 z^5$

Property 6 above can be extended to products of more than two functions. Once we've used Property 6 we can then use Property 8.

$$\ln x^3 y^4 z^5 = \ln x^3 + \ln y^4 + \ln z^5$$
$$= 3\ln x + 4\ln y + 5\ln z$$

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(b) $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$

When using property 7 above make sure that the logarithm that you subtract is the one that contains the denominator as its argument. Also, note that that we'll be converting the root to fractional exponents in the first step.

$$\log_3\left(\frac{9x^4}{\sqrt{y}}\right) = \log_3 9x^4 - \log_3 y^{\frac{1}{2}}$$

$$= \log_3 9 + \log_3 x^4 - \log_3 y^{\frac{1}{2}}$$

$$= 2 + 4\log_3 x - \frac{1}{2}\log_3 y$$

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(c) $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

The point to this problem is mostly the correct use of property 8 above.

$$\log\left(\frac{x^2+y^2}{(x-y)^3}\right)=\log\left(x^2+y^2\right)-\log\left(x-y\right)^3$$

$$=\log\left(x^2+y^2\right)-3\log\left(x-y\right)$$

You can use Property 8 on the second term because the WHOLE term was raised to the 3, but in the first logarithm, only the individual terms were squared and not the term as a whole so the 2's must stay where they are!

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The last topic that we need to look at in this section is the **change of base** formula for logarithms. The change of base formula is,

$$\log_b x=\frac{\log_a x}{\log_a b}$$

This is the most general change of base formula and will convert from base b to base a. However, the usual reason for using the change of base formula is to compute the value of a logarithm that is in a base that you can't easily deal with. Using the change of base formula means that you can write the logarithm in terms of a logarithm that you can deal with. The two most common change of base formulas are

$$\log_b x=\frac{\ln x}{\ln b}\qquad\text{and}\qquad \log_b x=\frac{\log x}{\log b}$$

In fact, often you will see one or the other listed as THE change of base formula!

In the first part of this section we computed the value of a few logarithms, but we could do these without the change of base formula because all the arguments could be written in terms of the base to a power. For instance,

$$\log_7 49=2\qquad\text{because}\qquad 7^2=49$$

However, this only works because 49 can be written as a power of 7! We would need the change of base formula to compute $\log_7 50$.

$$\log_7 50=\frac{\ln 50}{\ln 7}=\frac{3.91202300543}{1.94591014906}=2.0103821378$$

OR

$$\log_7 50=\frac{\log 50}{\log 7}=\frac{1.69897000434}{0.845098040014}=2.0103821378$$

So, it doesn't matter which we use, we will get the same answer regardless of the logarithm that we use in the change of base formula.

Note as well that we could use the change of base formula on $\log_7 49$ if we wanted to as well.

$$\log_7 49 = \frac{\ln 49}{\ln 7} = \frac{3.89182029811}{1.94591014906} = 2$$

This is a lot of work however, and is probably not the best way to deal with this.

So, in this section we saw how logarithms work and took a look at some of the properties of logarithms. We will run into logarithms on occasion so make sure that you can deal with them when we do run into them.

Review : Exponential and Logarithm Equations

In this section we'll take a look at solving equations with exponential functions or logarithms in them.

We'll start with equations that involve exponential functions. The main property that we'll need for these equations is,

$$\log_b b^x = x$$

Example 1 Solve $7 + 15e^{1-3z} = 10$.

Solution
The first step is to get the exponential all by itself on one side of the equation with a coefficient of one.

$$7 + 15e^{1-3z} = 10$$
$$15e^{1-3z} = 3$$
$$e^{1-3z} = \frac{1}{5}$$

Now, we need to get the z out of the exponent so we can solve for it. To do this we will use the property above. Since we have an **e** in the equation we'll use the natural logarithm. First we take the logarithm of both sides and then use the property to simplify the equation.

$$\ln\left(e^{1-3z}\right) = \ln\left(\frac{1}{5}\right)$$
$$1 - 3z = \ln\left(\frac{1}{5}\right)$$

All we need to do now is solve this equation for z.

$$1 - 3z = \ln\left(\frac{1}{5}\right)$$
$$-3z = -1 + \ln\left(\frac{1}{5}\right)$$
$$z = -\frac{1}{3}\left(-1 + \ln\left(\frac{1}{5}\right)\right) = 0.8698126372$$

Example 2 Solve $10^{t^2 - t} = 100$.

Solution
Now, in this case it looks like the best logarithm to use is the common logarithm since left hand side has a base of 10. There's no initial simplification to do, so just take the log of both sides and simplify.

$$\log 10^{t^2 - t} = \log 100$$
$$t^2 - t = 2$$
At this point, we've just got a quadratic that can be solved
$$t^2 - t - 2 = 0$$
$$(t - 2)(t + 1) = 0$$
So, it looks like the solutions in this case are $t = 2$ and $t = -1$.

Now that we've seen a couple of equations where the variable only appears in the exponent we need to see an example with variables both in the exponent and out of it.

Example 3 Solve $x - x\mathbf{e}^{5x+2} = 0$.

Solution
The first step is to factor an x out of both terms.

DO NOT DIVIDE AN x FROM BOTH TERMS!!!!

Note that it is very tempting to "simplify" the equation by dividing an x out of both terms. However, if you do that you'll miss a solution as we'll see.

$$x - x\mathbf{e}^{5x+2} = 0$$
$$x\left(1 - \mathbf{e}^{5x+2}\right) = 0$$

So, it's now a little easier to deal with. From this we can see that we get one of two possibilities.
$$x = 0 \qquad \text{OR}$$
$$1 - \mathbf{e}^{5x+2} = 0$$

The first possibility has nothing more to do, except notice that if we had divided both sides by an x we would have missed this one so be careful. In the second possibility we've got a little more to do. This is an equation similar to the first two that we did in this section.
$$\mathbf{e}^{5x+2} = 1$$
$$5x + 2 = \ln 1$$
$$5x + 2 = 0$$
$$x = -\frac{2}{5}$$
Don't forget that $\ln 1 = 0$!

So, the two solutions are $x = 0$ and $x = -\frac{2}{5}$.

The next equation is a more complicated (looking at least…) example similar to the previous one.

Example 4 Solve $5(x^2 - 4) = (x^2 - 4)e^{7-x}$.

Solution

As with the previous problem do NOT divide an $x^2 - 4$ out of both sides. Doing this will lose solutions even though it "simplifies" the equation. Note however, that if you can divide a term out then you can also factor it out if the equation is written properly.

So, the first step here is to move everything to one side of the equation and then to factor out the $x^2 - 4$.

$$5(x^2 - 4) - (x^2 - 4)e^{7-x} = 0$$
$$(x^2 - 4)(5 - e^{7-x}) = 0$$

At this point all we need to do is set each factor equal to zero and solve each.

$$x^2 - 4 = 0 \qquad 5 - e^{7-x} = 0$$
$$x = \pm 2 \qquad e^{7-x} = 5$$
$$7 - x = \ln(5)$$
$$x = 7 - \ln(5) = 5.390562088$$

The three solutions are then $x = \pm 2$ and $x = 5.3906$.

As a final example let's take a look at an equation that contains two different logarithms.

Example 5 Solve $4e^{1+3x} - 9e^{5-2x} = 0$.

Solution

The first step here is to get one exponential on each side and then we'll divide both sides by one of them (which doesn't matter for the most part) so we'll have a quotient of two exponentials. The quotient can then be simplified and we'll finally get both coefficients on the other side. Doing all of this gives,

$$4e^{1+3x} = 9e^{5-2x}$$
$$\frac{e^{1+3x}}{e^{5-2x}} = \frac{9}{4}$$
$$e^{1+3x-(5-2x)} = \frac{9}{4}$$
$$e^{5x-4} = \frac{9}{4}$$

Note that while we said that it doesn't really matter which exponential we divide out by doing it the way we did here we'll avoid a negative coefficient on the x. Not a major issue, but those minus signs on coefficients are really easy to lose on occasion.

This is now in a form that we can deal with so here's the rest of the solution.

$$\mathbf{e}^{5x-4} = \tfrac{9}{4}$$
$$5x - 4 = \ln\left(\tfrac{9}{4}\right)$$
$$5x = 4 + \ln\left(\tfrac{9}{4}\right)$$
$$x = \tfrac{1}{5}\left(4 + \ln\left(\tfrac{9}{4}\right)\right) = 0.9621860432$$

This equation has a single solution of $x = 0.9622$.

Now let's take a look at some equations that involve logarithms. The main property that we'll be using to solve these kinds of equations is,

$$b^{\log_b x} = x$$

Example 6 Solve $3 + 2\ln\left(\dfrac{x}{7} + 3\right) = -4$.

Solution
This first step in this problem is to get the logarithm by itself on one side of the equation with a coefficient of 1.

$$2\ln\left(\frac{x}{7} + 3\right) = -7$$
$$\ln\left(\frac{x}{7} + 3\right) = -\frac{7}{2}$$

Now, we need to get the x out of the logarithm and the best way to do that is to "exponentiate" both sides using **e**. In other word,

$$\mathbf{e}^{\ln\left(\frac{x}{7}+3\right)} = \mathbf{e}^{-\frac{7}{2}}$$

So using the property above with **e**, since there is a natural logarithm in the equation, we get,

$$\frac{x}{7} + 3 = \mathbf{e}^{-\frac{7}{2}}$$

Now all that we need to do is solve this for x.

$$\frac{x}{7} + 3 = \mathbf{e}^{-\frac{7}{2}}$$
$$\frac{x}{7} = -3 + \mathbf{e}^{-\frac{7}{2}}$$
$$x = 7\left(-3 + \mathbf{e}^{-\frac{7}{2}}\right) = -20.78861832$$

At this point we might be tempted to say that we're done and move on. However, we do need to be careful. Recall from the previous section that we can't plug a negative number into a logarithm. This, by itself, doesn't mean that our answer won't work since its negative. What we need to do is plug it into the logarithm and make sure that $\frac{x}{7} + 3$ will not be negative. I'll leave it to you to verify that this is in fact positive upon plugging our solution into the logarithm and so $x = -20.78861832$ is in fact a solution to the equation.

Let's now take a look at a more complicated equation. Often there will be more than one logarithm in the equation. When this happens we will need to use on or more of the following properties to combine all the logarithms into a single logarithm. Once this has been done we can proceed as we did in the previous example.

$$\log_b xy = \log_b x + \log_b y \qquad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \qquad \log_b\left(x^r\right) = r\log_b x$$

Example 7 Solve $2\ln\left(\sqrt{x}\right) - \ln(1-x) = 2$.

Solution
First get the two logarithms combined into a single logarithm.

$$2\ln\left(\sqrt{x}\right) - \ln(1-x) = 2$$
$$\ln\left(\left(\sqrt{x}\right)^2\right) - \ln(1-x) = 2$$
$$\ln(x) - \ln(1-x) = 2$$
$$\ln\left(\frac{x}{1-x}\right) = 2$$

Now, exponentiate both sides and solve for *x*.

$$\frac{x}{1-x} = \mathbf{e}^2$$
$$x = \mathbf{e}^2(1-x)$$
$$x = \mathbf{e}^2 - \mathbf{e}^2 x$$
$$x\left(1 + \mathbf{e}^2\right) = \mathbf{e}^2$$
$$x = \frac{\mathbf{e}^2}{1 + \mathbf{e}^2} = 0.8807970780$$

Finally, we just need to make sure that the solution, $x = 0.8807970780$, doesn't produce negative numbers in both of the original logarithms. It doesn't, so this is in fact our solution to this problem.

Let's take a look at one more example.

Example 8 Solve $\log x + \log(x-3) = 1$.

Solution
As with the last example, first combine the logarithms into a single logarithm.

$$\log x + \log(x-3) = 1$$
$$\log(x(x-3)) = 1$$

Now exponentiate, using 10 this time instead of **e** because we've got common logs in the equation, both sides.

$$10^{\log(x^2-3x)} = 10^1$$
$$x^2 - 3x = 10$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$

So, potential solutions are $x = 5$ and $x = -2$. Note, however that if we plug $x = -2$ into either of the two original logarithms we would get negative numbers so this can't be a solution. We can however, use $x = 5$.

Therefore, the solution to this equation is $x = 5$.

When solving equations with logarithms it is important to check your potential solutions to make sure that they don't generate logarithms of negative numbers or zero. It is also important to make sure that you do the checks in the **original** equation. If you check them in the second logarithm above (after we've combined the two logs) both solutions will appear to work! This is because in combining the two logarithms we've actually changed the problem. In fact, it is this change that introduces the extra solution that we couldn't use!

Also be careful in solving equations containing logarithms to not get locked into the idea that you will get two potential solutions and only one of these will work. It is possible to have problems where both are solutions and where neither are solutions.

Review : Common Graphs

The purpose of this section is to make sure that you're familiar with the graphs of many of the basic functions that you're liable to run across in a calculus class.

Example 1 Graph $y = -\dfrac{2}{5}x + 3$.

Solution
This is a line in the slope intercept form

$$y = mx + b$$

In this case the line has a *y* intercept of (0,*b*) and a slope of *m*. Recall that slope can be thought of as

$$m = \frac{\text{rise}}{\text{run}}$$

Note that if the slope is negative we tend to think of the rise as a fall.

The slope allows us to get a second point on the line. Once we have any point on the line and the slope we move right by *run* and up/down by *rise* depending on the sign. This will be a second point on the line.

In this case we know (0,3) is a point on the line and the slope is $-\dfrac{2}{5}$. So starting at (0,3) we'll move 5 to the right (*i.e.* $0 \rightarrow 5$) and down 2 (*i.e.* $3 \rightarrow 1$) to get (5,1) as a second point on the line. Once we've got two points on a line all we need to do is plot the two points and connect them with a line.

