Infinite Limits

In this section we will take a look at limits whose value is infinity or minus infinity. These kinds of limit will show up fairly regularly in later sections and in other courses and so you'll need to be able to deal with them when you run across them.

The first thing we should probably do here is to define just what we mean when we sat that a limit has a value of infinity or minus infinity.

Definition

We say

$$\lim_{x\to a} f\left(x\right) = \infty$$

if we can make *f(x)* arbitrarily large for all *x* sufficiently close to *x*=*a*, from both sides, without actually letting $x = a$.

We say

$$\lim_{x\to a} f\left(x\right) = -\infty$$

if we can make *f(x)* arbitrarily large and negative for all *x* sufficiently close to *x*=*a*, from both sides, without actually letting $x = a$.

These definitions can be appropriately modified for the one-sided limits as well. To see a more precise and mathematical definition of this kind of limit see the The Definition of the Limit section at the end of this chapter.

Let's start off with a fairly typical example illustrating infinite limits.

Example 1 Evaluate each of the following limits.

$$\lim_{x\to 0^+} \frac{1}{x} \qquad \lim_{x\to 0^-} \frac{1}{x} \qquad \lim_{x\to 0} \frac{1}{x}$$

Solution

So we're going to be taking a look at a couple of one-sided limits as well as the normal limit here. In all three cases notice that we can't just plug in $x = 0$. If we did we would get division by zero. Also recall that the definitions above can be easily modified to give similar definitions for the two one-sided limits which we'll be needing here.

Now, there are several ways we could proceed here to get values for these limits. One way is to plug in some points and see what value the function is approaching. In the proceeding section we said that we were no longer going to do this, but in this case it is a good way to illustrate just what's going on with this function.

So, here is a table of values of *x*'s from both the left and the right. Using these values we'll be able to estimate the value of the two one-sided limits and once we have that done we can use the fact that the normal limit will exist only if the two one-sided limits exist and have the same value.

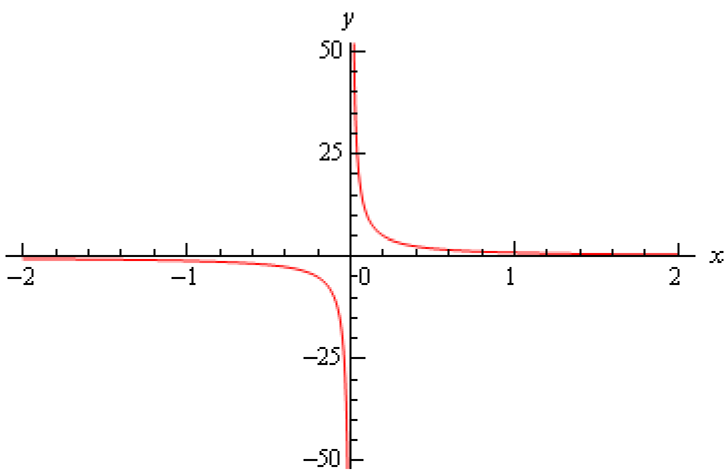
x	$\dfrac{1}{x}$	x	$\dfrac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	1000

From this table we can see that as we make x smaller and smaller the function $\frac{1}{x}$ gets larger and larger and will retain the same sign that x originally had. It should make sense that this trend will continue for any smaller value of x that we chose to use. The function is a constant (one in this case) divided by an increasingly small number. The resulting fraction should be an increasingly large number and as noted above the fraction will retain the same sign as x.

We can make the function as large and positive as we want for all x's sufficiently close to zero while staying positive (*i.e.* on the right). Likewise, we can make the function as large and negative as we want for all x's sufficiently close to zero while staying negative (*i.e.* on the left). So, from our definition above it looks like we should have the following values for the two one sided limits.

$$\lim_{x \to 0^+} \frac{1}{x} = -\infty \qquad\qquad \lim_{x \to 0^-} \frac{1}{x} = \infty$$

Another way to see the values of the two one sided limits here is to graph the function. Again, in the previous section we mentioned that we won't do this too often as most functions are not something we can just quickly sketch out as well as the problems with accuracy in reading values off the graph. In this case however, it's not too hard to sketch a graph of the function and, in this case as we'll see accuracy is not really going to be an issue. So, here is a quick sketch of the graph.



So, we can see from this graph that the function does behave much as we predicted that it would from our table values. The closer *x* gets to zero from the right the larger (in the positive sense) the function gets, while the closer *x* gets to zero from the left the larger (in the negative sense) the function gets.

Finally, the normal limit, in this case, will not exist since the two one-sided have different values.

So, in summary here are the values of the three limits for this example.

$$\lim_{x\to 0^+} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^-} \frac{1}{x} = \infty \qquad \lim_{x\to 0} \frac{1}{x} \text{ doesn't exist}$$

For most of the remaining examples in this section we'll attempt to "talk our way through" each limit. This means that we'll see if we can analyze what should happen to the function as we get very close to the point in question without actually plugging in any values into the function. For most of the following examples this kind of analysis shouldn't be all that difficult to do. We'll also verify our analysis with a quick graph.

So, let's do a couple more examples.

Example 2 Evaluate each of the following limits.

$$\lim_{x\to 0^+} \frac{6}{x^2} \qquad \lim_{x\to 0^-} \frac{6}{x^2} \qquad \lim_{x\to 0} \frac{6}{x^2}$$

Solution

As with the previous example let's start off by looking at the two one-sided limits. Once we have those we'll be able to determine a value for the normal limit.

So, let's take a look at the right-hand limit first and as noted above let's see if we can see if we can figure out what each limit will be doing without actually plugging in any values of *x* into the function. As we take smaller and smaller values of *x*, while staying positive, squaring them will only make them smaller (recall squaring a number between zero and one will make it smaller) and of course it will stay positive. So we have a positive constant divided by an increasingly small positive number. The result should then be an increasingly large positive number. It looks like we should have the following value for the right-hand limit in this case,

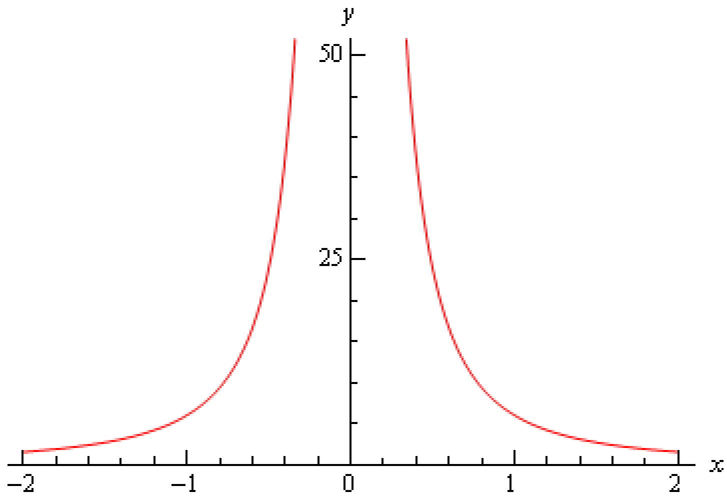
$$\lim_{x\to 0^+} \frac{6}{x^2} = \infty$$

Now, let's take a look at the left hand limit. In this case we're going to take smaller and smaller values of *x*, while staying negative this time. When we square them we'll get smaller, but upon squaring the result is now positive. So, we have a positive constant divided by an increasingly small positive number. The result, as with the right hand limit, will be an increasingly large positive number and so the left-hand limit will be,

$$\lim_{x\to 0^-} \frac{6}{x^2} = \infty$$

Now, in this example, unlike the first one, the normal limit will exist and be infinity since the two one-sided limits both exist and have the same value. So, in summary here are all the limits for this example as well as a quick graph verifying the limits.

$$\lim_{x\to 0^+}\frac{6}{x^2}=\infty \qquad \lim_{x\to 0^-}\frac{6}{x^2}=\infty \qquad \lim_{x\to 0}\frac{6}{x^2}=\infty$$



With this next example we'll move away from just an *x* in the denominator, but as we'll see in the next couple of examples they work pretty much the same way.

Example 3 Evaluate each of the following limits.

$$\lim_{x\to -2^+}\frac{-4}{x+2} \qquad \lim_{x\to -2^-}\frac{-4}{x+2} \qquad \lim_{x\to -2}\frac{-4}{x+2}$$

Solution

Let's again start with the right-hand limit. With the right hand limit we know that we have,

$$x>-2 \qquad \Rightarrow \qquad x+2>0$$

Also, as *x* gets closer and closer to -2 then $x+2$ will be getting closer and closer to zero, while staying positive as noted above. So, for the right-hand limit, we'll have a negative constant divided by an increasingly small positive number. The result will be an increasingly large and negative number. So, it looks like the right-hand limit will be negative infinity.