Here's the sketch for this line.
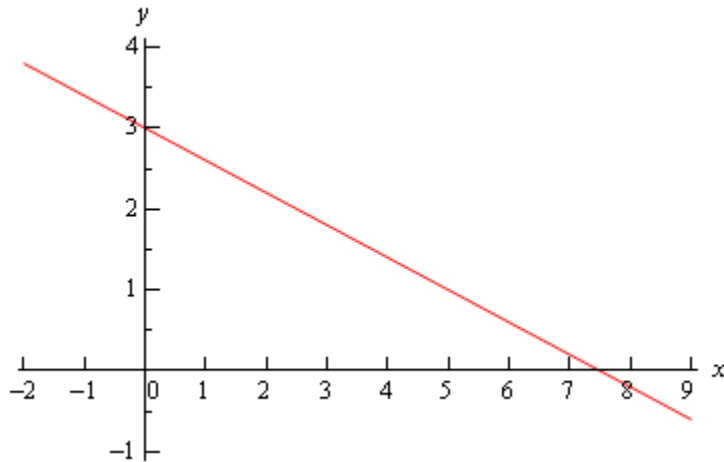
Example 2 Graph $f(x) = |x|$

Solution
There really isn't much to this problem outside of reminding ourselves of what absolute value is. Recall that the absolute value function is defined as,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph is then,
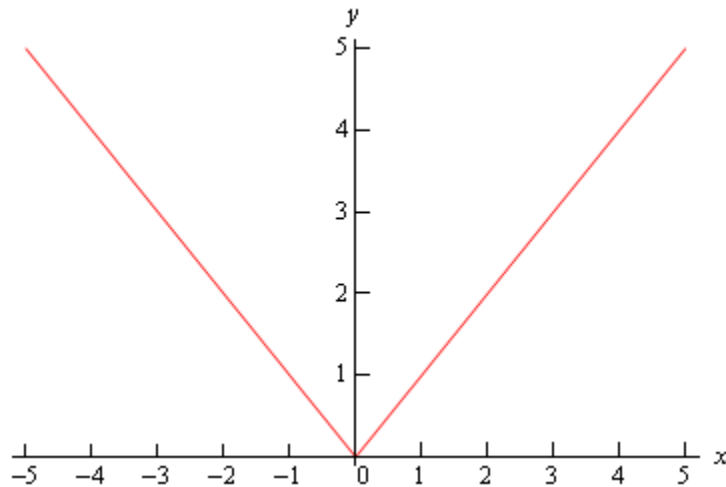
Example 3 Graph $f(x) = -x^2 + 2x + 3$.

Solution
This is a parabola in the general form.

$$f(x) = ax^2 + bx + c$$

In this form, the *x*-coordinate of the vertex (the highest or lowest point on the parabola) is $x = -\dfrac{b}{2a}$ and we get the *y*-coordinate is $y = f\left(-\dfrac{b}{2a}\right)$. So, for our parabola the coordinates of the vertex will be.

$$x = -\frac{2}{2(-1)} = 1$$

$$y = f(1) = -(1)^2 + 2(1) + 3 = 4$$

So, the vertex for this parabola is (1,4).

We can also determine which direction the parabola opens from the sign of *a*. If *a* is positive the parabola opens up and if *a* is negative the parabola opens down. In our case the parabola opens down.

Now, because the vertex is above the *x*-axis and the parabola opens down we know that we'll have *x*-intercepts (*i.e.* values of *x* for which we'll have $f(x) = 0$) on this graph. So, we'll solve the following.

$$-x^2 + 2x + 3 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$

So, we will have *x*-intercepts at $x = -1$ and $x = 3$. Notice that to make our life easier in the solution process we multiplied everything by -1 to get the coefficient of the x^2 positive. This made the factoring easier.

Here's a sketch of this parabola.
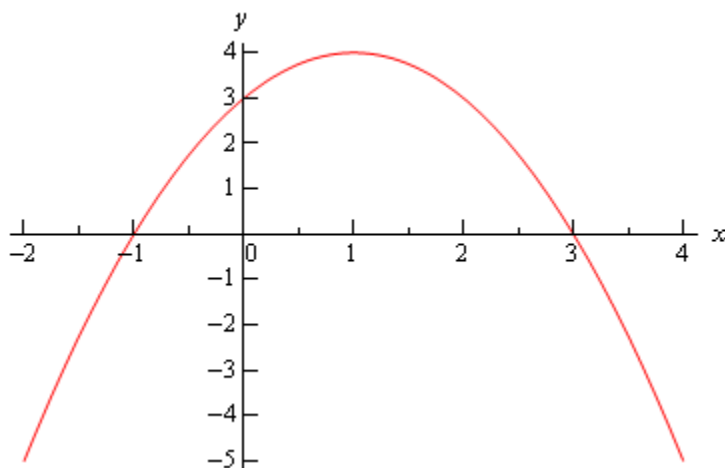
Example 4 Graph $f(y) = y^2 - 6y + 5$

Solution
Most people come out of an Algebra class capable of dealing with functions in the form $y = f(x)$. However, many functions that you will have to deal with in a Calculus class are in the form $x = f(y)$ and can only be easily worked with in that form. So, you need to get used to working with functions in this form.

The nice thing about these kinds of function is that if you can deal with functions in the form $y = f(x)$ then you can deal with functions in the form $x = f(y)$ even if you aren't that familiar with them.

Let's first consider the equation.

$$y = x^2 - 6x + 5$$

This is a parabola that opens up and has a vertex of (3,-4), as we know from our work in the previous example.

For our function we have essentially the same equation except the *x* and *y*'s are switched around. In other words, we have a parabola in the form,

$$x = ay^2 + by + c$$

This is the general form of this kind of parabola and this will be a parabola that opens left or right depending on the sign of *a*. The *y*-coordinate of the vertex is given by $y = -\dfrac{b}{2a}$ and we find the *x*-coordinate by plugging this into the equation. So, you can see that this is very similar to the type of parabola that you're already used to dealing with.

Now, let's get back to the example. Our function is a parabola that opens to the right (*a* is positive) and has a vertex at (-4,3). The vertex is to the left of the *y*-axis and opens to the right so we'll need the *y*-intercepts (*i.e.* values of *y* for which we'll have $f(y) = 0$)). We find these just like we found *x*-intercepts in the previous problem.

$$y^2 - 6y + 5 = 0$$
$$(y-5)(y-1) = 0$$

So, our parabola will have *y*-intercepts at $y = 1$ and $y = 5$. Here's a sketch of the graph.
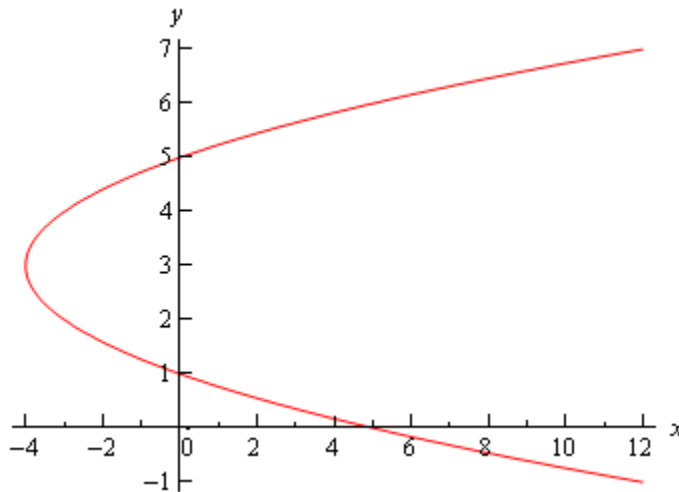
Example 5 Graph $x^2 + 2x + y^2 - 8y + 8 = 0$.

Solution
To determine just what kind of graph we've got here we need complete the square on both the *x* and the *y*.

$$x^2 + 2x + y^2 - 8y + 8 = 0$$
$$x^2 + 2x + 1 - 1 + y^2 - 8y + 16 - 16 + 8 = 0$$
$$(x+1)^2 + (y-4)^2 = 9$$

Recall that to complete the square we take the half of the coefficient of the *x* (or the *y*), square this

and then add and subtract it to the equation.

Upon doing this we see that we have a circle and it's now written in standard form.

$$(x-h)^2 + (y-k)^2 = r^2$$

When circles are in this form we can easily identify the center : (h, k) and radius : r. Once we have these we can graph the circle simply by starting at the center and moving right, left, up and down by r to get the rightmost, leftmost, top most and bottom most points respectively.
Our circle has a center at (-1, 4) and a radius of 3. Here's a sketch of this circle.
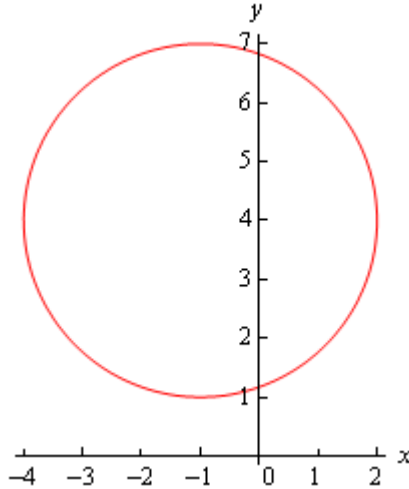
Example 6 Graph $\dfrac{(x-2)^2}{9} + 4(y+2)^2 = 1$

Solution
This is an ellipse. The standard form of the ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

This is an ellipse with center (h, k) and the right most and left most points are a distance of a away from the center and the top most and bottom most points are a distance of b away from the center.
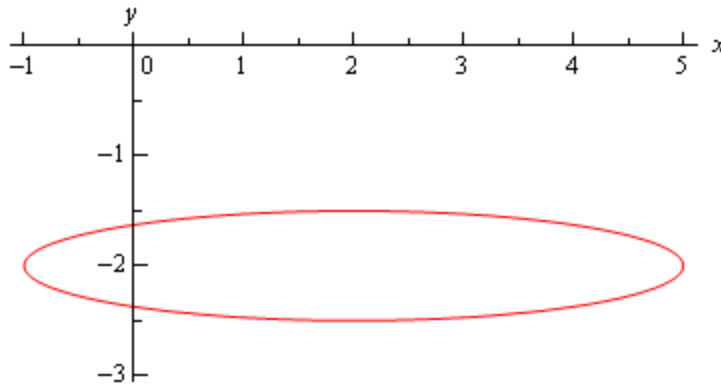
The ellipse for this problem has center (2, -2) and has $a = 3$ and $b = \dfrac{1}{2}$. Note that to get the b we're really rewriting the equation as,

$$\frac{(x-2)^2}{9} + \frac{(y+2)^2}{1/4} = 1$$

to get it into standard from.

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Here's a sketch of the ellipse.

Example 7 Graph $\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{4} = 1$

Solution
This is a hyperbola. There are actually two standard forms for a hyperbola. Here are the basics for each form.

Form	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$
Center	(h, k)	(h, k)
Opens	Opens right and left	Opens up and down
Vertices	a units right and left of center.	b units up and down from center.
Slope of Asymptotes	$\pm\dfrac{b}{a}$	$\pm\dfrac{b}{a}$

So, what does all this mean? First, notice that one of the terms is positive and the other is negative. This will determine which direction the two parts of the hyperbola open. If the *x* term is positive the hyperbola opens left and right. Likewise, if the *y* term is positive the parabola opens up and down.

Both have the same "center". Note that hyperbolas don't really have a center in the sense that circles and ellipses have centers. The center is the starting point in graphing a hyperbola. It tells up how to get to the vertices and how to get the asymptotes set up.

The asymptotes of a hyperbola are two lines that intersect at the center and have the slopes listed

above. As you move farther out from the center the graph will get closer and closer to the asymptotes.

For the equation listed here the hyperbola will open left and right. Its center is (-1, 2). The two vertices are (-4, 2) and (2, 2). The asymptotes will have slopes $\pm\dfrac{2}{3}$.

Here is a sketch of this hyperbola. Note that the asymptotes are denoted by the two dashed lines.
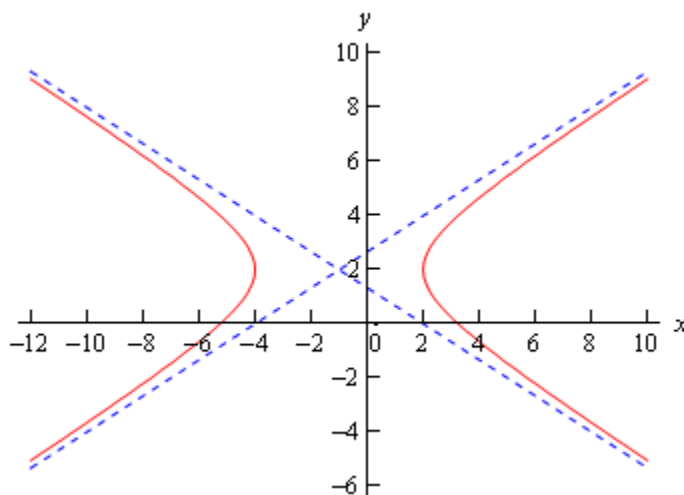
Example 8 Graph $f(x) = \mathbf{e}^x$ and $g(x) = \mathbf{e}^{-x}$.

Solution
There really isn't a lot to this problem other than making sure that both of these exponentials are graphed somewhere.
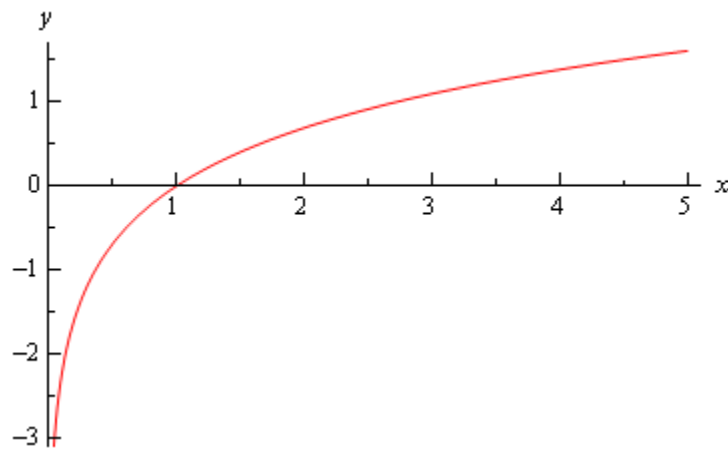
These will both show up with some regularity in later sections and their behavior as x goes to both plus and minus infinity will be needed and from this graph we can clearly see this behavior.

$g(x) = e^{-x}$ y $f(x) = e^{x}$
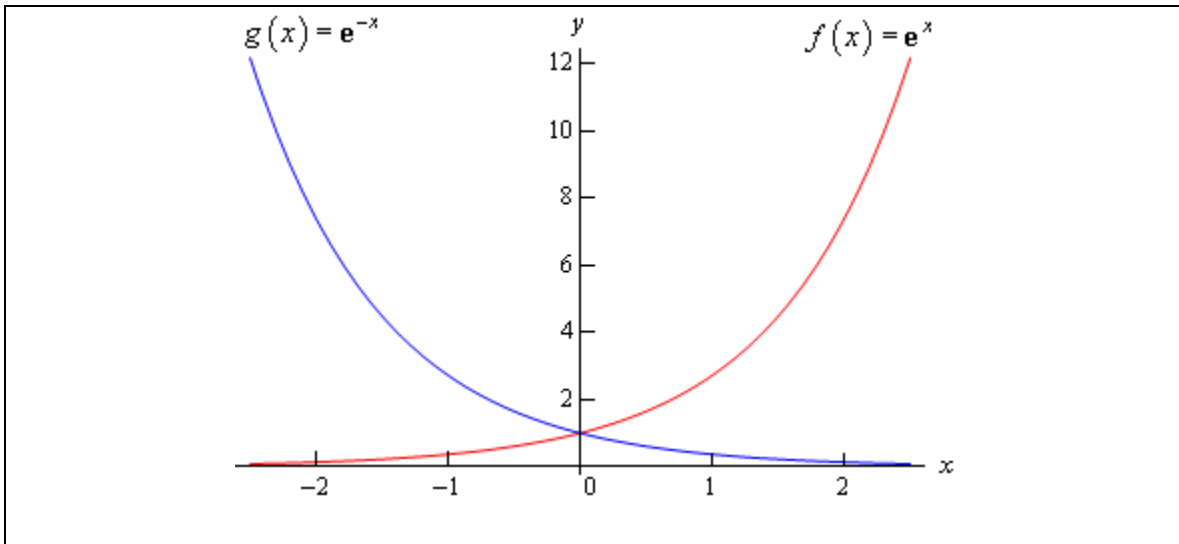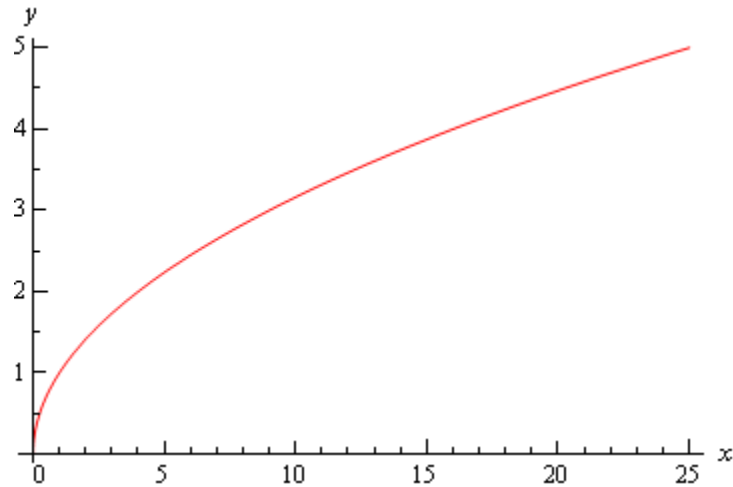
Example 9 Graph $f(x) = \ln(x)$.

Solution
This has already been graphed once in this review, but this puts it here with all the other "important" graphs.

Example 10 Graph $y = \sqrt{x}$.

Solution
This one is fairly simple, we just need to make sure that we can graph it when need be.
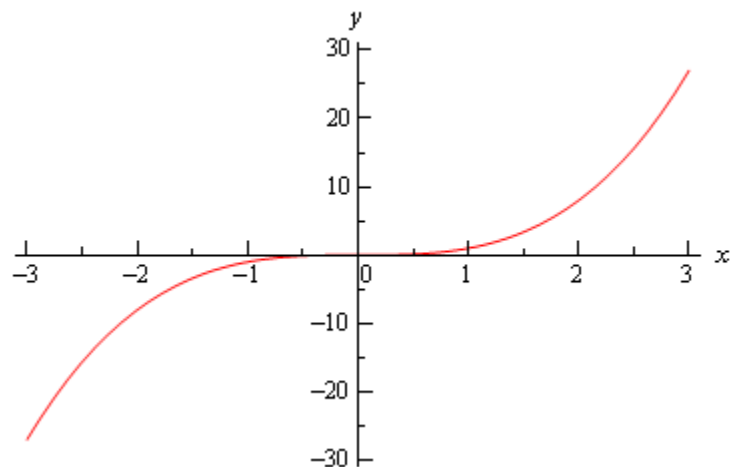

Remember that the domain of the square root function is $x \geq 0$.

Example 11 Graph $y = x^3$

Solution
Again, there really isn't much to this other than to make sure it's been graphed somewhere so we can say we've done it.
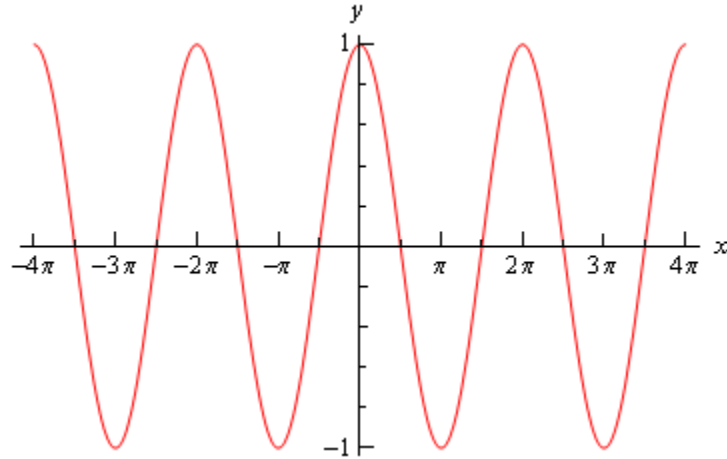

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Example 12 Graph $y = \cos(x)$

Solution
There really isn't a whole lot to this one. Here's the graph for $-4\pi \le x \le 4\pi$.

Let's also note here that we can put all values of x into cosine (which won't be the case for most of the trig functions) and so the domain is all real numbers. Also note that
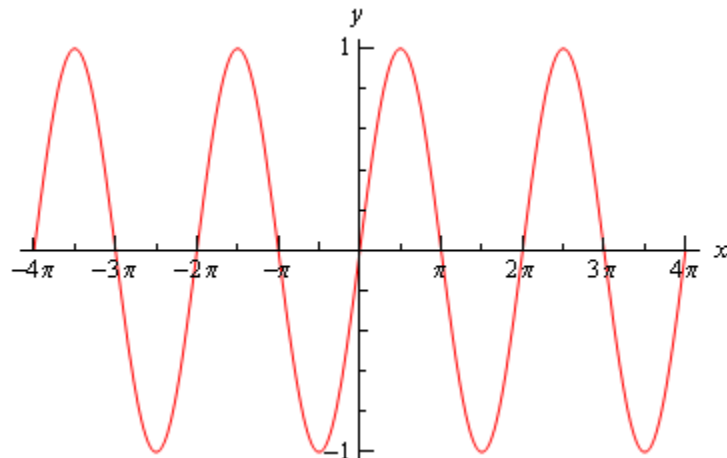
$$-1 \le \cos(x) \le 1$$

It is important to notice that cosine will never be larger than 1 or smaller than -1. This will be useful on occasion in a calculus class. In general we can say that

$$-R \le R\cos(\omega x) \le R$$

Example 13 Graph $y = \sin(x)$

Solution
As with the first problem in this section there really isn't a lot to do other than graph it. Here is the graph.

From this graph we can see that sine has the same range that cosine does. In general
$$-R \leq R\sin(\omega x) \leq R$$
As with cosine, sine itself will never be larger than 1 and never smaller than -1. Also the domain of sine is all real numbers.

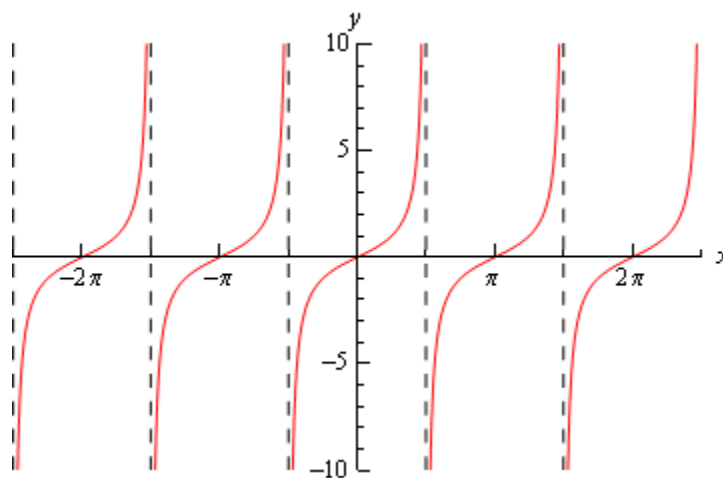***Example 14*** Graph $y = \tan(x)$.

Solution
In the case of tangent we have to be careful when plugging x's in since tangent doesn't exist wherever cosine is zero (remember that $\tan x = \dfrac{\sin x}{\cos x}$). Tangent will not exist at
$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$
and the graph will have asymptotes at these points. Here is the graph of tangent on the range $-\dfrac{5\pi}{2} < x < \dfrac{5\pi}{2}$.

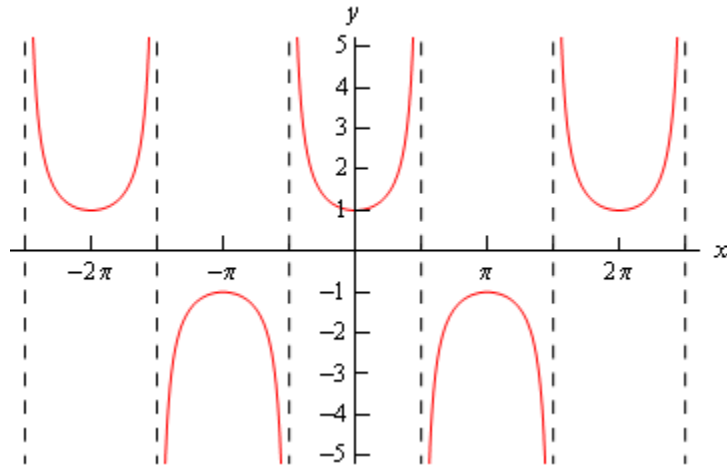***Example 15*** Graph $y = \sec(x)$

Solution
As with tangent we will have to avoid x's for which cosine is zero (remember that $\sec x = \dfrac{1}{\cos x}$). Secant will not exist at
$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$
and the graph will have asymptotes at these points. Here is the graph of secant on the range $-\dfrac{5\pi}{2} < x < \dfrac{5\pi}{2}$.

Notice that the graph is always greater than 1 and less than -1. This should not be terribly surprising. Recall that $-1 \leq \cos(x) \leq 1$. So, one divided by something less than one will be greater than 1. Also, $\frac{1}{\pm 1} = \pm 1$ and so we get the following ranges for secant.

$$\sec(\omega x) \geq 1 \qquad \text{and} \qquad \sec(\omega x) \leq -1$$

Limits

Introduction

The topic that we will be examining in this chapter is that of Limits. This is the first of three major topics that we will be covering in this course. While we will be spending the least amount of time on limits in comparison to the other two topics limits are very important in the study of Calculus. We will be seeing limits in a variety of places once we move out of this chapter. In particular we will see that limits are part of the formal definition of the other two major topics.

Here is a quick listing of the material that will be covered in this chapter.

Tangent Lines and Rates of Change – In this section we will take a look at two problems that we will see time and again in this course. These problems will be used to introduce the topic of limits.

The Limit – Here we will take a conceptual look at limits and try to get a grasp on just what they are and what they can tell us.

One-Sided Limits – A brief introduction to one-sided limits.

Limit Properties – Properties of limits that we'll need to use in computing limits. We will also compute some basic limits in this section

Computing Limits – Many of the limits we'll be asked to compute will not be "simple" limits. In other words, we won't be able to just apply the properties and be done. In this section we will look at several types of limits that require some work before we can use the limit properties to compute them.

Infinite Limits – Here we will take a look at limits that have a value of infinity or negative infinity. We'll also take a brief look at vertical asymptotes.

Limits At Infinity, Part I – In this section we'll look at limits at infinity. In other words, limits in which the variable gets very large in either the positive or negative sense. We'll also take a brief look at horizontal asymptotes in this section. We'll be concentrating on polynomials and rational expression involving polynomials in this section.

Limits At Infinity, Part II – We'll continue to look at limits at infinity in this section, but this time we'll be looking at exponential, logarithms and inverse tangents.

Continuity – In this section we will introduce the concept of continuity and how it relates to limits. We will also see the Mean Value Theorem in this section.

The Definition of the Limit – We will give the exact definition of several of the limits covered in this section. We'll also give the exact definition of continuity.

Rates of Change and Tangent Lines

In this section we are going to take a look at two fairly important problems in the study of calculus. There are two reasons for looking at these problems now.

First, both of these problems will lead us into the study of limits, which is the topic of this chapter after all. Looking at these problems here will allow us to start to understand just what a limit is and what it can tell us about a function.

Secondly, the rate of change problem that we're going to be looking at is one of the most important concepts that we'll encounter in the second chapter of this course. In fact, it's probably one of the most important concepts that we'll encounter in the whole course. So looking at it now will get us to start thinking about it from the very beginning.

Tangent Lines

The first problem that we're going to take a look at is the tangent line problem. Before getting into this problem it would probably be best to define a tangent line.

A tangent line to the function $f(x)$ at the point $x = a$ is a line that just touches the graph of the function at the point in question and is "parallel" (in some way) to the graph at that point. Take a look at the graph below.
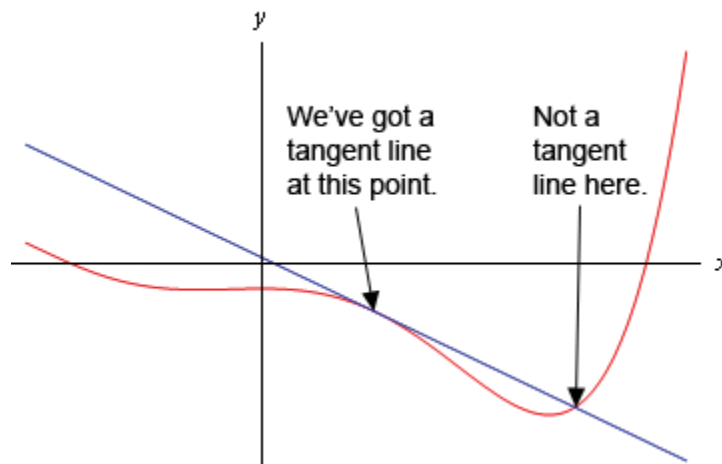
In this graph the line is a tangent line at the indicated point because it just touches the graph at that point and is also "parallel" to the graph at that point. Likewise, at the second point shown, the line does just touch the graph at that point, but it is not "parallel" to the graph at that point and so it's not a tangent line to the graph at that point.

At the second point shown (the point where the line isn't a tangent line) we will sometimes call the line a **secant line**.

We've used the word parallel a couple of times now and we should probably be a little careful with it. In general, we will think of a line and a graph as being parallel at a point if they are both

moving in the same direction at that point. So, in the first point above the graph and the line are moving in the same direction and so we will say they are parallel at that point. At the second point, on the other hand, the line and the graph are not moving in the same direction and so they aren't parallel at that point.