For the left hand limit we have,

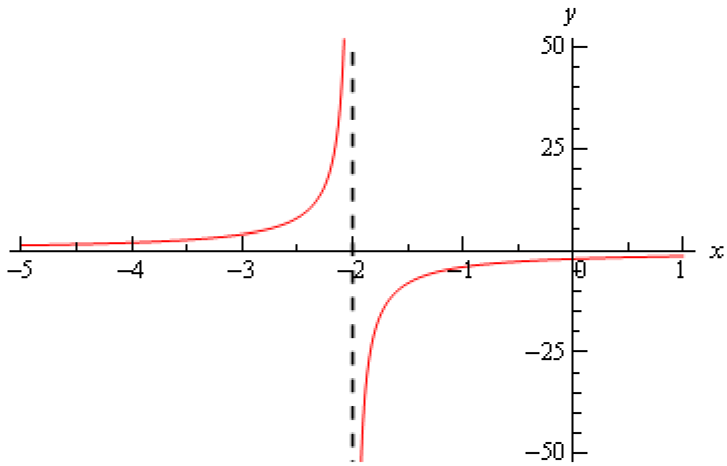
$$x<-2 \qquad \Rightarrow \qquad x+2<0$$

and $x+2$ will get closer and closer to zero (and be negative) as *x* gets closer and closer to -2. In this case then we'll have a negative constant divided by an increasingly small negative number. The result will then be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

Finally, since two one sided limits are not the same the normal limit won't exist.

Here are the official answers for this example as well as a quick graph of the function for verification purposes.

$$\lim_{x\to -2^+} \frac{-4}{x+2} = -\infty \qquad \lim_{x\to -2^-} \frac{-4}{x+2} = \infty \qquad \lim_{x\to -2} \frac{-4}{x+2} \text{ doesn't exist}$$



At this point we should briefly acknowledge the idea of vertical asymptotes. Each of the three previous graphs have had one. Recall from an Algebra class that a vertical asymptote is a vertical line (the dashed line at $x = -2$ in the previous example) in which the graph will go towards infinity and/or minus infinity on one or both sides of the line.

In an Algebra class they are a little difficult to define other than to say pretty much what we just said. Now that we have infinite limits under our belt we can easily define a vertical asymptote as follows,

Definition

The function *f(x)* will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$.

$$\lim_{x\to a^-} f(x) = \pm\infty \qquad \lim_{x\to a^+} f(x) = \pm\infty \qquad \lim_{x\to a} f(x) = \pm\infty$$

Note that it only requires one of the above limits for a function to have a vertical asymptote at $x = a$.

Using this definition we can see that the first two examples had vertical asymptotes at $x = 0$ while the third example had a vertical asymptote at $x = -2$.

We aren't really going to do a lot with vertical asymptotes here, but wanted to mention them at this since we'd reached a good point to do that.

Let's now take a look at a couple more examples of infinite limits that can cause some problems on occasion.

Example 4 Evaluate each of the following limits.

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} \qquad \lim_{x \to 4^-} \frac{3}{(4-x)^3} \qquad \lim_{x \to 4} \frac{3}{(4-x)^3}$$

Solution

Let's start with the right-hand limit. For this limit we have,

$$x > 4 \qquad \Rightarrow \qquad 4 - x < 0 \qquad \Rightarrow \qquad (4-x)^3 < 0$$

also, $4 - x \to 0$ as $x \to 4$. So, we have a positive constant divided by an increasingly small negative number. The results will be an increasingly large negative number and so it looks like the right-hand limit will be negative infinity.

For the left-handed limit we have,

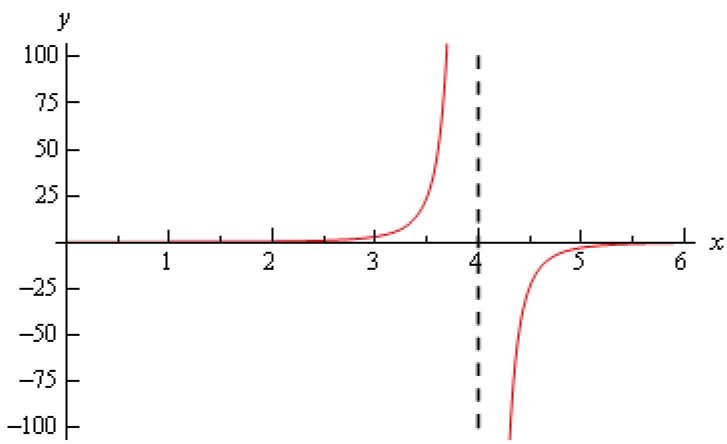
$$x < 4 \qquad \Rightarrow \qquad 4 - x > 0 \qquad \Rightarrow \qquad (4-x)^3 > 0$$

and we still have, $4 - x \to 0$ as $x \to 4$. In this case we have a positive constant divided by an increasingly small positive number. The results will be an increasingly large positive number and so it looks like the right-hand limit will be positive infinity.

The normal limit will not exist since the two one-sided limits are not the same. The official answers to this example are then,

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} = -\infty \qquad \lim_{x \to 4^-} \frac{3}{(4-x)^3} = \infty \qquad \lim_{x \to 4} \frac{3}{(4-x)^3} \text{ doesn't exist}$$

Here is a quick sketch to verify our limits.



All the examples to this point have had a constant in the numerator and we should probably take a quick look at an example that doesn't have a constant in the numerator.

Example 5 Evaluate each of the following limits.

$$\lim_{x \to 3^+} \frac{2x}{x-3} \qquad \lim_{x \to 3^-} \frac{2x}{x-3} \qquad \lim_{x \to 3} \frac{2x}{x-3}$$

Solution

Let's take a look at the right-handed limit first. For this limit we'll have,

$$x > 3 \qquad \Rightarrow \qquad x - 3 > 0$$

The main difference here with this example is the behavior of the numerator as we let *x* get closer and closer to 3. In this case we have the following behavior for both the numerator and denominator.

$$x - 3 \to 0 \text{ and } 2x \to 6 \text{ as } x \to 3$$

So, as we let *x* get closer and closer to 3 (always staying on the right of course) the numerator, while not a constant, is getting closer and closer to a positive constant while the denominator is getting closer and closer to zero, and will be positive since we are on the right side.

This means that we'll have a numerator that is getting closer and closer to a non-zero and positive constant divided by an increasingly smaller positive number and so the result should be an increasingly larger positive number. The right-hand limit should then be positive infinity.

For the left-hand limit we'll have,

$$x < 3 \qquad \Rightarrow \qquad x - 3 < 0$$

As with the right-hand limit we'll have the following behaviors for the numerator and the denominator,

$$x - 3 \to 0 \text{ and } 2x \to 6 \text{ as } x \to 3$$

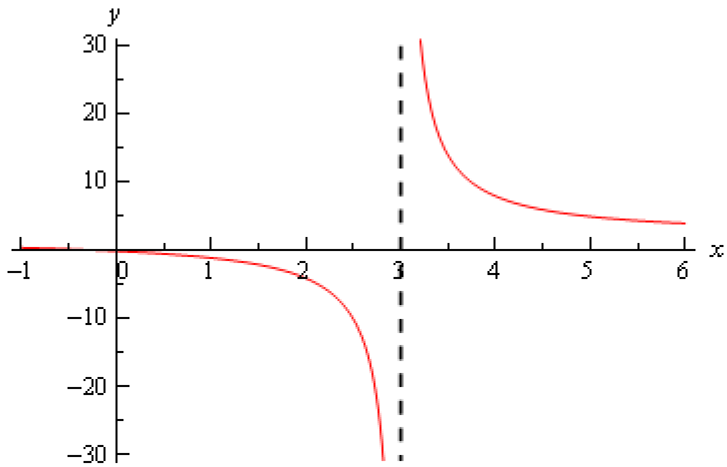
The main difference in this case is that the denominator will now be negative. So, we'll have a numerator that is approaching a positive, non-zero constant divided by an increasingly small negative number. The result will be an increasingly large and negative number.

The formal answers for this example are then,

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty \qquad \lim_{x \to 3^-} \frac{2x}{x-3} = -\infty \qquad \lim_{x \to 3} \frac{2x}{x-3} \text{ doesn't exist}$$

As with most of the examples in this section the normal limit does not exist since the two one-sided limits are not the same.

Here's a quick graph to verify our limits.