Okay, now that we've gotten the definition of a tangent line out of the way let's move on to the tangent line problem. That's probably best done with an example.

Example 1 Find the tangent line to $f(x) = 15 - 2x^2$ at $x = 1$.

Solution
We know from algebra that to find the equation of a line we need either two points on the line or a single point on the line and the slope of the line. Since we know that we are after a tangent line we do have a point that is on the line. The tangent line and the graph of the function must touch at $x = 1$ so the point $(1, f(1)) = (1, 13)$ must be on the line.

Now we reach the problem. This is all that we know about the tangent line. In order to find the tangent line we need either a second point or the slope of the tangent line. Since the only reason for needing a second point is to allow us to find the slope of the tangent line let's just concentrate on seeing if we can determine the slope of the tangent line.

At this point in time all that we're going to be able to do is to get an estimate for the slope of the tangent line, but if we do it correctly we should be able to get an estimate that is in fact the actual slope of the tangent line. We'll do this by starting with the point that we're after, let's call it $P = (1, 13)$. We will then pick another point that lies on the graph of the function, let's call that point $Q = (x, f(x))$.

For the sake of argument let's take choose $x = 2$ and so the second point will be $Q = (2, 7)$. Below is a graph of the function, the tangent line and the secant line that connects P and Q.

We can see from this graph that the secant and tangent lines are somewhat similar and so the slope of the secant line should be somewhat close to the actual slope of the tangent line. So, as an estimate of the slope of the tangent line we can use the slope of the secant line, let's call it m_{PQ}, which is,

$$m_{PQ} = \frac{f(2) - f(1)}{2 - 1} = \frac{7 - 13}{1} = -6$$
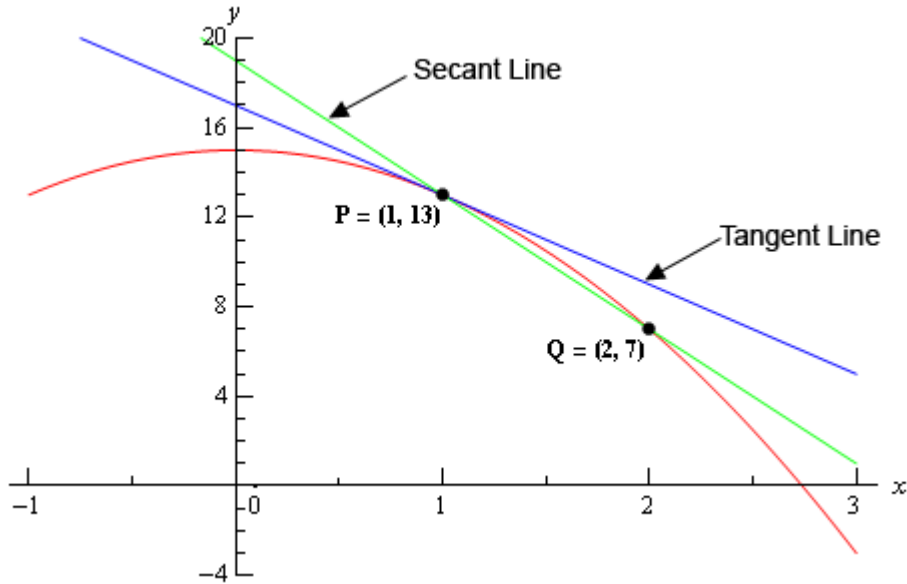
Now, if we weren't too interested in accuracy we could say this is good enough and use this as an estimate of the slope of the tangent line. However, we would like an estimate that is at least somewhat close the actual value. So, to get a better estimate we can take an x that is closer to $x = 1$ and redo the work above to get a new estimate on the slope. We could then take a third value of x even closer yet and get an even better estimate.

In other words, as we take Q closer and closer to P the slope of the secant line connecting Q and P should be getting closer and closer to the slope of the tangent line. If you are viewing this on the web, the image below shows this process.
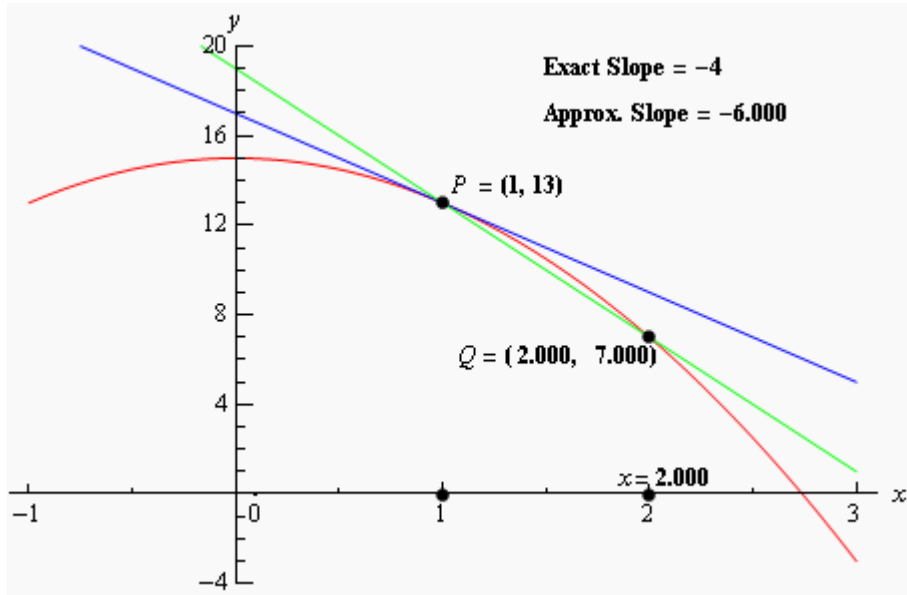
As you can see (if you're reading this on the web) as we moved Q in closer and closer to P the secant lines does start to look more and more like the tangent line and so the approximate slopes (*i.e.* the slopes of the secant lines) are getting closer and closer to the exact slope. Also, do now

worry about how I got the exact or approximate slopes. We'll be computing the approximate slopes shortly and we'll be able to compute the exact slope in a few sections.

In this figure we only looked at Q's that were to the right of P, but we could have just as easily used Q's that were to the left of P and we would have received the same results. In fact, we should always take a look at Q's that are on both sides of P. In this case the same thing is happening on both sides of P. However, we will eventually see that doesn't have to happen. Therefore we should always take a look at what is happening on both sides of the point in question when doing this kind of process.

So, let's see if we can come up with the approximate slopes I showed above, and hence an estimation of the slope of the tangent line. In order to simplify the process a little let's get a formula for the slope of the line between P and Q, m_{PQ}, that will work for any x that we choose to work with. We can get a formula by finding the slope between P and Q using the "general" form of $Q = (x, f(x))$.

$$m_{PQ} = \frac{f(x) - f(1)}{x - 1} = \frac{15 - 2x^2 - 13}{x - 1} = \frac{2 - 2x^2}{x - 1}$$

Now, let's pick some values of x getting closer and closer to $x = 1$, plug in and get some slopes.

x	m_{PQ}	x	m_{PQ}
2	-6	0	-2
1.5	-5	0.5	-3
1.1	-4.2	0.9	-3.8
1.01	-4.02	0.99	-3.98
1.001	-4.002	0.999	-3.998
1.0001	-4.0002	0.9999	-3.9998

So, if we take x's to the right of 1 and move them in very close to 1 it appears that the slope of the secant lines appears to be approaching -4. Likewise, if we take x's to the left of 1 and move them in very close to 1 the slope of the secant lines again appears to be approaching -4.

Based on this evidence it seems that the slopes of the secant lines are approaching -4 as we move in towards $x = 1$, so we will estimate that the slope of the tangent line is also -4. As noted above, this is the correct value and we will be able to prove this eventually.

Now, the equation of the line that goes through $(a, f(a))$ is given by

$$y = f(a) + m(x - a)$$

Therefore, the equation of the tangent line to $f(x) = 15 - 2x^2$ at $x = 1$ is

$$y = 13 - 4(x - 1) = -4x + 17$$

There are a couple of important points to note about our work above. First, we looked at points that were on both sides of $x = 1$. In this kind of process it is important to never assume that what is happening on one side of a point will also be happening on the other side as well. We should always look at what is happening on both sides of the point. In this example we could sketch a graph and from that guess that what is happening on one side will also be happening on the other, but we will usually not have the graphs in front of us or be able to easily get them.

Next, notice that when we say we're going to move in close to the point in question we do mean that we're going to move in very close and we also used more than just a couple of points. We should never try to determine a trend based on a couple of points that aren't really all that close to the point in question.

The next thing to notice is really a warning more than anything. The values of m_{PQ} in this example were fairly "nice" and it was pretty clear what value they were approaching after a couple of computations. In most cases this will not be the case. Most values will be far "messier" and you'll often need quite a few computations to be able to get an estimate.

Last, we were after something that was happening at $x = 1$ and we couldn't actually plug $x = 1$ into our formula for the slope. Despite this limitation we were able to determine some information about what was happening at $x = 1$ simply by looking at what was happening around $x = 1$. This is more important than you might at first realize and we will be discussing this point in detail in later sections.

Before moving on let's do a quick review of just what we did in the above example. We wanted the tangent line to $f(x)$ at a point $x = a$. First, we know that the point $P = (a, f(a))$ will be on the tangent line. Next, we'll take a second point that is on the graph of the function, call it $Q = (x, f(x))$ and compute the slope of the line connecting P and Q as follows,

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

We then take values of x that get closer and closer to $x = a$ (making sure to look at x's on both sides of $x = a$ and use this list of values to estimate the slope of the tangent line, m.

The tangent line will then be,

$$y = f(a) + m(x - a)$$

Rates of Change
The next problem that we need to look at is the rate of change problem. This will turn out to be one of the most important concepts that we will look at throughout this course.

Here we are going to consider a function, *f(x)*, that represents some quantity that varies as *x* varies. For instance, maybe *f(x)* represents the amount of water in a holding tank after *x* minutes. Or maybe *f(x)* is the distance traveled by a car after *x* hours. In both of these example we used *x* to represent time. Of course *x* doesn't have to represent time, but it makes for examples that are easy to visualize.

What we want to do here is determine just how fast *f(x)* is changing at some point, say $x = a$. This is called the **instantaneous rate of change** or sometimes just **rate of change** of *f(x)* at $x = a$.

As with the tangent line problem all that we're going to be able to do at this point is to estimate the rate of change. So let's continue with the examples above and think of *f(x)* as something that is changing in time and *x* being the time measurement. Again *x* doesn't have to represent time but it will make the explanation a little easier. While we can't compute the instantaneous rate of change at this point we can find the average rate of change.

To compute the average rate of change of *f(x)* at $x = a$ all we need to do is to choose another point, say *x*, and then the average rate of change will be,

$$A.R.C. = \frac{\text{change in } f(x)}{\text{change in } x}$$
$$= \frac{f(x) - f(a)}{x - a}$$

Then to estimate the instantaneous rate of change at $x = a$ all we need to do is to choose values of *x* getting closer and closer to $x = a$ (don't forget to chose them on both sides of $x = a$) and compute values of *A.R.C.* We can then estimate the instantaneous rate of change form that.

Let's take a look at an example.

Example 2 Suppose that the amount of air in a balloon after *t* hours is given by
$$V(t) = t^3 - 6t^2 + 35$$
Estimate the instantaneous rate of change of the volume after 5 hours.

Solution
Okay. The first thing that we need to do is get a formula for the average rate of change of the volume. In this case this is,

$$A.R.C. = \frac{V(t) - V(5)}{t - 5} = \frac{t^3 - 6t^2 + 35 - 10}{t - 5} = \frac{t^3 - 6t^2 + 25}{t - 5}$$

To estimate the instantaneous rate of change of the volume at $t = 5$ we just need to pick values of *t* that are getting closer and closer to $t = 5$. Here is a table of values of *t* and the average rate of change for those values.

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t	A.R.C.	t	A.R.C.
6	25.0	4	7.0
5.5	19.75	4.5	10.75
5.1	15.91	4.9	14.11
5.01	15.0901	4.99	14.9101
5.001	15.009001	4.999	14.991001
5.0001	15.00090001	4.9999	14.99910001

So, from this table it looks like the average rate of change is approaching 15 and so we can estimate that the instantaneous rate of change is 15 at this point.

So, just what does this tell us about the volume at this point? Let's put some units on the answer from above. This might help us to see what is happening to the volume at this point. Let's suppose that the units on the volume were in cm^3. The units on the rate of change (both average and instantaneous) are then cm^3/hr.

We have estimated that at $t = 5$ the volume is changing at a rate of 15 cm^3/hr. This means that at $t = 5$ the volume is changing in such a way that, if the rate were constant, then an hour later there would be 15 cm^3 more air in the balloon than there was at $t = 5$.

We do need to be careful here however. In reality there probably won't be 15 cm^3 more air in the balloon after an hour. The rate at which the volume is changing is generally not constant and so we can't make any real determination as to what the volume will be in another hour. What we can say is that the volume is increasing, since the instantaneous rate of change is positive, and if we had rates of change for other values of *t* we could compare the numbers and see if the rate of change is faster or slower at the other points.

For instance, at $t = 4$ the instantaneous rate of change is 0 cm^3/hr and at $t = 3$ the instantaneous rate of change is -9 cm^3/hr. I'll leave it to you to check these rates of change. In fact, that would be a good exercise to see if you can build a table of values that will support my claims on these rates of change.

Anyway, back to the example. At $t = 4$ the rate of change is zero and so at this point in time the volume is not changing at all. That doesn't mean that it will not change in the future. It just means that exactly at $t = 4$ the volume isn't changing. Likewise at $t = 3$ the volume is decreasing since the rate of change at that point is negative. We can also say that, regardless of the increasing/decreasing aspects of the rate of change, the volume of the balloon is changing faster at $t = 5$ than it is at $t = 3$ since 15 is larger than 9.

We will be talking a lot more about rates of change when we get into the next chapter.

Velocity Problem

Let's briefly look at the velocity problem. Many calculus books will treat this as its own problem. I however, like to think of this as a special case of the rate of change problem. In the velocity problem we are given a position function of an object, *f(t)*, that gives the position of an object at time *t*. Then to compute the instantaneous velocity of the object we just need to recall that the velocity is nothing more than the rate at which the position is changing.

In other words, to estimate the instantaneous velocity we would first compute the average velocity,

$$A.V. = \frac{\text{change in position}}{\text{time traveled}}$$
$$= \frac{f(t) - f(a)}{t - a}$$

and then take values of *t* closer and closer to $t = a$ and use these values to estimate the instantaneous velocity.