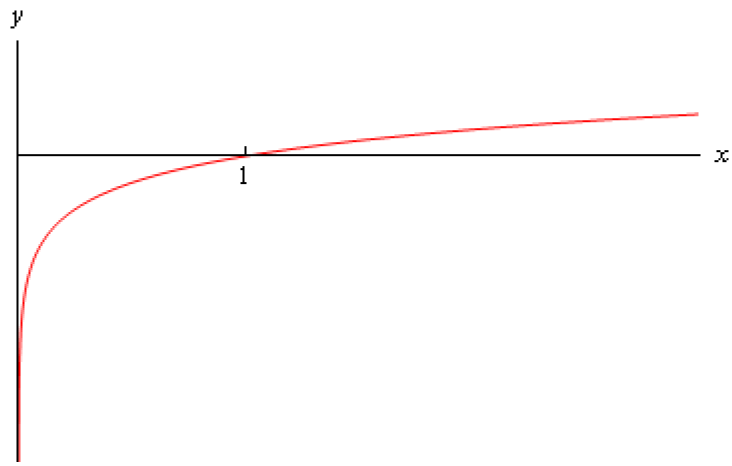
So far all we've done is look at limits of rational expressions, let's do a couple of quick examples with some different functions.

Example 6 Evaluate $\lim\limits_{x \to 0^+} \ln\left(x\right)$

Solution

First, notice that we can only evaluate the right-handed limit here. We know that the domain of any logarithm is only the positive numbers and so we can't even talk about the left-handed limit because that would necessitate the use of negative numbers. Likewise, since we can't deal with the left-handed limit then we can't talk about the normal limit.

This limit is pretty simple to get from a quick sketch of the graph.



From this we can see that,

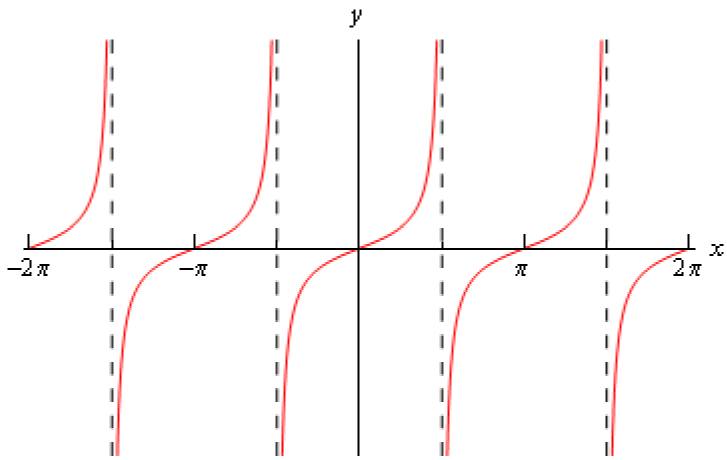
$$\lim_{x \to 0^+} \ln\left(x\right) = -\infty$$

Example 7 Evaluate both of the following limits.

$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) \qquad\qquad \lim_{x \to \frac{\pi}{2}^-} \tan(x)$$

Solution

Here's a quick sketch of the graph of the tangent function.



From this it's easy to see that we have the following values for each of these limits,

$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty \qquad\qquad \lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$$

Note that the normal limit will not exist because the two one-sided limits are not the same.

Limits At Infinity, Part I

In the previous section we saw limits that were infinity and it's now time to take a look at limits at infinity. By limits at infinity we mean one of the following two limits.

$$\lim_{x\to\infty} f(x) \qquad\qquad \lim_{x\to-\infty} f(x)$$

In other words, we are going to be looking at what happens to a function if we let *x* get very large in either the positive or negative sense. Also, as well soon see, these limits may also have infinity as a value.

For many of the limits that we're going to be looking at we will need the following facts.

Fact 1

1. If *r* is a positive rational number and *c* is any real number then,

$$\lim_{x\to\infty} \frac{c}{x^r} = 0$$

2. If *r* is a positive rational number, *c* is any real number and x^r is defined for $x < 0$ then,

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0$$

The first part of this fact should make sense if you think about it. Because we are requiring $r > 0$ we know that x^r will stay in the denominator. Next as we increase *x* then x^r will also increase. So, we have a constant divided by an increasingly large number and so the result will be increasingly small. Or, in the limit we will get zero.

The second part is nearly identical except we need to worry about x^r being defined for negative *x*. This condition is here to avoid cases such as $r = \frac{1}{2}$. If this *r* were allowed then we'd be taking the square root of negative numbers which would be complex and we want to avoid that at this level.

Note as well that the sign of *c* will not affect the answer. Regardless of the sign of *c* we'll still have a constant divided by a very large number which will result in a very small number and the larger *x* get the smaller the fraction gets. The sign of *c* will affect which direction the fraction approaches zero (*i.e.* from the positive or negative side) but it still approaches zero.

To see the proof of this fact see the Proof of Various Limit Properties section in the Extras chapter.

Let's start the off the examples with one that will lead us to a nice idea that we'll use on a regular basis about limits at infinity for polynomials.

Example 1 Differentiate each of the following functions.

(a) $\lim_{x\to\infty} 2x^4 - x^2 - 8x$ [Solution]

(b) $\lim_{t\to-\infty} \frac{1}{3}t^5 + 2t^3 - t^2 + 8$ [Solution]

Solution

(a) $\lim_{x\to\infty} 2x^4 - x^2 - 8x$

Our first thought here is probably to just "plug" infinity into the polynomial and "evaluate" each term to determine the value of the limit. It is pretty simple to see what each term will do in the limit and so this seems like an obvious step, especially since we've been doing that for other limits in previous sections.

So, let's see what we get if we do that. As x approaches infinity, then x to a power can only get larger and the coefficient on each term (the first and third) will only make the term even larger. So, if we look at what each term is doing in the limit we get the following,

$$\lim_{x\to\infty} 2x^4 - x^2 - 8x = \infty - \infty - \infty$$

Now, we've got a small, but easily fixed, problem to deal with. We are probably tempted to say that the answer is zero (because we have an infinity minus an infinity) or maybe $-\infty$ (because we're subtracting two infinities off of one infinity). However, in both cases we'd be wrong. This is one of those **indeterminate forms** that we first started seeing in a previous section.

Infinities just don't always behave as real numbers do when it comes to arithmetic. Without more work there is simply no way to know what $\infty - \infty$ will be and so we really need to be careful with this kind of problem. To read a little more about this see the Types of Infinity section in the Extras chapter.

So, we need a way to get around this problem. What we'll do here is factor the largest power of x out of the whole polynomial as follows,

$$\lim_{x\to\infty} 2x^4 - x^2 - 8x = \lim_{x\to\infty} x^4\left(2 - \frac{1}{x^2} - \frac{8}{x^3}\right)$$

If you're not sure you agree with the factoring above (there's a chance you haven't really been asked to do this kind of factoring prior to this) then recall that to check all you need to do is multiply the x^4 back through the parenthesis to verify it was done correctly. Also, an easy way to remember how to do this kind of factoring is to note that the second term is just the original polynomial divided by x^4. This will always work when factoring a power of x out of a polynomial.

Next, from our properties of limits we know that the limit of a product is the product of the limits so we can further write this as,

$$\lim_{x\to\infty} 2x^4 - x^2 - 8x = \left(\lim_{x\to\infty} x^4\right)\left(\lim_{x\to\infty} 2 - \frac{1}{x^2} - \frac{8}{x^3}\right)$$

The first limit is clearly infinity and for the second limit we'll use the fact above on the last two terms and so we'll arrive at the following value of the limit,

$$\lim_{x\to\infty} 2x^4 - x^2 - 8x = (\infty)(2) = \infty$$

Note that while we can't give a value for $\infty - \infty$, if we multiply an infinity by a constant we will still have an infinity no matter how large or small the constant is. The only thing that we need to be careful of is signs. If the constant had been negative then we'd have gotten negative infinity for a value.

[Return to Problems]

(b) $\lim_{t\to-\infty} \frac{1}{3}t^5 + 2t^3 - t^2 + 8$

We'll work this part much quicker than the previous part. All we need to do is factor out the largest power of *t* to get the following,

$$\lim_{t\to-\infty} \tfrac{1}{3}t^5 + 2t^3 - t^2 + 8 = \lim_{t\to-\infty} t^5\left(\frac{1}{3} + \frac{2}{t^2} - \frac{1}{t^3} + \frac{8}{t^5}\right)$$

Remember that all you need to do to get the factoring correct is divide the original polynomial by the power of *t* we're factoring out, t^5 in this case.

Now all we need to do is take the limit of the two terms. In the first don't forget that since we're going out towards $-\infty$ and we're raising *t* to the 5th power that the limit will be negative (negative number raised to an odd power is still negative). In the second term well again make heavy use of the fact above.

So, taking the limits of the two terms gives,

$$\lim_{t\to-\infty} \tfrac{1}{3}t^5 + 2t^3 - t^2 + 8 = (-\infty)\left(\frac{1}{3}\right) = -\infty$$

Note that dividing an infinity (positive or negative) by a constant will still give an infinity.

[Return to Problems]

Okay, now that we've seen how a couple of polynomials work we can give a simple fact about polynomials in general.

Fact 2

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree *n* (*i.e.* $a_n \neq 0$) then,

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n \qquad\qquad \lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$$

What this fact is really saying is that when we go to take a limit at infinity for a polynomial then all we need to really do is look at the term with the largest power and ask what that term is doing in the limit since the polynomial will have the same behavior.

You can see the proof in the Proof of Various Limit Properties section in the Extras chapter.

Let's now move into some more complicated limits.

Example 2 Evaluate both of the following limits.

$$\lim_{x\to\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} \qquad\qquad \lim_{x\to-\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$$

Solution

First, the only difference between these two is that one is going to positive infinity and the other is going to negative infinity. Sometimes this small difference will affect then value of the limit and at other times it won't.

Let's start with the first limit and as with our first set of examples it might be tempting to just "plug" in the infinity. Since both the numerator and denominator are polynomials we can use the above fact to determine the behavior of each. Doing this gives,

$$\lim_{x\to\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \frac{\infty}{-\infty}$$

This is yet another indeterminate form. In this case we might be tempted to say that the limit is infinity (because of the infinity in the numerator), zero (because of the infinity in the denominator) or -1 (because something divided by itself is one). There are three separate arithmetic "rules" at work here and without work there is no way to know which "rule" will be correct and to make matters worse it's possible that none of them may work and we might get a completely different answer, say $-\frac{2}{5}$ to pick a number completely at random.

So, when we have a polynomial divided by a polynomial we're going to proceed much as we did with only polynomials. We first identify the largest power of *x* in the denominator (and yes, we only look at the denominator for this) and we then factor this out of both the numerator and denominator. Doing this for the first limit gives,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\lim_{x\to\infty}\frac{x^4\left(2-\frac{1}{x^2}+\frac{8}{x^3}\right)}{x^4\left(-5+\frac{7}{x^4}\right)}$$

Once we've done this we can cancel the x^4 from both the numerator and the denominator and then use the Fact 1 above to take the limit of all the remaining terms. This gives,

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}&=\lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{8}{x^3}}{-5+\frac{7}{x^4}}\\&=\frac{2+0+0}{-5+0}\\&=-\frac{2}{5}\end{aligned}$$

In this case the indeterminate form was neither of the "obvious" choices of infinity, zero, or -1 so be careful with make these kinds of assumptions with this kind of indeterminate forms.

The second limit is done in a similar fashion. Notice however, that nowhere in the work for the first limit did we actually use the fact that the limit was going to plus infinity. In this case it doesn't matter which infinity we are going towards we will get the same value for the limit.

$$\lim_{x\to-\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=-\frac{2}{5}$$

In the previous example the infinity that we were using in the limit didn't change the answer. This will not always be the case so don't make the assumption that this will always be the case.

Let's take a look at an example where we get different answers for each limit.

Example 3 Evaluate each of the following limits.

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}\qquad\qquad\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}$$

Solution

The square root in this problem won't change our work, but it will make the work a little messier.

Let's start with the first limit. In this case the largest power of x in the denominator is just an x. So we need to factor an x out of the numerator and the denominator. When we are done factoring the x out we will need an x in both of the numerator and the denominator. To get this in the numerator we will have to factor an x^2 out of the square root so that after we take the square root we will get an x.

This is probably not something you're used to doing, but just remember that when it comes out of the square root it needs to be an *x* and the only way have an *x* come out of a square is to take the square root of x^2 and so that is what we'll need to factor out of the term under the radical. Here's the factoring work for this part,

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{\sqrt{x^2\left(3+\frac{6}{x^2}\right)}}{x\left(\frac{5}{x}-2\right)}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

This is where we need to be really careful with the square root in the problem. Don't forget that

$$\sqrt{x^2}=|x|$$

Square roots are ALWAYS positive and so we need the absolute value bars on the *x* to make sure that it will give a positive answer. This is not something that most people every remember seeing in an Algebra class and in fact it's not always given in an Algebra class. However, at this point it becomes absolutely vital that we know and use this fact. Using this fact the limit becomes,

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

Now, we can't just cancel the *x*'s. We first will need to get rid of the absolute value bars. To do this let's recall the definition of absolute value.

$$|x|=\begin{cases}x & \text{if } x\geq 0\\ -x & \text{if } x<0\end{cases}$$

In this case we are going out to plus infinity so we can safely assume that the *x* will be positive and so we can just drop the absolute value bars. The limit is then,

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$=\lim_{x\to\infty}\frac{\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}=\frac{\sqrt{3+0}}{0-2}=-\frac{\sqrt{3}}{2}$$

Let's now take a look at the second limit (the one with negative infinity). In this case we will need to pay attention to the limit that we are using. The initial work will be the same up until we reach the following step.

$$\lim_{x \to -\infty} \frac{\sqrt{3x^2+6}}{5-2x} = \lim_{x \to -\infty} \frac{\left|x\right|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

In this limit we are going to minus infinity so in this case we can assume that x is negative. So, in order to drop the absolute value bars in this case we will need to tack on a minus sign as well. The limit is then,

$$\begin{aligned}\lim_{x \to -\infty} \frac{\sqrt{3x^2+6}}{5-2x} &= \lim_{x \to -\infty} \frac{-x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)} \\ &= \lim_{x \to -\infty} \frac{-\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

So, as we saw in the last two examples sometimes the infinity in the limit will affect the answer and other times it won't. Note as well that it doesn't always just change the sign of the number. It can on occasion completely change the value. We'll see an example of this later in this section.

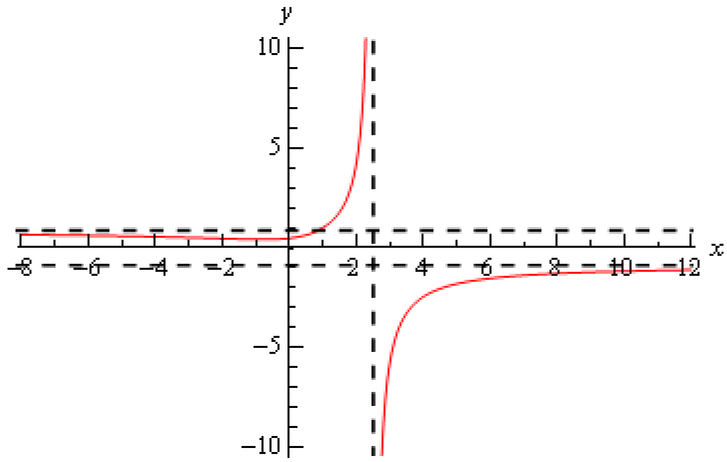
Before moving on to a couple of more examples let's revisit the idea of asymptotes that we first saw in the previous section. Just as we can have vertical asymptotes defined in terms of limits we can also have horizontal asymptotes defined in terms of limits.

Definition

The function *f(x)* will have a horizontal asymptote at *y*=*L* if either of the following are true.

$$\lim_{x \to \infty} f\left(x\right) = L \qquad\qquad \lim_{x \to -\infty} f\left(x\right) = L$$

We're not going to be doing much with asymptotes here, but it's an easy fact to give and we can use the previous example to illustrate all the asymptote ideas we've seen in the both this section and the previous section. The function in the last example will have two horizontal asymptotes. It will also have a vertical asymptote. Here is a graph of the function showing these.



Let's work another couple of examples involving of rational expressions.

Example 4 Evaluate each of the following limits.

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3} \qquad\qquad \lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}$$

Solution

Let's do the first limit and in this case it looks like we will factor a z^3 out of both the numerator and denominator. Remember that we only look at the denominator when determining the largest power of z here. There is a larger power of z in the numerator but we ignore it. We ONLY look at the denominator when doing this! So doing the factoring gives,

$$\begin{aligned}\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3} &= \lim_{z\to\infty}\frac{z^3\left(\frac{4}{z}+z^3\right)}{z^3\left(\frac{1}{z^3}-5\right)} \\ &= \lim_{z\to\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}\end{aligned}$$

When we take the limit we'll need to be a little careful. The first term in the numerator and denominator will both be zero. However, the z^3 in the numerator will be going to plus infinity in the limit and so the limit is,

$$\lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3} = \frac{\infty}{-5} = -\infty$$

The final limit is negative because we have a quotient of positive quantity and a negative quantity.