Change of Notation

There is one last thing that we need to do in this section before we move on. The main point of this section was to introduce us to a couple of key concepts and ideas that we will see throughout the first portion of this course as well as get us started down the path towards limits.

Before we move into limits officially let's go back and do a little work that will relate both (or all three if you include velocity as a separate problem) problems to a more general concept.

First, notice that whether we wanted the tangent line, instantaneous rate of change, or instantaneous velocity each of these came down to using exactly the same formula. Namely,

$$\frac{f(x) - f(a)}{x - a} \tag{1}$$

This should suggest that all three of these problems are then really the same problem. In fact this is the case as we will see in the next chapter. We are really working the same problem in each of these cases the only difference is the interpretation of the results.

In preparation for the next section where we will discuss this in much more detail we need to do a quick change of notation. It's easier to do here since we've already invested a fair amount of time into these problems.

In all of these problems we wanted to determine what was happening at $x = a$. To do this we chose another value of *x* and plugged into (1). For what we were doing here that is probably most intuitive way of doing it. However, when we start looking at these problems as a single problem (1) will not be the best formula to work with.

What we'll do instead is to first determine how far from $x = a$ we want to move and then define our new point based on that decision. So, if we want to move a distance of h from $x = a$ the new point would be $x = a + h$.

As we saw in our work above it is important to take values of x that are both sides of $x = a$. This way of choosing new value of x will do this for us. If $h>0$ we will get value of x that are to the right of $x = a$ and if $h<0$ we will get values of x that are to the left of $x = a$.

Now, with this new way of getting a second x, (1) will become,

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h} \qquad (2)$$

On the surface it might seem that (2) is going to be an overly complicated way of dealing with this stuff. However, as we will see it will often be easier to deal with (2) than it will be to deal with (1).

The Limit

In the previous section we looked at a couple of problems and in both problems we had a function (slope in the tangent problem case and average rate of change in the rate of change problem) and we wanted to know how that function was behaving at some point $x = a$. At this stage of the game we no longer care where the functions came from and we no longer care if we're going to see them down the road again or not. All that we need to know or worry about is that we've got these functions and we want to know something about them.

To answer the questions in the last section we choose values of *x* that got closer and closer to $x = a$ and we plugged these into the function. We also made sure that we looked at values of *x* that were on both the left and the right of $x = a$. Once we did this we looked at our table of function values and saw what the function values were approaching as *x* got closer and closer to $x = a$ and used this to guess the value that we were after.

This process is called **taking a limit** and we have some notation for this. The limit notation for the two problems from the last section is,

$$\lim_{x \to 1} \frac{2 - 2x^2}{x - 1} = -4 \qquad\qquad \lim_{t \to 5} \frac{t^3 - 6t^2 + 25}{t - 5} = 15$$

In this notation we will note that we always give the function that we're working with and we also give the value of *x* (or *t*) that we are moving in towards.

In this section we are going to take an intuitive approach to limits and try to get a feel for what they are and what they can tell us about a function. With that goal in mind we are not going to get into how we actually compute limits yet. We will instead rely on what we did in the previous section as well as another approach to guess the value of the limits.

Both of the approaches that we are going to use in this section are designed to help us understand just what limits are. In general we don't typically use the methods in this section to compute limits and in many cases can be very difficult to use to even estimate the value of a limit and/or will give the wrong value on occasion. We will look at actually computing limits in a couple of sections.

Let's first start off with the following "definition" of a limit.

Definition

We say that the limit of *f(x)* is *L* as *x* approaches *a* and write this as

$$\lim_{x \to a} f\left(x\right) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a*, from both sides, without actually letting *x* be *a*.

This is not the exact, precise definition of a limit. If you would like to see the more precise and mathematical definition of a limit you should check out the The Definition of a Limit section at the end of this chapter. The definition given above is more of a "working" definition. This definition helps us to get an idea of just what limits are and what they can tell us about functions.

So just what does this definition mean? Well let's suppose that we know that the limit does in fact exist. According to our "working" definition we can then decide how close to *L* that we'd like to make *f(x)*. For sake of argument let's suppose that we want to make *f(x)* no more that 0.001 away from *L*. This means that we want one of the following

$$f(x) - L < 0.001 \qquad \text{if } f(x) \text{ is larger than L}$$
$$L - f(x) < 0.001 \qquad \text{if } f(x) \text{ is smaller than L}$$

Now according to the "working" definition this means that if we get *x* sufficiently close to we can make one of the above true. However, it actually says a little more. It actually says that somewhere out there in the world is a value of *x*, say *X*, so that for all *x's* that are closer to *a* than *X* then one of the above statements will be true.

This is actually a fairly important idea. There are many functions out there in the work that we can make as close to *L* for specific values of *x* that are close to *a*, but there will other values of *x* closer to *a* that give functions values that are nowhere near close to *L*. In order for a limit to exist once we get *f(x)* as close to *L* as we want for some *x* then it will need to stay in that close to *L* (or get closer) for all values of *x* that are closer to *a*. We'll see an example of this later in this section.

In somewhat simpler terms the definition says that as *x* gets closer and closer to *x=a* (from both sides of course…) then *f(x)* **must** be getting closer and closer to *L*. Or, as we move in towards *x=a* then *f(x)* **must** be moving in towards *L*.

It is important to note once again that we must look at values of *x* that are on both sides of *x=a*. We should also note that we are not allowed to use *x=a* in the definition. We will often use the information that limits give us to get some information about what is going on right at *x=a*, but the limit itself is not concerned with what is actually going on at *x=a*. The limit is only concerned with what is going on around the point *x=a*. This is an important concept about limits that we need to keep in mind.

An alternative notation that we will occasionally use in denoting limits is
$$f(x) \to L \qquad \text{as} \qquad x \to a$$

How do we use this definition to help us estimate limits? We do exactly what we did in the previous section. We take *x's* on both sides of *x=a* that move in closer and closer to *a* and we plug these into our function. We then look to see if we can determine what number the function values are moving in towards and use this as our estimate.

Let's work an example.

Example 1 Estimate the value of the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Solution

Notice that I did say estimate the value of the limit. Again, we are not going to directly compute limits in this section. The point of this section is to give us a better idea of how limits work and what they can tell us about the function.

So, with that in mind we are going to work this in pretty much the same way that we did in the last section. We will choose values of *x* that get closer and closer to *x=2* and plug these values into the function. Doing this gives the following table of values.

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

Note that we made sure and picked values of *x* that were on both sides of $x = 2$ and that we moved in very close to $x = 2$ to make sure that any trends that we might be seeing are in fact correct.

Also notice that we can't actually plug in $x = 2$ into the function as this would give us a division by zero error. This is not a problem since the limit doesn't care what is happening at the point in question.

From this table it appears that the function is going to 4 as *x* approaches 2, so

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4.$$

Let's think a little bit more about what's going on here. Let's graph the function from the last example. The graph of the function in the range of *x*'s that were interested in is shown below.

First, notice that there is a rather large open dot at $x = 2$. This is there to remind us that the function (and hence the graph) doesn't exist at $x = 2$.

As we were plugging in values of *x* into the function we are in effect moving along the graph in towards the point as $x = 2$. This is shown in the graph by the two arrows on the graph that are moving in towards the point.
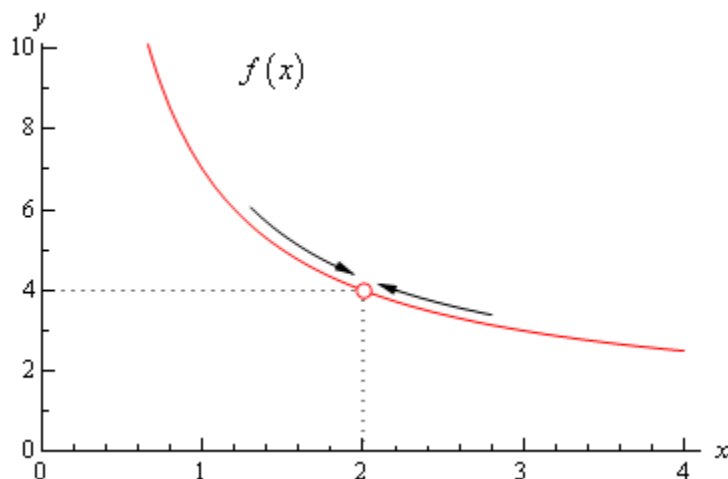
When we are computing limits the question that we are really asking is what *y* value is our graph approaching as we move in towards $x = a$ on our graph. We are **NOT** asking what *y* value the graph takes at the point in question. In other words, we are asking what the graph is doing **around** the point $x = a$. In our case we can see that as *x* moves in towards 2 (from both sides) the function is approaching $y = 4$ even though the function itself doesn't even exist at $x = 2$. Therefore we can say that the limit is in fact 4.

So what have we learned about limits? Limits are asking what the function is doing **around** $x = a$ and are **not** concerned with what the function is actually doing at $x = a$. This is a good thing as many of the functions that we'll be looking at won't even exist at $x = a$ as we saw in our last example.

Let's work another example to drive this point home.

Example 2 Estimate the value of the following limit.

$$\lim_{x \to 2} g(x) \qquad \text{where,} \qquad g(x) = \begin{cases} \dfrac{x^2 + 4x - 12}{x^2 - 2x} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$$

Solution
The first thing to note here is that this is exactly the same function as the first example with the exception that we've now given it a value for $x = 2$. So, let's first note that

$$g(2) = 6$$

As far as estimating the value of this limit goes, nothing has changed in comparison to the first example. We could build up a table of values as we did in the first example or we could take a quick look at the graph of the function. Either method will give us the value of the limit.

Lets' first take a look at a table of values and see what that tells us. Notice that the presence of the value for the function at $x = 2$ will not change our choices for *x*. We only choose values of *x*

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that are getting closer to $x = 2$ but we never take $x = 2$. In other words the table of values that we used in the first example will be exactly the same table that we'll use here. So, since we've already got it down once there is no reason to redo it here.

From this table it is again clear that the limit is,

$$\lim_{x \to 2} g(x) = 4$$

The limit is **NOT** 6! Remember from the discussion after the first example that limits do not care what the function is actually doing at the point in question. Limits are only concerned with what is going on **around** the point. Since the only thing about the function that we actually changed was its behavior at $x = 2$ this will not change the limit.

Let's also take a quick look at this functions graph to see if this says the same thing.
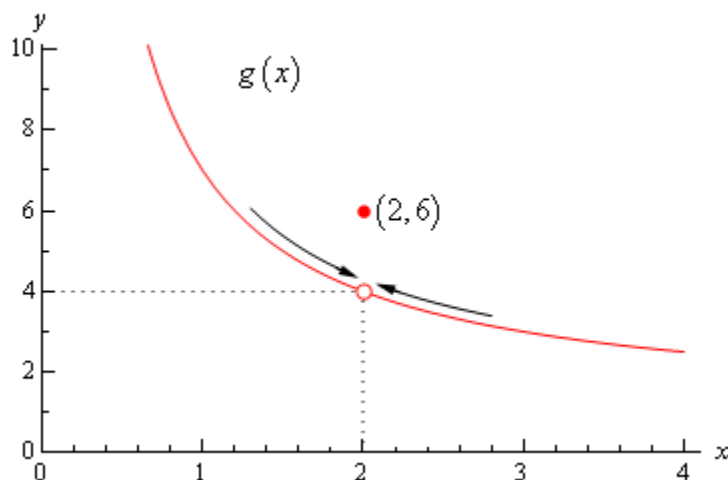
Again, we can see that as we move in towards $x = 2$ on our graph the function is still approaching a *y* value of 4. Remember that we are only asking what the function is doing **around** $x = 2$ and we don't care what the function is actually doing at $x = 2$. The graph then also supports the conclusion that the limit is,

$$\lim_{x \to 2} g(x) = 4$$

Let's make the point one more time just to make sure we've got it. Limits are **not** concerned with what is going on at $x = a$. Limits are only concerned with what is going on **around** $x = a$. We keep saying this, but it is a very important concept about limits that we must always keep in mind. So, we will take every opportunity to remind ourselves of this idea.

Since limits aren't concerned with what is actually happening at $x = a$ we will, on occasion, see situations like the previous example where the limit at a point and the function value at a point are different. This won't always happen of course. There are times where the function value and the limit at a point are the same and we will eventually see some examples of those. It is important however, to not get excited about things when the function and the limit do not take the same

value at a point. It happens sometimes and so we will need to be able to deal with those cases when they arise.

Let's take a look another example to try and beat this idea into the ground.

Example 3 Estimate the value of the following limit.
$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta}$$

Solution
First don't get excited about the θ in function. It's just a letter, just like x is a letter! It's a Greek letter, but it's a letter and you will be asked to deal with Greek letters on occasion so it's a good idea to start getting used to them at this point.

Now, also notice that if we plug in $\theta = 0$ that we will get division by zero and so the function doesn't exist at this point. Actually, we get 0/0 at this point, but because of the division by zero this function does not exist at $\theta = 0$.

So, as we did in the first example let's get a table of values and see what if we can guess what value the function is heading in towards.

θ	$f(\theta)$	θ	$f(\theta)$
1	0.45969769	-1	-0.45969769
0.1	0.04995835	-0.1	-0.04995835
0.01	0.00499996	-0.01	-0.00499996
0.001	0.00049999	-0.001	-0.00049999

Okay, it looks like the function is moving in towards a value of zero as θ moves in towards 0, from both sides of course.

Therefore, the we will guess that the limit has the value,
$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$$

So, once again, the limit had a value even though the function didn't exist at the point we were interested in.

It's now time to work a couple of more examples that will lead us into the next idea about limits that we're going to want to discuss.

Example 4 Estimate the value of the following limit.

$$\lim_{t \to 0} \cos\left(\frac{\pi}{t}\right)$$

Solution

Let's build up a table of values and see what's going on with our function in this case.

t	$f(t)$	t	$f(t)$
1	-1	-1	-1
0.1	1	-0.1	1
0.01	1	-0.01	1
0.001	1	-0.001	1

Now, if we were to guess the limit from this table we would guess that the limit is 1. However, if we did make this guess we would be wrong. Consider any of the following function evaluations.

$$f\left(\frac{1}{2001}\right) = -1 \qquad f\left(\frac{2}{2001}\right) = 0 \qquad f\left(\frac{4}{4001}\right) = \frac{\sqrt{2}}{2}$$

In all three of these function evaluations we evaluated the function at a number that is less that 0.001 and got three totally different numbers. Recall that the definition of the limit that we're working with requires that the function be approaching a single value (our guess) as t gets closer and closer to the point in question. It doesn't say that only some of the function values must be getting closer to the guess. It says that all the function values must be getting closer and closer to our guess.