Now, let's take a look at the second limit. Note that the only different in the work is at the final "evaluation" step and so we'll pick up the work there.

$$\lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}=\lim_{z\to-\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}=\frac{-\infty}{-5}=\infty$$

In this case the z^3 in the numerator gives negative infinity in the limit since we are going out to minus infinity and the power is odd. The answer is positive since we have a quotient of two negative numbers.

Example 5 Evaluate the following limit.

$$\lim_{t\to-\infty}\frac{t^2-5t-9}{2t^4+3t^3}$$

Solution

In this case it looks like we will factor a t^4 out of both the numerator and denominator. Doing this gives,

$$\begin{aligned}\lim_{t\to-\infty}\frac{t^2-5t-9}{2t^4+3t^3}&=\lim_{t\to-\infty}\frac{t^4\left(\frac{1}{t^2}-\frac{5}{t^3}-\frac{9}{t^4}\right)}{t^4\left(2+\frac{3}{t}\right)}\\&=\lim_{t\to-\infty}\frac{\frac{1}{t^2}-\frac{5}{t^3}-\frac{9}{t^4}}{2+\frac{3}{t}}\\&=\frac{0}{2}\\&=0\end{aligned}$$

In this case using Fact 1 we can see that the numerator is zero and so since the denominator is also not zero the fraction, and hence the limit, will be zero.

In this section we concentrated on limits at infinity with functions that only involved polynomials and/or rational expression involving polynomials. There are many more types of functions that we could use here. That is the subject of the next section.

To see a precise and mathematical definition of this kind of limit see the The Definition of the Limit section at the end of this chapter.

Limits At Infinity, Part II

In the previous section we look at limit at infinity of polynomials and/or rational expression involving polynomials. In this section we want to take a look at some other types of functions that often show up in limits at infinity. The functions we'll be looking at here are exponentials, natural logarithms and inverse tangents.

Let's start by taking a look at a some of very basic examples involving exponential functions.

Example 1 Evaluate each of the following limits.

$$\lim_{x\to\infty} \mathbf{e}^{x} \qquad \lim_{x\to-\infty} \mathbf{e}^{x} \qquad \lim_{x\to\infty} \mathbf{e}^{-x} \qquad \lim_{x\to-\infty} \mathbf{e}^{-x}$$

Solution

There are really just restatements of facts given in the basic exponential section of the review so we'll leave it to you to go back and verify these.

$$\lim_{x\to\infty} \mathbf{e}^{x} = \infty \qquad \lim_{x\to-\infty} \mathbf{e}^{x} = 0 \qquad \lim_{x\to\infty} \mathbf{e}^{-x} = 0 \qquad \lim_{x\to-\infty} \mathbf{e}^{-x} = \infty$$

The main point of this example was to point out that if the exponent of an exponential goes to infinity in the limit then the exponential function will also go to infinity in the limit. Likewise, if the exponent goes to minus infinity in the limit then the exponential will go to zero in the limit.

Here's a quick set of examples to illustrate these ideas.

Example 2 Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \mathbf{e}^{2-4x-8x^2}$ [Solution]

(b) $\lim_{t\to-\infty} \mathbf{e}^{t^4-5t^2+1}$ [Solution]

(c) $\lim_{z\to 0^+} \mathbf{e}^{\frac{1}{z}}$ [Solution]

Solution

(a) $\lim_{x\to\infty} \mathbf{e}^{2-4x-8x^2}$

In this part what we need to note (using Fact 2 above) is that in the limit the exponent of the exponential does the following,

$$\lim_{x\to\infty} 2-4x-8x^2 = -\infty$$

So, the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous set of examples. So, the answer here is,

$$\lim_{x\to\infty} \mathbf{e}^{2-4x-8x^2} = 0$$

[Return to Problems]

(b) $\lim\limits_{t \to -\infty} \mathbf{e}^{t^4 - 5t^2 + 1}$

Here let's first note that,

$$\lim_{t \to -\infty} t^4 - 5t^2 + 1 = \infty$$

The exponent goes to infinity in the limit and so the exponential will also need to go to infinity in the limit. Or,

$$\lim_{t \to -\infty} \mathbf{e}^{t^4 - 5t^2 + 1} = \infty$$

[Return to Problems]

(c) $\lim\limits_{z \to 0^+} \mathbf{e}^{\frac{1}{z}}$

On the surface this part doesn't appear to belong in this section since it isn't a limit at infinity. However, it does fit into the ideas we're examining in this set of examples.

So, let's first note that using the idea from the previous section we have,

$$\lim_{z \to 0^+} \frac{1}{z} = \infty$$

Remember that in order to do this limit here we do need to do a right hand limit.

So, the exponent goes to infinity in the limit and so the exponential must also go to infinity.

Here's the answer to this part.

$$\lim_{z \to 0^+} \mathbf{e}^{\frac{1}{z}} = \infty$$

[Return to Problems]

Let's work some more complicated examples involving exponentials. In the following set of examples it won't be that the exponents are more complicated, but instead that there will be more than one exponential function to deal with.

Example 3 Evaluate each of the following limits.

(a) $\lim\limits_{x \to \infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}$ [Solution]

(b) $\lim\limits_{x \to -\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}$ [Solution]

Solution

So, the only difference between these two limits is the fact that in the first we're taking the limit as we go to plus infinity and in the second we're going to minus infinity. To this point we've been able to "reuse" work from the first limit in the at least a portion of the second limit. With exponentials that will often not be the case we we're going to treat each of these as separate problems.

(a) $\lim\limits_{x\to\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}$

Let's start by just taking the limit of each of the pieces and see what we get.

$$\lim_{x\to\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x} = \infty - \infty + \infty + 0 - 0$$

The last two terms aren't any problem (they will be in the next part however, do you see that?). The first three are a problem however as they present us with another indeterminate form.

When dealing with polynomials we factored out the term with the largest exponent in it. Let's do the same thing here. However, we now have to deal with both positive and negative exponents and just what do we mean by the "largest" exponent. When dealing with these here we look at the terms that are causing the problems and ask which is the largest exponent in those terms. So, since only the first three terms are causing us problems (*i.e.* they all evaluate to an infinity in the limit) we'll look only at those.

So, since $10x$ is the largest of the three exponents there we'll "factor" an $\mathbf{e}^{10x}$ out of the whole thing. Just as with polynomials we do the factoring by, in essence, dividing each term by $\mathbf{e}^{10x}$ and remembering that to simply the division all we need to do is subtract the exponents. For example, let's just take a look at the last term,

$$\frac{-9\mathbf{e}^{-15x}}{\mathbf{e}^{10x}} = -9\mathbf{e}^{-15x-10x} = -9\mathbf{e}^{-25x}$$

Doing factoring on all terms then gives,

$$\lim_{x\to\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x} = \lim_{x\to\infty} \mathbf{e}^{10x}\left(1 - 4\mathbf{e}^{-4x} + 3\mathbf{e}^{-9x} + 2\mathbf{e}^{-12x} - 9\mathbf{e}^{-25x}\right)$$

Notice that in doing this factoring all the remaining exponentials now have negative exponents and we know that for this limit (*i.e.* going out to positive infinity) these will all be zero in the limit and so will no longer cause problems.

We can now take the limit and doing so gives,

$$\lim_{x\to\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x} = (\infty)(1) = \infty$$

To simplify the work here a little all we really needed to do was factor the $\mathbf{e}^{10x}$ out of the "problem" terms (the first three in this case) as follows,

$$\begin{aligned}\lim_{x\to\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} &= \lim_{x\to\infty} \mathbf{e}^{10x}\left(1 - 4\mathbf{e}^{-4x} + 3\mathbf{e}^{-9x}\right) + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x} \\ &= (\infty)(1) + 0 - 0 \\ &= \infty\end{aligned}$$

We factored the $\mathbf{e}^{10x}$ out of all terms for the practice of doing the factoring and to avoid any issues with having the extra terms at the end.

[Return to Problems]

(b) $\lim\limits_{x\to-\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}$

Let's start this one off in the same manner as the first part. Let's take the limit of each of the pieces. This time note that because our limit is going to negative infinity the first three exponentials will in fact go to zero (because their exponents go to minus infinity in the limit). The final two exponentials will go to infinity in the limit (because their exponents go to plus infinity in the limit).

Taking the limits gives,

$$\lim_{x\to-\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x} = 0 - 0 + 0 + \infty - \infty$$

So, the last two terms are the problem here as they once again leave us with an indeterminate form. As with the first example we're going to factor out the "largest" exponent in the last two terms. This time however, "largest" doesn't refer to the bigger of the two numbers (-2 is bigger than -15). Instead we're going to use "largest" to refer to the exponent that is farther away from zero. Using this definition of "largest" means that we're going to factor an $\mathbf{e}^{-15x}$ out.