To see what's happening here a graph of the function would be convenient.
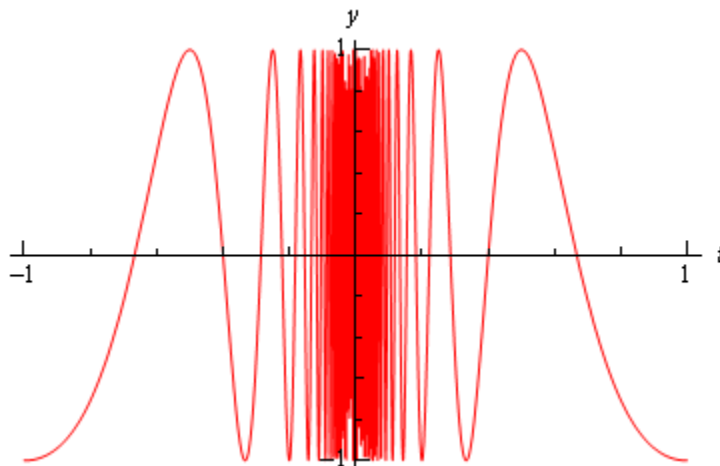
From this graph we can see that as we move in towards $t = 0$ the function starts oscillating wildly and in fact the oscillations increases in speed the closer to $t = 0$ that we get. Recall from our definition of the limit that in order for a limit to exist the function must be settling down in towards a single value as we get closer to the point in question.

This function clearly does not settle in towards a single number and so this limit **does not exist**!

This last example points out the drawback of just picking values of *x* using a table of function values to estimate the value of a limit. The values of *x* that we chose in the previous example were valid and in fact were probably values that many would have picked. In fact they were exactly the same values we used in the problem before this one and they worked in that problem!

When using a table of values there will always be the possibility that we aren't choosing the correct values and that we will guess incorrectly for our limit. This is something that we should always keep in mind when doing this to guess the value of limits. In fact, this is such a problem that after this section we will never use a table of values to guess the value of a limit again.

This last example also has shown us that limits do not have to exist. To this point we've only seen limits that have existed, but that just doesn't always have to be the case.

Let's take a look at one more example in this section.

Example 5 Estimate the value of the following limit.

$$\lim_{t \to 0} H(t) \quad \text{where,} \quad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Solution
This function is often called either the **Heaviside** or **step** function. We could use a table of values to estimate the limit, but it's probably just as quick in this case to use the graph so let's do that. Below is the graph of this function.
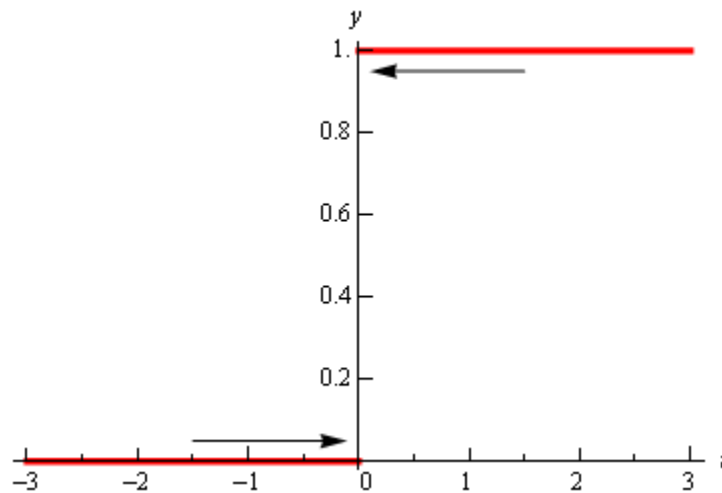
We can see from the graph that if we approach $t = 0$ from the right side the function is moving in towards a *y* value of 1. Well actually it's just staying at 1, but in the terminology that we've been using in this section it's moving in towards 1…

Also, if we move in towards $t = 0$ from the left the function is moving in towards a *y* value of 0.

According to our definition of the limit the function needs to move in towards a single value as

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we move in towards $t = a$ (from both sides). This isn't happening in this case and so in this example we will also say that the limit doesn't exist.

Note that the limit in this example is a little different from the previous example. In the previous example the function did not settle down to a single number as we moved in towards $t = 0$. In this example however, the function does settle down to a single number as $t = 0$ on either side. The problem is that the number is different on each side of $t = 0$. This is an idea that we'll look at in a little more detail in the next section.

Let's summarize what we (hopefully) learned in this section. In the first three examples we saw that limits do not care what the function is actually doing at the point in question. They only are concerned with what is happening around the point. In fact, we can have limits at $x = a$ even if the function itself does not exist at that point. Likewise, even if a function exists at a point there is no reason (at this point) to think that the limit will have the same value as the function at that point. Sometimes the limit and the function will have the same value at a point and other times they won't have the same value.

Next, in the third and fourth examples we saw the main reason for not using a table of values to guess the value of a limit. In those examples we used exactly the same set of values, however they only worked in one of the examples. Using tables of values to guess the value of limits is simply not a good way to get the value of a limit. This is the only section in which we will do this. Tables of values should always be your last choice in finding values of limits.

The last two examples showed us that not all limits will in fact exist. We should not get locked into the idea that limits will always exist. In most calculus courses we work with limits that almost always exist and so it's easy to start thinking that limits always exist. Limits don't always exist and so don't get into the habit of assuming that they will.

Finally, we saw in the fourth example that the only way to deal with the limit was to graph the function. Sometimes this is the only way, however this example also illustrated the drawback of using graphs. In order to use a graph to guess the value of the limit you need to be able to actually sketch the graph. For many functions this is not that easy to do.

There is another drawback in using graphs. Even if you actually have the graph it's only going to be useful if the y value is approaching an integer. If the y value is approaching say $\dfrac{-15}{123}$ there is no way that you're going to be able to guess that value from the graph and we are usually going to want exact values for our limits.

So while graphs of functions can, on occasion, make your life easier in guessing values of limits they are again probably not the best way to get values of limits. They are only going to be useful if you can get your hands on it and the value of the limit is a "nice" number.

The natural question then is why did we even talk about using tables and/or graphs to estimate limits if they aren't the best way. There were a couple of reasons.

First, they can help us get a better understanding of what limits are and what they can tell us. If we don't do at least a couple of limits in this way we might not get all that good of an idea on just what limits are.

The second reason for doing limits in this way is to point out their drawback so that we aren't tempted to use them all the time!

We will eventually talk about how we really do limits. However, there is one more topic that we need to discuss before doing that. Since this section has already gone on for a while we will talk about this in the next section.

One-Sided Limits

In the final two examples in the previous section we saw two limits that did not exist. However, the reason for each of the limits not existing was different for each of the examples.

We saw that

$$\lim_{t \to 0} \cos\left(\frac{\pi}{t}\right)$$

did not exist because the function did not settle down to a single value as t approached $t = 0$. The closer to $t = 0$ we moved the more wildly the function oscillated and in order for a limit to exist the function must settle down to a single value.

However we saw that

$$\lim_{t \to 0} H(t) \quad \text{where,} \quad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

did not exist not because the function didn't settle down to a single number as we moved in towards $t = 0$, but instead because it settled into two different numbers depending on which side of $t = 0$ we were on.

In this case the function was a very well behaved function, unlike the first function. The only problem was that, as we approached $t = 0$, the function was moving in towards different numbers on each side. We would like a way to differentiate between these two examples.

We do this with **one-sided limits**. As the name implies, with one-sided limits we will only be looking at one side of the point in question. Here are the definitions for the two one sided limits.

Right-handed limit

We say

$$\lim_{x \to a^+} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and *x>a* without actually letting *x* be *a*.

Left-handed limit

We say

$$\lim_{x \to a^-} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and *x<a* without actually letting *x* be *a*.

Note that the change in notation is very minor and in fact might be missed if you aren't paying attention. The only difference is the bit that is under the "lim" part of the limit. For the right-handed limit we now have $x \to a^+$ (note the "+") which means that we know will only look at

$x>a$. Likewise for the left-handed limit we have $x \to a^-$ (note the "-") which means that we will only be looking at $x<a$.

Also, note that as with the "normal" limit (*i.e.* the limits from the previous section) we still need the function to settle down to a single number in order for the limit to exist. The only difference this time is that the function only needs to settle down to a single number on either the right side of $x = a$ or the left side of $x = a$ depending on the one-sided limit we're dealing with.

So when we are looking at limits it's now important to pay very close attention to see whether we are doing a normal limit or one of the one-sided limits. Let's now take a look at the some of the problems from the last section and look at one-sided limits instead of the normal limit.

Example 1 Estimate the value of the following limits.

$$\lim_{t \to 0^+} H(t) \quad \text{and} \quad \lim_{t \to 0^-} H(t) \quad \text{where,} \quad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Solution
To remind us what this function looks like here's the graph.
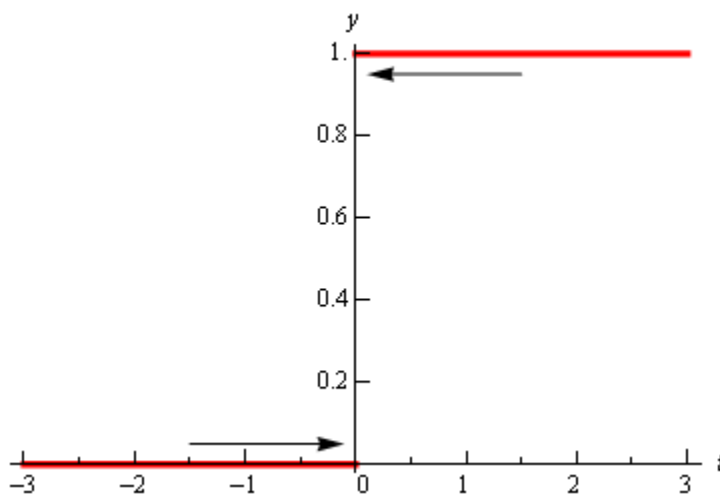
So, we can see that if we stay to the right of $t = 0$ (*i.e.* $t > 0$) then the function is moving in towards a value of 1 as we get closer and closer to $t = 0$, but staying to the right. We can therefore say that the right-handed limit is,

$$\lim_{t \to 0^+} H(t) = 1$$

Likewise, if we stay to the left of $t = 0$ (*i.e* $t < 0$) the function is moving in towards a value of 0 as we get closer and closer to $t = 0$, but staying to the left. Therefore the left-handed limit is,
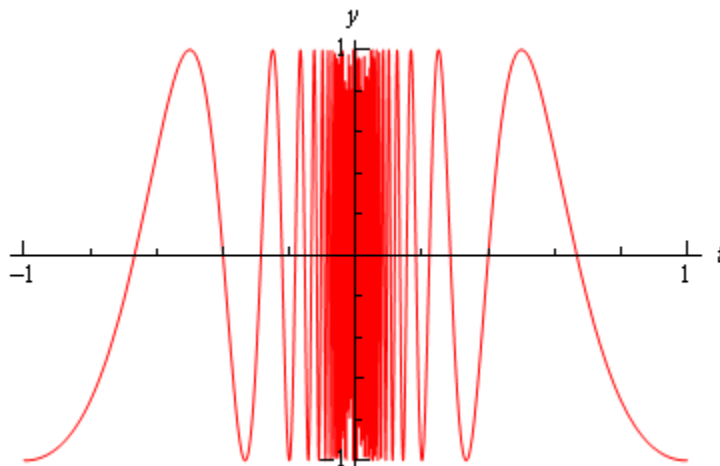
$$\lim_{t \to 0^-} H(t) = 0$$

In this example we do get one-sided limits even though the normal limit itself doesn't exist.

Example 2 Estimate the value of the following limits.

$$\lim_{t\to0^+}\cos\left(\frac{\pi}{t}\right) \qquad \lim_{t\to0^-}\cos\left(\frac{\pi}{t}\right)$$

Solution
From the graph of this function shown below,

we can see that both of the one-sided limits suffer the same problem that the normal limit did in the previous section. The function does not settle down to a single number on either side of $t=0$. Therefore, neither the left-handed nor the right-handed limit will exist in this case.

So, one-sided limits don't have to exist just as normal limits aren't guaranteed to exist.
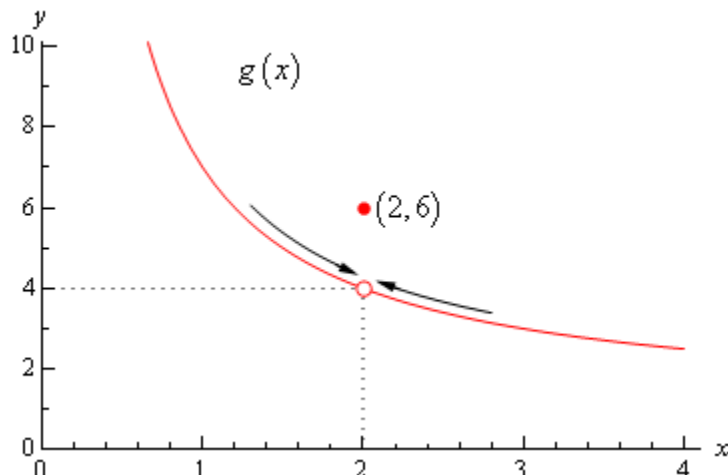
Let's take a look at another example from the previous section.

Example 3 Estimate the value of the following limits.

$$\lim_{x\to2^+}g(x)\quad\text{and}\qquad\lim_{x\to2^-}g(x)\qquad\text{where, } g(x)=\begin{cases}\dfrac{x^2+4x-12}{x^2-2x} & \text{if } x\neq2\\ 6 & \text{if } x=2\end{cases}$$

Solution
So as we've done with the previous two examples, let's remind ourselves of the graph of this function.

In this case regardless of which side of $x = 2$ we are on the function is always approaching a value of 4 and so we get,

$$\lim_{x \to 2^+} g(x) = 4 \qquad\qquad \lim_{x \to 2^-} g(x) = 4$$

Note that one-sided limits do not care about what's happening at the point any more than normal limits do. They are still only concerned with what is going on around the point. The only real difference between one-sided limits and normal limits is the range of x's that we look at when determining the value of the limit.

Now let's take a look at the first and last example in this section to get a very nice fact about the relationship between one-sided limits and normal limits. In the last example the one-sided limits as well as the normal limit existed and all three had a value of 4. In the first example the two one-sided limits both existed, but did not have the same value and the normal limit did not exist.

The relationship between one-sided limits and normal limits can be summarized by the following fact.

Fact

Given a function $f(x)$ if,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

then the normal limit will exist and

$$\lim_{x \to a} f(x) = L$$

Likewise, if

$$\lim_{x \to a} f(x) = L$$

then,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

This fact can be turned around to also say that if the two one-sided limits have different values, *i.e.*,

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

then the normal limit will not exist.