Again, remember that to factor this out all we really are doing is dividing each term by $\mathbf{e}^{-15x}$ and then subtracting exponents. Here's the work for the first term as an example,

$$\frac{\mathbf{e}^{10x}}{\mathbf{e}^{-15x}} = \mathbf{e}^{10x-(-15x)} = \mathbf{e}^{25x}$$

As with the first part we can either factor it out of only the "problem" terms (*i.e.* the last two terms), or all the terms. For the practice we'll factor it out of all the terms. Here is the factoring work for this limit,

$$\lim_{x\to-\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x} = \lim_{x\to-\infty} \mathbf{e}^{-15x}\left(\mathbf{e}^{25x} - 4\mathbf{e}^{21x} + 3\mathbf{e}^{16x} + 2\mathbf{e}^{13x} - 9\right)$$

Finally, all we need to do is take the limit.

$$\lim_{x\to-\infty} \mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x} = (\infty)(-9) = -\infty$$

[Return to Problems]

So, when dealing with sums and/or differences of exponential functions we look for the exponential with the "largest" exponent and remember here that "largest" means the exponent farthest from zero. Also remember that if we're looking at a limit at plus infinity only the exponentials with positive exponents are going to cause problems so those are the only terms we look at in determining the largest exponent. Likewise, if we are looking at a limit at minus infinity then only exponentials with negative exponents are going to cause problems and so only those are looked at in determining the largest exponent.

Finally, as you might have been able to guess from the previous example when dealing with a sum and/or difference of exponentials all we need to do is look at the largest exponent to determine the behavior of the whole expression. Again, remembering that if the limit is at plus infinity we only look at exponentials with positive exponents and if we're looking at a limit at minus infinity we only look at exponentials with negative exponents.

Let's next take a look at some rational functions involving exponentials.

Example 4 Evaluate each of the following limits.

(a) $\displaystyle \lim_{x\to\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}}$ [Solution]

(b) $\displaystyle \lim_{x\to-\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}}$ [Solution]

(c) $\displaystyle \lim_{t\to-\infty} \frac{\mathbf{e}^{6t} - 4\mathbf{e}^{-6t}}{2\mathbf{e}^{3t} - 5\mathbf{e}^{-9t} + \mathbf{e}^{-3t}}$ [Solution]

Solution

As with the previous example, the only difference between the first two parts is that one of the limits is going to plus infinity and the other is going to minus infinity and just as with the previous example each will need to be worked differently.

(a) $\displaystyle \lim_{x\to\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}}$

The basic concept involved in working this problem is the same as with rational expressions in the previous section. We look at the denominator and determine the exponential function with the "largest" exponent which we will then factor out from both numerator and denominator. We will use the same reasoning as we did with the previous example to determine the "largest" exponent. In the case since we are looking at a limit at plus infinity we only look at exponentials with positive exponents.

So, we'll factor an $\mathbf{e}^{4x}$ out of both then numerator and denominator. Once that is done we can cancel the $\mathbf{e}^{4x}$ and then take the limit of the remaining terms. Here is the work for this limit,

$$\begin{aligned}
\lim_{x\to\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}} &= \lim_{x\to\infty} \frac{\mathbf{e}^{4x}\left(6 - \mathbf{e}^{-6x}\right)}{\mathbf{e}^{4x}\left(8 - \mathbf{e}^{-6x} + 3\mathbf{e}^{-5x}\right)} \\
&= \lim_{x\to\infty} \frac{6 - \mathbf{e}^{-6x}}{8 - \mathbf{e}^{-6x} + 3\mathbf{e}^{-5x}} \\
&= \frac{6-0}{8-0+0} \\
&= \frac{2}{3}
\end{aligned}$$

[Return to Problems]

(b) $\displaystyle \lim_{x\to-\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}}$

In this case we're going to minus infinity in the limit and so we'll look at exponentials in the denominator with negative exponents in determining the "largest" exponent. There's only one however in this problem so that is what we'll use.

Again, remember to only look at the denominator. Do NOT use the exponential from the numerator, even though that one is "larger" than the exponential in then denominator. We always look only at the denominator when determining what term to factor out regardless of what is going on in the numerator.

Here is the work for this part.

$$\begin{aligned} \lim_{x\to-\infty} \frac{6\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{8\mathbf{e}^{4x} - \mathbf{e}^{2x} + 3\mathbf{e}^{-x}} &= \lim_{x\to-\infty} \frac{\mathbf{e}^{-x}\left(6\mathbf{e}^{5x} - \mathbf{e}^{-x}\right)}{\mathbf{e}^{-x}\left(8\mathbf{e}^{5x} - \mathbf{e}^{3x} + 3\right)} \\ &= \lim_{x\to-\infty} \frac{6\mathbf{e}^{5x} - \mathbf{e}^{-x}}{8\mathbf{e}^{5x} - \mathbf{e}^{3x} + 3} \\ &= \frac{0-\infty}{0-0+3} \\ &= -\infty \end{aligned}$$

[Return to Problems]

(c) $\displaystyle \lim_{t\to-\infty} \frac{\mathbf{e}^{6t} - 4\mathbf{e}^{-6t}}{2\mathbf{e}^{3t} - 5\mathbf{e}^{-9t} + \mathbf{e}^{-3t}}$

We'll do the work on this part with much less detail.

$$\begin{aligned} \lim_{t\to-\infty} \frac{\mathbf{e}^{6t} - 4\mathbf{e}^{-6t}}{2\mathbf{e}^{3t} - 5\mathbf{e}^{-9t} + \mathbf{e}^{-3t}} &= \lim_{t\to-\infty} \frac{\mathbf{e}^{-9t}\left(\mathbf{e}^{15t} - 4\mathbf{e}^{3t}\right)}{\mathbf{e}^{-9t}\left(2\mathbf{e}^{12t} - 5 + \mathbf{e}^{6t}\right)} \\ &= \lim_{t\to-\infty} \frac{\mathbf{e}^{15t} - 4\mathbf{e}^{3t}}{2\mathbf{e}^{12t} - 5 + \mathbf{e}^{6t}} \\ &= \frac{0-0}{0-5+0} \\ &= 0 \end{aligned}$$

[Return to Problems]

Next, let's take a quick look at some basic limits involving logarithms.

Example 5 Evaluate each of the following limits.

$$\lim_{x\to0^+} \ln x \qquad\qquad \lim_{x\to\infty} \ln x$$

Solution

As with the last example I'll leave it to you to verify these restatements from the basic logarithm section.

$$\lim_{x \to 0^+} \ln x = -\infty \qquad\qquad \lim_{x \to \infty} \ln x = \infty$$

Note that we had to do a right-handed limit for the first one since we can't plug negative *x*'s into a logarithm. This means that the normal limit won't exist since we must look at *x*'s from both sides of the point in question and *x*'s to the left of zero are negative.

From the previous example we can see that if the argument of a log (the stuff we're taking the log of) goes to zero from the right (*i.e.* always positive) then the log goes to negative infinity in the limit while if the argument goes to infinity then the log also goes to infinity in the limit.

Note as well that we can't look at a limit of a logarithm as *x* approaches minus infinity since we can't plug negative numbers into the logarithm.

Let's take a quick look at some logarithm examples.

Example 6 Evaluate each of the following limits.

(a) $\lim_{x \to \infty} \ln\left(7x^3 - x^2 + 1\right)$ [Solution]

(b) $\lim_{t \to -\infty} \ln\left(\frac{1}{t^2 - 5t}\right)$ [Solution]

Solution

(a) $\lim_{x \to \infty} \ln\left(7x^3 - x^2 + 1\right)$

So, let's first look to see what the argument of log is doing,

$$\lim_{x \to \infty} 7x^3 - x^2 + 1 = \infty$$

The argument of the log is going to infinity and so the log must also be going to infinity in the limit. The answer to this part is then,

$$\lim_{x \to \infty} \ln\left(7x^3 - x^2 + 1\right) = \infty$$

[Return to Problems]

(b) $\lim_{t \to -\infty} \ln\left(\frac{1}{t^2 - 5t}\right)$

First, note that the limit going to negative infinity here isn't a violation (necessarily) of the fact that we can't plug negative numbers into the logarithm. The real issue is whether or not the argument of the log will be negative or not.

Using the techniques from earlier in this section we can see that,

$$\lim_{t \to -\infty} \frac{1}{t^2 - 5t} = 0$$

and let's also not that for negative numbers (which we can assume we've got since we're going to minus infinity in the limit) the denominator will always be positive and so the quotient will also always be positive. Therefore, not only does the argument go to zero, it goes to zero from the right. This is exactly what we need to do this limit.

So, the answer here is,

$$\lim_{t \to -\infty} \ln\left(\frac{1}{t^2 - 5t}\right) = -\infty$$

[Return to Problems]

As a final set of examples let's take a look at some limits involving inverse tangents.

Example 7 Evaluate each of the following limits.

(a) $\lim_{x \to \infty} \tan^{-1} x$ [Solution]

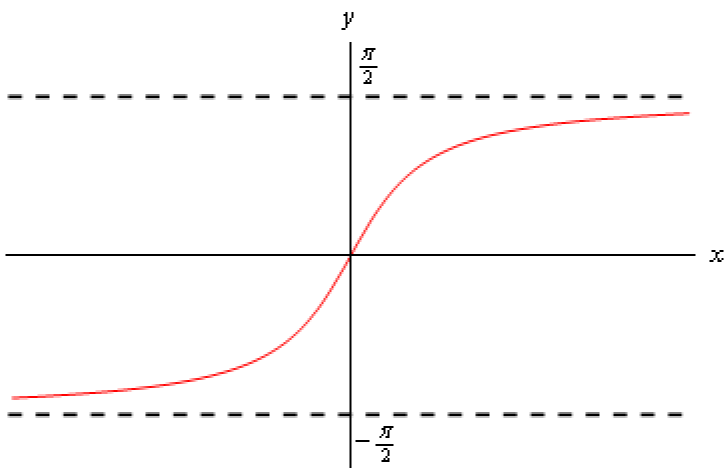
(b) $\lim_{x \to -\infty} \tan^{-1} x$ [Solution]

(c) $\lim_{x \to \infty} \tan^{-1}\left(x^3 - 5x + 6\right)$ [Solution]

(d) $\lim_{x \to 0^-} \tan^{-1}\left(\frac{1}{x}\right)$ [Solution]

Solution

The first two parts here are really just the basic limits involving inverse tangents and can easily be found by examining the following sketch of inverse tangents. The remaining two parts are more involved but as with the exponential and logarithm limits really just refer back to the first two parts as we'll see.



(a) $\lim_{x \to \infty} \tan^{-1} x$

As noted above all we really need to do here is look at the graph of the inverse tangent. Doing this shows us that we have the following value of the limit.

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

[Return to Problems]

(b) $\lim\limits_{x\to-\infty} \tan^{-1} x$

Again, not much to do here other than examine the graph of the inverse tangent.

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

[Return to Problems]

(c) $\lim\limits_{x\to\infty} \tan^{-1}\left(x^3 - 5x + 6\right)$

Okay, in part (a) above we saw that if the argument of the inverse tangent function (the stuff inside the parenthesis) goes to plus infinity then we know the value of the limit. In this case (using the techniques from the previous section) we have,

$$\lim_{x\to\infty} x^3 - 5x + 6 = \infty$$

So, this limit is,

$$\lim_{x\to\infty} \tan^{-1}\left(x^3 - 5x + 6\right) = \frac{\pi}{2}$$

[Return to Problems]

(d) $\lim\limits_{x\to 0^-} \tan^{-1}\left(\frac{1}{x}\right)$

Even though this limit is not a limit at infinity we're still looking at the same basic idea here. We'll use part (b) from above as a guide for this limit. We know from the Infinite Limits section that we have the following limit for the argument of this inverse tangent,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

So, since the argument goes to minus infinity in the limit we know that this limit must be,

$$\lim_{x\to 0^-} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

[Return to Problems]

To see a precise and mathematical definition of this kind of limit see the The Definition of the Limit section at the end of this chapter.

Continuity

Over the last few sections we've been using the term "nice enough" to define those functions that we could evaluate limits by just evaluating the function at the point in question. It's now time to formally define what we mean by "nice enough".

Definition

A function $f(x)$ is said to be **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

A function is said to be continuous on the interval [*a*, *b*] if it is continuous at each point in the interval.

This definition can be turned around into the following fact.

Fact 1

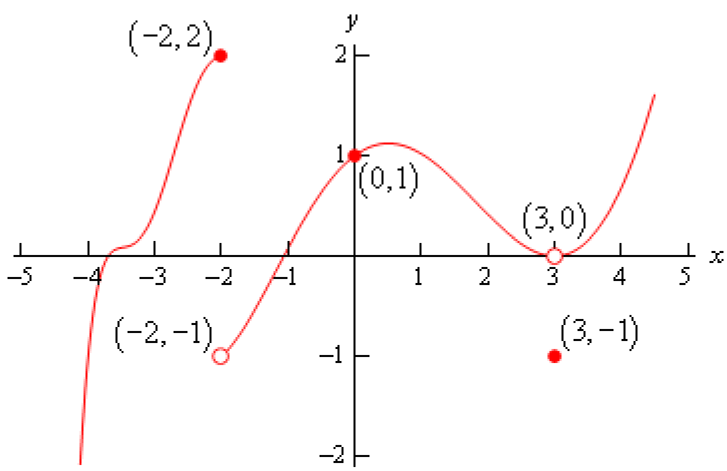
If $f(x)$ is continuous at $x = a$ then,

$$\lim_{x \to a} f(x) = f(a) \qquad \lim_{x \to a^-} f(x) = f(a) \qquad \lim_{x \to a^+} f(x) = f(a)$$

This is exactly the same fact that we first put down back when we started looking at limits with the exception that we have replaced the phrase "nice enough" with continuous.

It's nice to finally know what we mean by "nice enough", however, the definition doesn't really tell us just what it means for a function to be continuous. Let's take a look at an example to help us understand just what it means for a function to be continuous.

Example 1 Given the graph of *f(x)*, shown below, determine if *f(x)* is continuous at $x = -2$, $x = 0$, and $x = 3$.



Solution

To answer the question for each point we'll need to get both the limit at that point and the function value at that point. If they are equal the function is continuous at that point and if they aren't equal the function isn't continuous at that point.

First $x = -2$.

$$f\left(-2\right) = 2 \qquad\qquad \lim_{x \to -2} f\left(x\right) \text{ doesn't exist}$$

The function value and the limit aren't the same and so the function is not continuous at this point. This kind of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities occur where the graph has a break in it is as this graph does.

Now $x = 0$.

$$f\left(0\right) = 1 \qquad\qquad \lim_{x \to 0} f\left(x\right) = 1$$

The function is continuous at this point since the function and limit have the same value.

Finally $x = 3$.

$$f\left(3\right) = -1 \qquad\qquad \lim_{x \to 3} f\left(x\right) = 0$$

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

From this example we can get a quick "working" definition of continuity. A function is continuous on an interval if we can draw the graph from start to finish without ever once picking up our pencil. The graph in the last example has only two discontinuities since there are only two places where we would have to pick up our pencil in sketching it.
In other words, a function is continuous if its graph has no holes or breaks in it.

For many functions it's easy to determine where it won't be continuous. Functions won't be continuous where we have things like division by zero or logarithms of zero. Let's take a quick look at an example of determining where a function is not continuous.

Example 2 Determine where the function below is not continuous.

$$h\left(t\right) = \frac{4t + 10}{t^2 - 2t - 15}$$

Solution

Rational functions are continuous everywhere except where we have division by zero. So all that we need to is determine where the denominator is zero. That's easy enough to determine by setting the denominator equal to zero and solving.

$$t^2 - 2t - 15 = \left(t - 5\right)\left(t + 3\right) = 0$$

So, the function will not be continuous at *t=-3* and *t=5*.

A nice consequence of continuity is the following fact.

Fact 2

If $f(x)$ is continuous at $x = b$ and $\lim_{x \to a} g(x) = b$ then,

$$\lim_{x \to a} f\left(g(x)\right) = f\left(\lim_{x \to a} g(x)\right)$$

To see a proof of this fact see the Proof of Various Limit Properties section in the Extras chapter. With this fact we can now do limits like the following example.