This should make some sense. If the normal limit did exist then by the fact the two one-sided limits would have to exist and have the same value by the above fact. So, if the two one-sided limits have different values (or don't even exist) then the normal limit simply can't exist.

Let's take a look at one more example to make sure that we've got all the ideas about limits down that we've looked at in the last couple of sections.

Example 4 Given the following graph,
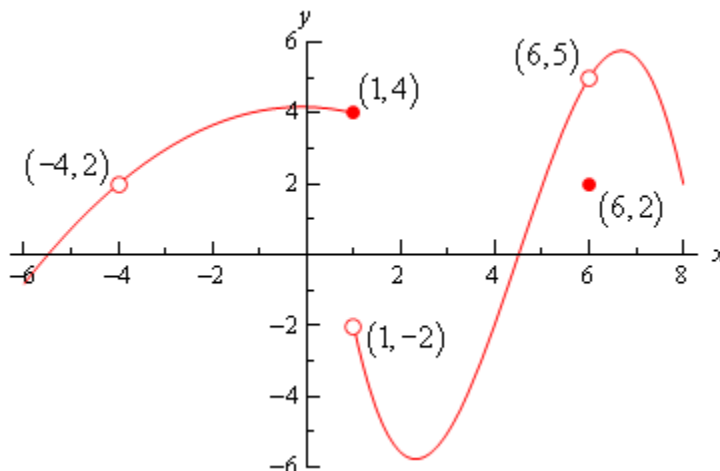
compute each of the following.

(a) $f(-4)$	**(b)** $\lim\limits_{x \to -4^-} f(x)$	**(c)** $\lim\limits_{x \to -4^+} f(x)$	**(d)** $\lim\limits_{x \to -4} f(x)$
(e) $f(1)$	**(f)** $\lim\limits_{x \to 1^-} f(x)$	**(g)** $\lim\limits_{x \to 1^+} f(x)$	**(h)** $\lim\limits_{x \to 1} f(x)$
(i) $f(6)$	**(j)** $\lim\limits_{x \to 6^-} f(x)$	**(k)** $\lim\limits_{x \to 6^+} f(x)$	**(l)** $\lim\limits_{x \to 6} f(x)$

Solution
(a) $f(-4)$ doesn't exist. There is no closed dot for this value of x and so the function doesn't exist at this point.

(b) $\lim\limits_{x \to -4^-} f(x) = 2$ The function is approaching a value of 2 as x moves in towards -4 from the left.

(c) $\lim\limits_{x \to -4^+} f(x) = 2$ The function is approaching a value of 2 as x moves in towards -4 from the right.

(d) $\lim\limits_{x \to -4} f(x) = 2$ We can do this one of two ways. Either we can use the fact here and notice that the two one-sided limits are the same and so the normal limit must exist and have the same value as the one-sided limits or just get the answer from the graph.

Also recall that a limit can exist at a point even if the function doesn't exist at that point.

(e) $f(1) = 4$. The function will take on the y value where the closed dot is.

(f) $\lim\limits_{x \to 1^-} f(x) = 4$ The function is approaching a value of 4 as x moves in towards 1 from the left.

(g) $\lim\limits_{x \to 1^+} f(x) = -2$ The function is approaching a value of -2 as x moves in towards 1 from the right. Remember that the limit does NOT care about what the function is actually doing at the point, it only cares about what the function is doing around the point. In this case, always staying to the right of $x = 1$, the function is approaching a value of -2 and so the limit is -2. The limit is not 4, as that is value of the function at the point and again the limit doesn't care about that!

(h) $\lim\limits_{x \to 1} f(x)$ doesn't exist. The two one-sided limits both exist, however they are different and so the normal limit doesn't exist.

(i) $f(6) = 2$. The function will take on the y value where the closed dot is.

(j) $\lim\limits_{x \to 6^-} f(x) = 5$ The function is approaching a value of 5 as x moves in towards 6 from the left.

(k) $\lim\limits_{x \to 6^+} f(x) = 5$ The function is approaching a value of 5 as x moves in towards 6 from the right.

(l) $\lim\limits_{x \to 6} f(x) = 5$ Again, we can use either the graph or the fact to get this. Also, once more remember that the limit doesn't care what is happening at the point and so it's possible for the limit to have a different value than the function at a point. When dealing with limits we've always got to remember that limits simply do not care about what the function is doing at the point in question. Limits are only concerned with what the function is doing around the point.

Hopefully over the last couple of sections you've gotten an idea on how limits work and what they can tell us about functions. Some of these ideas will be important in later sections so it's important that you have a good grasp on them.

Limit Properties

The time has almost come for us to actually compute some limits. However, before we do that we will need some properties of limits that will make our life somewhat easier. So, let's take a look at those first. The proof of some of these properties can be found in the Proof of Various Limit Properties section of the Extras chapter.

Properties

First we will assume that $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist and that c is any constant. Then,

1. $\lim\limits_{x \to a} \left[cf(x) \right] = c \lim\limits_{x \to a} f(x)$

    In other words we can "factor" a multiplicative constant out of a limit.

2. $\lim\limits_{x \to a} \left[f(x) \pm g(x) \right] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

    So to take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign. This is also not limited to two functions. This fact will work no matter how many functions we've got separated by "+" or "-".

3. $\lim\limits_{x \to a} \left[f(x) g(x) \right] = \lim\limits_{x \to a} f(x) \; \lim\limits_{x \to a} g(x)$

    We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

4. $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)} \right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)},$ \qquad provided $\lim\limits_{x \to a} g(x) \neq 0$

    As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

5. $\lim\limits_{x \to a} \left[f(x) \right]^n = \left[\lim\limits_{x \to a} f(x) \right]^n,$ \qquad where n is any real number

    In this property n can be any real number (positive, negative, integer, fraction, irrational, zero, *etc.*). In the case that n is an integer this rule can be thought of as an extended case of **3**.

    For example consider the case of $n = 2$.

$$\lim_{x \to a}\left[f(x)\right]^2 = \lim_{x \to a}\left[f(x)f(x)\right]$$
$$= \lim_{x \to a} f(x)\lim_{x \to a} f(x) \qquad \text{using property 3}$$
$$= \left[\lim_{x \to a} f(x)\right]^2$$

The same can be done for any integer n.

6. $\lim_{x \to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x \to a} f(x)}$

This is just a special case of the previous example.

$$\lim_{x \to a}\left[\sqrt[n]{f(x)}\right] = \lim_{x \to a}\left[f(x)\right]^{\frac{1}{n}}$$
$$= \left[\lim_{x \to a} f(x)\right]^{\frac{1}{n}}$$
$$= \sqrt[n]{\lim_{x \to a} f(x)}$$

7. $\lim_{x \to a} c = c, \qquad c$ is any real number

In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of $f(x) = c$.

8. $\lim_{x \to a} x = a$

As with the last one you should be able to convince yourself of this by drawing the graph of $f(x) = x$.

9. $\lim_{x \to a} x^n = a^n$

This is really just a special case of property **5** using $f(x) = x$.

Note that all these properties also hold for the two one-sided limits as well we just didn't write them down with one sided limits to save on space.

Let's compute a limit or two using these properties. The next couple of examples will lead us to some truly useful facts about limits that we will use on a continual basis.

Example 1 Compute the value of the following limit.
$$\lim_{x \to -2}\left(3x^2 + 5x - 9\right)$$
Solution
This first time through we will use only the properties above to compute the limit.

First we will use property **2** to break up the limit into three separate limits. We will then use property **1** to bring the constants out of the first two limits. Doing this gives us,

$$\lim_{x \to -2}\left(3x^2 + 5x - 9\right) = \lim_{x \to -2} 3x^2 + \lim_{x \to -2} 5x - \lim_{x \to -2} 9$$
$$= 3\lim_{x \to -2} x^2 + 5\lim_{x \to -2} x - \lim_{x \to -2} 9$$

We can now use properties **7** through **9** to actually compute the limit.

$$\lim_{x \to -2}\left(3x^2 + 5x - 9\right) = 3\lim_{x \to -2} x^2 + 5\lim_{x \to -2} x - \lim_{x \to -2} 9$$
$$= 3(-2)^2 + 5(-2) - 9$$
$$= -7$$

Now, let's notice that if we had defined

$$p(x) = 3x^2 + 5x - 9$$

then the proceeding example would have been,

$$\lim_{x \to -2} p(x) = \lim_{x \to -2}\left(3x^2 + 5x - 9\right)$$
$$= 3(-2)^2 + 5(-2) - 9$$
$$= -7$$
$$= p(-2)$$

In other words, in this case we were the limit is the same value that we'd get by just evaluating the function at the point in question. This seems to violate one of the main concepts about limits that we've seen to this point.

In the previous two sections we made a big deal about the fact that limits do not care about what is happening at the point in question. They only care about what is happening around the point. So how does the previous example fit into this since it appears to violate this main idea about limits?

Despite appearances the limit still doesn't care about what the function is doing at $x = -2$. In this case the function that we've got is simply "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. <u>Eventually</u> we will formalize up just what is meant by "nice enough". At this point let's not worry too much about what "nice enough" is. Let's just take advantage of the fact that some functions will be "nice enough", whatever that means.

The function in the last example was a polynomial. It turns out that all polynomials are "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. This leads to the following fact.

Fact

If *p(x)* is a polynomial then,

$$\lim_{x \to a} p(x) = p(a)$$

By the end of this section we will generalize this out considerably to most of the functions that we'll be seeing throughout this course.

Let's take a look at another example.

Example 2 Evaluate the following limit.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1}$$

Solution

First notice that we can use property **4)** to write the limit as,

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{\lim_{z \to 1} 6 - 3z + 10z^2}{\lim_{z \to 1} -2z^4 + 7z^3 + 1}$$

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn't zero. As we will see however, it isn't in this case so we're okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{6 - 3(1) + 10(1)^2}{-2(1)^4 + 7(1)^3 + 1}$$

$$= \frac{13}{6}$$

Notice that the limit of the denominator wasn't zero and so our use of property **4** was legitimate.

Notice in this last example that again all we really did was evaluate the function at the point in question. So it appears that there is a fairly large class of functions for which this can be done. Let's generalize the fact from above a little.

Fact

Provided *f(x)* is "nice enough" we have,

$$\lim_{x \to a} f(x) = f(a) \qquad \lim_{x \to a^-} f(x) = f(a) \qquad \lim_{x \to a^+} f(x) = f(a)$$

Again, we will formalize up just what we mean by "nice enough" eventually. At this point all we want to do is worry about which functions are "nice enough". Some functions are "nice enough" for all *x* while others will only be "nice enough" for certain values of *x*. It will all depend on the function.

As noted in the statement, this fact also holds for the two one-sided limits as well as the normal limit.

Here is a list of some of the more common functions that are "nice enough".

- Polynomials are nice enough for all *x*'s.
- If $f(x) = \dfrac{p(x)}{q(x)}$ then *f(x)* will be nice enough provided both *p(x)* and *q(x)* are nice enough and if we don't get division by zero at the point we're evaluating at.
- $\cos(x)$, $\sin(x)$ are nice enough for all *x*'s
- $\sec(x)$, $\tan(x)$ are nice enough provided $x \neq \ldots, -\dfrac{5\pi}{2}, -\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$ In other words secant and tangent are nice enough everywhere cosine isn't zero. To see why recall that these are both really rational functions and that cosine is in the denominator of both then go back up and look at the second bullet above.
- $\csc(x)$, $\cot(x)$ are nice enough provided $x \neq \ldots, -3\pi, -\pi, 0, \pi, 3\pi, \ldots$ In other words cosecant and cotangent are nice enough everywhere sine isn't zero.
- $\sqrt[n]{x}$ is nice enough for all *x* if *n* is odd.
- $\sqrt[n]{x}$ is nice enough for $x \geq 0$ if *n* is even. Here we require $x \geq 0$ to avoid having to deal with complex values.
- a^x, $\mathbf{e}^x$ are nice enough for all *x*'s.
- $\log_b x$, $\ln x$ are nice enough for *x>0*. Remember we can only plug positive numbers into logarithms and not zero or negative numbers.
- Any sum, difference or product of the above functions will also be nice enough. Quotients will be nice enough provided we don't get division by zero upon evaluating the limit.

The last bullet is important. This means that for any combination of these functions all we need to do is evaluate the function at the point in question, making sure that none of the restrictions are violated. This means that we can now do a large number of limits.

Example 3 Evaluate the following limit.

$$\lim_{x \to 3}\left(-\sqrt[5]{x} + \frac{\mathbf{e}^x}{1 + \ln(x)} + \sin(x)\cos(x) \right)$$

Solution
This is a combination of several of the functions listed above and none of the restrictions are violated so all we need to do is plug in $x = 3$ into the function to get the limit.

$$\lim_{x \to 3}\left(-\sqrt[5]{x} + \frac{\mathbf{e}^x}{1 + \ln(x)} + \sin(x)\cos(x)\right) = -\sqrt[5]{3} + \frac{\mathbf{e}^3}{1 + \ln(3)} + \sin(3)\cos(3)$$

$$= 8.185427271$$

Not a very pretty answer, but we can now do the limit.

Computing Limits

In the previous section we saw that there is a large class of function that allows us to use

$$\lim_{x \to a} f(x) = f(a)$$

to compute limits. However, there are also many limits for which this won't work easily. The purpose of this section is to develop techniques for dealing with some of these limits that will not allow us to just use this fact.

Let's first got back and take a look at one of the first limits that we looked at and compute its exact value and verify our guess for the limit.

Example 1 Evaluate the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Solution
First let's notice that if we try to plug in $x = 2$ we get,

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \frac{0}{0}$$

So, we can't just plug in $x = 2$ to evaluate the limit. So, we're going to have to do something else.

The first thing that we should always do when evaluating limits is to simplify the function as much as possible. In this case that means factoring both the numerator and denominator. Doing this gives,

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \to 2} \frac{(x - 2)(x + 6)}{x(x - 2)}$$

$$= \lim_{x \to 2} \frac{x + 6}{x}$$

So, upon factoring we saw that we could cancel an $x - 2$ from both the numerator and the denominator. Upon doing this we now have a new rational expression that we can plug $x = 2$ into because we lost the division by zero problem. Therefore, the limit is,

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \to 2} \frac{x + 6}{x} = \frac{8}{2} = 4$$

Note that this is in fact what we guessed the limit to be.

On a side note, the 0/0 we initially got in the previous example is called an **indeterminate form**. This means that we don't really know what it will be until we do some more work. Typically zero in the denominator means it's undefined. However that will only be true if the numerator isn't also zero. Also, zero in the numerator usually means that the fraction is zero, unless the

denominator is also zero. Likewise anything divided by itself is 1, unless we're talking about zero.

So, there are really three competing "rules" here and it's not clear which one will win out. It's also possible that none of them will win out and we will get something totally different from undefined, zero, or one. We might, for instance, get a value of 4 out of this, to pick a number completely at random.

There are many more kinds of indeterminate forms and we will be discussing indeterminate forms at length in the next chapter.

Let's take a look at a couple of more examples.

Example 2 Evaluate the following limit.

$$\lim_{h \to 0} \frac{2(-3+h)^2 - 18}{h}$$

Solution

In this case we also get 0/0 and factoring is not really an option. However, there is still some simplification that we can do.

$$\lim_{h \to 0} \frac{2(-3+h)^2 - 18}{h} = \lim_{h \to 0} \frac{2(9 - 6h + h^2) - 18}{h}$$

$$= \lim_{h \to 0} \frac{18 - 12h + 2h^2 - 18}{h}$$

$$= \lim_{h \to 0} \frac{-12h + 2h^2}{h}$$

So, upon multiplying out the first term we get a little cancellation and now notice that we can factor an *h* out of both terms in the numerator which will cancel against the *h* in the denominator and the division by zero problem goes away and we can then evaluate the limit.

$$\lim_{h \to 0} \frac{2(-3+h)^2 - 18}{h} = \lim_{h \to 0} \frac{-12h + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(-12 + 2h)}{h}$$

$$= \lim_{h \to 0} -12 + 2h = -12$$

Example 3 Evaluate the following limit.

$$\lim_{t \to 4} \frac{t - \sqrt{3t + 4}}{4 - t}$$

Solution

This limit is going to be a little more work than the previous two. Once again however note that we get the indeterminate form 0/0 if we try to just evaluate the limit. Also note that neither of the

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two examples will be of any help here, at least initially. We can't factor and we can't just multiply something out to get the function to simplify.

When there is a square root in the numerator or denominator we can try to rationalize and see if that helps. Recall that rationalizing makes use of the fact that
$$(a+b)(a-b)=a^2-b^2$$
So, if either the first and/or the second term have a square root in them the rationalizing will eliminate the root(s). This *might* help in evaluating the limit.

Let's try rationalizing the numerator in this case.
$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{\left(t-\sqrt{3t+4}\right)\left(t+\sqrt{3t+4}\right)}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$

Remember that to rationalize we just take the numerator (since that's what we're rationalizing), change the sign on the second term and multiply the numerator and denominator by this new term.

Next, we multiply the numerator out being careful to watch minus signs.
$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{t^2-\left(3t+4\right)}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$
$$=\lim_{t\to 4}\frac{t^2-3t-4}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$

Notice that we didn't multiply the denominator out as well. Most students come out of an Algebra class having it beaten into their heads to always multiply this stuff out. However, in this case multiplying out will make the problem very difficult and in the end you'll just end up factoring it back out anyway.

At this stage we are almost done. Notice that we can factor the numerator so let's do that.
$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{\left(t-4\right)\left(t+1\right)}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$

Now all we need to do is notice that if we factor a "-1"out of the first term in the denominator we can do some canceling. At that point the division by zero problem will go away and we can evaluate the limit.

$$\lim_{t \to 4} \frac{t - \sqrt{3t+4}}{4-t} = \lim_{t \to 4} \frac{(t-4)(t+1)}{-(t-4)(t+\sqrt{3t+4})}$$

$$= \lim_{t \to 4} \frac{t+1}{-(t+\sqrt{3t+4})}$$

$$= -\frac{5}{8}$$

Note that if we had multiplied the denominator out we would not have been able to do this canceling and in all likelihood would not have even seen that some canceling could have been done.

So, we've taken a look at a couple of limits in which evaluation gave the indeterminate form 0/0 and we now have a couple of things to try in these cases.

Let's take a look at another kind of problem that can arise in computing some limits involving piecewise functions.

Example 4 Given the function,

$$g(y) = \begin{cases} y^2 + 5 & \text{if } y < -2 \\ 1 - 3y & \text{if } y \geq -2 \end{cases}$$

Compute the following limits.

         (a) $\lim_{y \to 6} g(y)$ [Solution]

         (b) $\lim_{y \to -2} g(y)$ [Solution]

Solution

(a) $\lim_{y \to 6} g(y)$

In this case there really isn't a whole lot to do. In doing limits recall that we must always look at what's happening on both sides of the point in question as we move in towards it. In this case $y = 6$ is completely inside the second interval for the function and so there are values of y on both sides of $y = 6$ that are also inside this interval. This means that we can just use the fact to evaluate this limit.

$$\lim_{y \to 6} g(y) = \lim_{y \to 6} 1 - 3y$$

$$= -17$$

[Return to Problems]

(b) $\lim_{y \to -2} g(y)$

This part is the real point to this problem. In this case the point that we want to take the limit for is the cutoff point for the two intervals. In other words we can't just plug $y = -2$ into the second

portion because this interval does not contain values of y to the left of $y = -2$ and we need to know what is happening on both sides of the point.

To do this part we are going to have to remember the fact from the section on <u>one-sided limits</u> that says that if the two one-sided limits exist and are the same then the normal limit will also exist and have the same value.

Notice that both of the one sided limits can be done here since we are only going to be looking at one side of the point in question. So let's do the two one-sided limits and see what we get.

$$\lim_{y \to -2^-} g(y) = \lim_{y \to -2^-} y^2 + 5 \qquad \text{since } y \to 2^- \text{ implies } y < -2$$
$$= 9$$

$$\lim_{y \to -2^+} g(y) = \lim_{y \to -2^+} 1 - 3y \qquad \text{since } y \to 2^+ \text{ implies } y > -2$$
$$= 7$$

So, in this case we can see that,

$$\lim_{y \to -2^-} g(y) = 9 \neq 7 = \lim_{y \to -2^+} g(y)$$

and so since the two one sided limits aren't the same

$$\lim_{y \to -2} g(y)$$

doesn't exist.

Note that a very simple change to the function will make the limit at $y = -2$ exist so don't get in into your head that limits at these cutoff points in piecewise function don't ever exist.

Example 5 Evaluate the following limit.

$$\lim_{y \to -2} g(y) \qquad \text{where, } g(y) = \begin{cases} y^2 + 5 & \text{if } y < -2 \\ 3 - 3y & \text{if } y \geq -2 \end{cases}$$

Solution
The two one-sided limits this time are,

$$\lim_{y \to -2^-} g(y) = \lim_{y \to -2^-} y^2 + 5 \qquad \text{since } y \to 2^- \text{ implies } y < -2$$
$$= 9$$

$$\lim_{y \to -2^+} g(y) = \lim_{y \to -2^-} 3 - 3y \qquad \text{since } y \to 2^+ \text{ implies } y > -2$$
$$= 9$$

The one-sided limits are the same so we get,

$$\lim_{y \to -2} g(y) = 9$$

There is one more limit that we need to do. However, we will need a new fact about limits that will help us to do this.

Fact

If $f(x) \le g(x)$ for all x on [a, b] (except possibly at $x = c$) and $a \le c \le b$ then,

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x)$$

Note that this fact should make some sense to you if we assume that both functions are nice enough. If both of the functions are "nice enough" to use the limit evaluation fact then we have,

$$\lim_{x \to c} f(x) = f(c) \le g(c) = \lim_{x \to c} g(x)$$

The inequality is true because we know that c is somewhere between a and b and in that range we also know $f(x) \le g(x)$.

Note that we don't really need the two functions to be nice enough for the fact to be true, but it does provide a nice way to give a quick "justification" for the fact.

Also, note that we said that we assumed that $f(x) \le g(x)$ for all x on [a, b] (except possibly at $x = c$). Because limits do not care what is actually happening at $x = c$ we don't really need the inequality to hold at that specific point. We only need it to hold around $x = c$ since that is what the limit is concerned about.

We can take this fact one step farther to get the following theorem.

Squeeze Theorem

Suppose that for all x on [a, b] (except possibly at $x = c$) we have,

$$f(x) \le h(x) \le g(x)$$

Also suppose that,

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

for some $a \le c \le b$. Then,

$$\lim_{x \to c} h(x) = L$$

As with the previous fact we only need to know that $f(x) \le h(x) \le g(x)$ is true around $x = c$ because we are working with limits and they are only concerned with what is going on around $x = c$ and not what is actually happening at $x = c$.

Now, if we again assume that all three functions are nice enough (again this isn't required to make the Squeeze Theorem true, it only helps with the visualization) then we can get a quick

sketch of what the Squeeze Theorem is telling us. The following figure illustrates what is happening in this theorem.
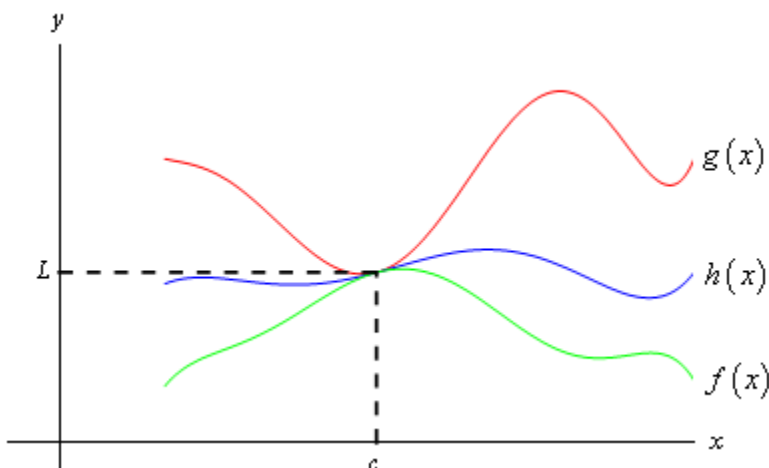
From the figure we can see that if the limits of *f(x)* and *g(x)* are equal at $x = c$ then the function values must also be equal at $x = c$ (this is where we're using the fact that we assumed the functions where "nice enough", which isn't really required for the Theorem). However, because *h(x)* is "squeezed" between *f(x)* and *g(x)* at this point then *h(x)* must have the same value. Therefore, the limit of *h(x)* at this point must also be the same.

The Squeeze theorem is also known as the **Sandwich Theorem** and the **Pinching Theorem**.

So, how do we use this theorem to help us with limits? Let's take a look at the following example to see the theorem in action.

Example 6 Evaluate the following limit.
$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Solution
In this example none of the previous examples can help us. There's no factoring or simplifying to do. We can't rationalize and one-sided limits won't work. There's even a question as to whether this limit will exist since we have division by zero inside the cosine at *x=0*.

The first thing to notice is that we know the following fact about cosine.
$$-1 \le \cos(x) \le 1$$
Our function doesn't have just an *x* in the cosine, but as long as we avoid $x = 0$ we can say the same thing for our cosine.
$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$
It's okay for us to ignore $x = 0$ here because we are taking a limit and we know that limits don't care about what's actually going on at the point in question, $x = 0$ in this case.

Now if we have the above inequality for our cosine we can just multiply everything by an x^2 and get the following.

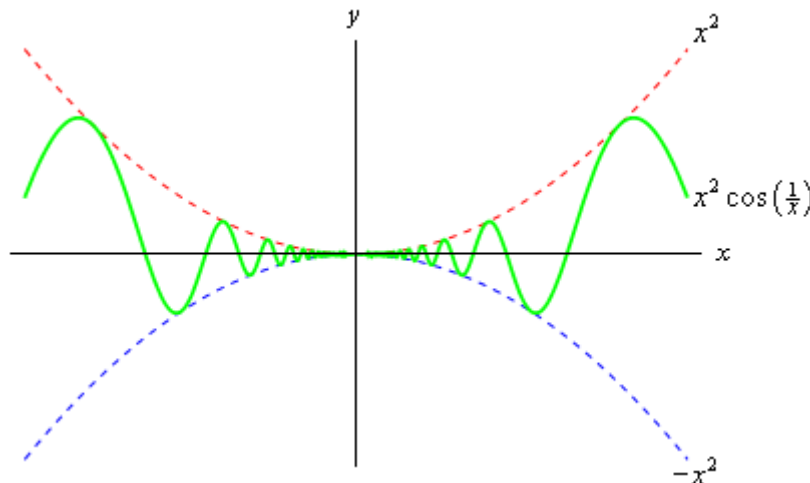$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$$

In other words we've managed to squeeze the function that we were interested in between two other functions that are very easy to deal with. So, the limits of the two outer functions are.

$$\lim_{x \to 0} x^2 = 0 \qquad\qquad \lim_{x \to 0}\left(-x^2\right) = 0$$

These are the same and so by the Squeeze theorem we must also have,

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

We can verify this with the graph of the three functions. This is shown below.

In this section we've seen several tools that we can use to help us to compute limits in which we can't just evaluate the function at the point in question. As we will see many of the limits that we'll be doing in later sections will require one or more of these tools.

Infinite Limits

In this section we will take a look at limits whose value is infinity or minus infinity. These kinds of limit will show up fairly regularly in later sections and in other courses and so you'll need to be able to deal with them when you run across them.

The first thing we should probably do here is to define just what we mean when we sat that a limit has a value of infinity or minus infinity.

Definition

We say
$$\lim_{x \to a} f(x) = \infty$$
if we can make *f(x)* arbitrarily large for all *x* sufficiently close to *x=a*, from both sides, without actually letting $x = a$.

We say
$$\lim_{x \to a} f(x) = -\infty$$
if we can make *f(x)* arbitrarily large and negative for all *x* sufficiently close to *x=a*, from both sides, without actually letting $x = a$.

These definitions can be appropriately modified for the one-sided limits as well. To see a more precise and mathematical definition of this kind of limit see the [The Definition of the Limit](#) section at the end of this chapter.

Let's start off with a fairly typical example illustrating infinite limits.

Example 1 Evaluate each of the following limits.
$$\lim_{x \to 0^+} \frac{1}{x} \qquad \lim_{x \to 0^-} \frac{1}{x} \qquad \lim_{x \to 0} \frac{1}{x}$$

Solution
So we're going to be taking a look at a couple of one-sided limits as well as the normal limit here. In all three cases notice that we can't just plug in $x = 0$. If we did we would get division by zero. Also recall that the definitions above can be easily modified to give similar definitions for the two one-sided limits which we'll be needing here.

Now, there are several ways we could proceed here to get values for these limits. One way is to plug in some points and see what value the function is approaching. In the proceeding section we said that we were no longer going to do this, but in this case it is a good way to illustrate just what's going on with this function.

So, here is a table of values of *x*'s from both the left and the right. Using these values we'll be able to estimate the value of the two one-sided limits and once we have that done we can use the

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