Example 3 Evaluate the following limit.

$$\lim_{x \to 0} \mathbf{e}^{\sin x}$$

Solution

Since we know that exponentials are continuous everywhere we can use the fact above.

$$\lim_{x \to 0} \mathbf{e}^{\sin x} = \mathbf{e}^{\lim_{x \to 0} \sin x} = \mathbf{e}^0 = 1$$

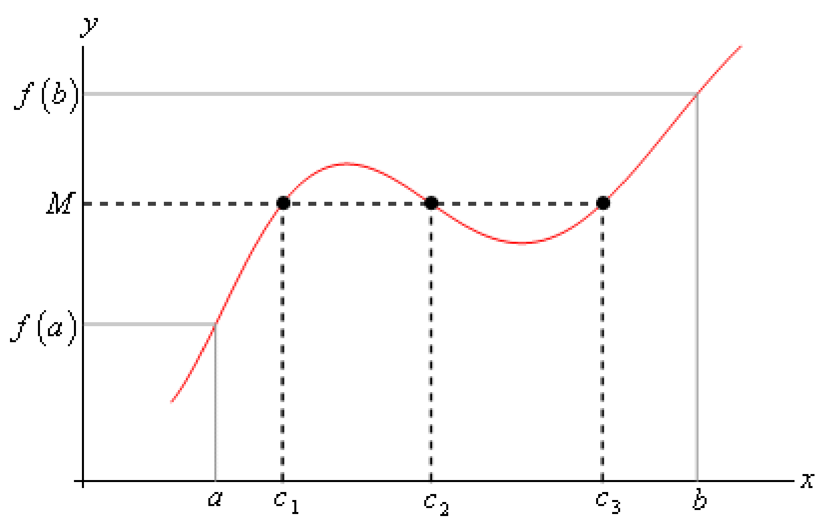
Another very nice consequence of continuity is the Intermediate Value Theorem.

Intermediate Value Theorem

Suppose that *f(x)* is continuous on [*a*, *b*] and let *M* be any number between *f(a)* and *f(b)*. Then there exists a number *c* such that,

1. $a < c < b$
2. $f(c) = M$

All the Intermediate Value Theorem is really saying is that a continuous function will take on all values between *f(a)* and *f(b)*. Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



As we can see from this image if we pick any value, *M*, that is between the value of *f(a)* and the value of *f(b)* and draw a line straight out from this point the line will hit the graph in at least one point. In other words somewhere between *a* and *b* the function will take on the value of *M*. Also, as the figure shows the function may take on the value at more than one place.

It's also important to note that the Intermediate Value Theorem only says that the function will take on the value of *M* somewhere between *a* and *b*. It doesn't say just what that value will be. It only says that it exists.

So, the Intermediate Value Theorem tells us that a function will take the value of *M* somewhere between *a* and *b* but it doesn't tell us where it will take the value nor does it tell us how many times it will take the value. There are important idea to remember about the Intermediate Value Theorem.

A nice use of the Intermediate Value Theorem is to prove the existence of roots of equations as the following example shows.

Example 4 Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval [-1,2].

Solution
What we're really asking here is whether or not the function will take on the value

$$p(x) = 0$$

somewhere between -1 and 2. In other words, we want to show that there is a number *c* such that $-1 < c < 2$ and $p(c) = 0$. However if we define $M = 0$ and acknowledge that $a = -1$ and $b = 2$ we can see that these two condition on *c* are exactly the conclusions of the Intermediate Value Theorem.

So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function is continuous and that $M = 0$ is between $p(-1)$ and $p(2)$ (*i.e.* $p(-1) < 0 < p(2)$ or $p(2) < 0 < p(-1)$ and we'll be done.

To do this all we need to do is compute,

$$p(-1) = 8 \qquad\qquad p(2) = -19$$

So we have,

$$-19 = p(2) < 0 < p(-1) = 8$$

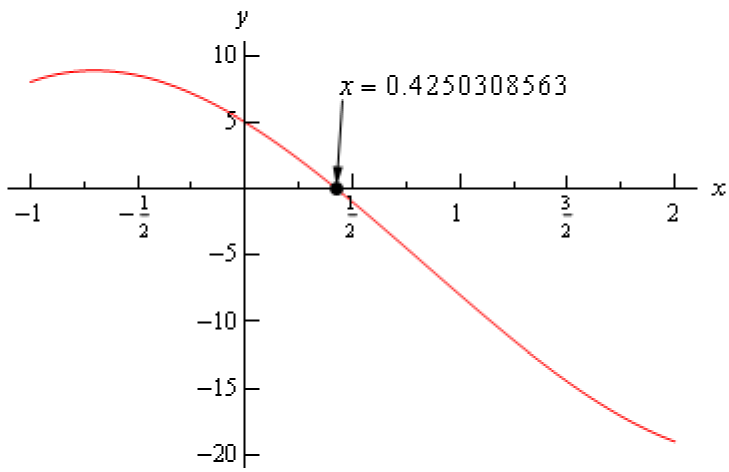
Therefore $M = 0$ is between $p(-1)$ and $p(2)$ and since $p(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval [-1,2]. So by the Intermediate

Value Theorem there must be a number $-1 < c < 2$ so that,

$$p(c) = 0$$

Therefore the polynomial does have a root between -1 and 2.

For the sake of completeness here is a graph showing the root that we just proved existed. Note that we used a computer program to actually find the root and that the Intermediate Value Theorem did not tell us what this value was.



Let's take a look at another example of the Intermediate Value Theorem.

Example 5 If possible, determine if $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ takes the following values in the interval [0,5].

(a) Does $f(x) = 10$? [Solution]

(b) Does $f(x) = -10$? [Solution]

Solution

Okay, so much as the previous example we're being asked to determine, if possible, if the function takes on either of the two values above in the interval [0,5]. First, let's notice that this is a continuous function and so we know that we can use the Intermediate Value Theorem to do this problem.

Now, for each part we will let M be the given value for that part and then we'll need to show that M lives between $f(0)$ and $f(5)$. If it does then we can use the Intermediate Value Theorem to prove that the function will take the given value.

So, since we'll need the two function evaluations for each part let's give them here,

$$f(0) = 2.8224 \qquad\qquad f(5) = 19.7436$$

Now, let's take a look at each part.

(a) Okay, in this case we'll define $M = 10$ and we can see that,

$$f(0) = 2.8224 < 10 < 19.7436 = f(5)$$

So, by the Intermediate Value Theorem there must be a number $0 \le c \le 5$ such that

$$f(c) = 10$$

[Return to Problems]

(b) In this part we'll define $M = -10$. We now have a problem. In this part *M* does not live between $f(0)$ and $f(5)$. So, what does this mean for us? Does this mean that $f(x) \ne -10$ in [0,5]?

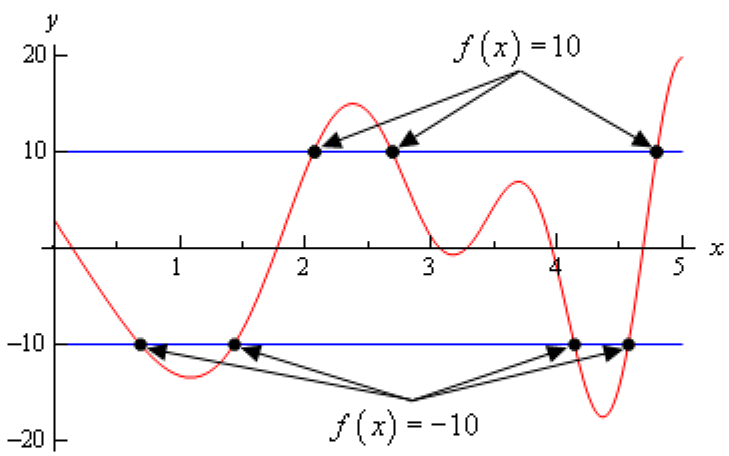
Unfortunately for us, this doesn't mean anything. It is possible that $f(x) \ne -10$ in [0,5], but is it also possible that $f(x) = -10$ in [0,5]. The Intermediate Value Theorem will only tell us that *c*'s will exist. The theorem will NOT tell us that *c*'s don't exist.

In this case it is not possible to determine if $f(x) = -10$ in [0,5] using the Intermediate Value Theorem.

[Return to Problems]

Okay, as the previous example has shown, the Intermediate Value Theorem will not always be able to tell us what we want to know. Sometimes we can use it to verify that a function will take some value in a given interval and in other cases we won't be able to use it.

For completeness sake here is the graph of $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ in the interval [0,5].



From this graph we can see that not only does $f(x) = -10$ in [0,5] it does so a total of 4 times! Also note that as we verified in the first part of the previous example $f(x) = 10$ in [0,5] and in fact it does so a total of 3 times.

So, remember that the Intermediate Value Theorem will only verify that a function will take on a given value. It will never exclude a value from being taken by the function. Also, if we can use the Intermediate Value Theorem to verify that a function will take on a value it never tells us how many times the function will the value, it only tells us that it does take the value.

The Definition of the Limit

In this section we're going to be taking a look at the precise, mathematical definition of the three kinds of limits we looked at in this chapter. We'll be looking at the precise definition of limits at finite points that have finite values, limits that are infinity and limits at infinity. We'll also give the precise, mathematical definition of continuity.

Let's start this section out with the definition of a limit at a finite point that has a finite value.

Definition 1 Let *f(x)* be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

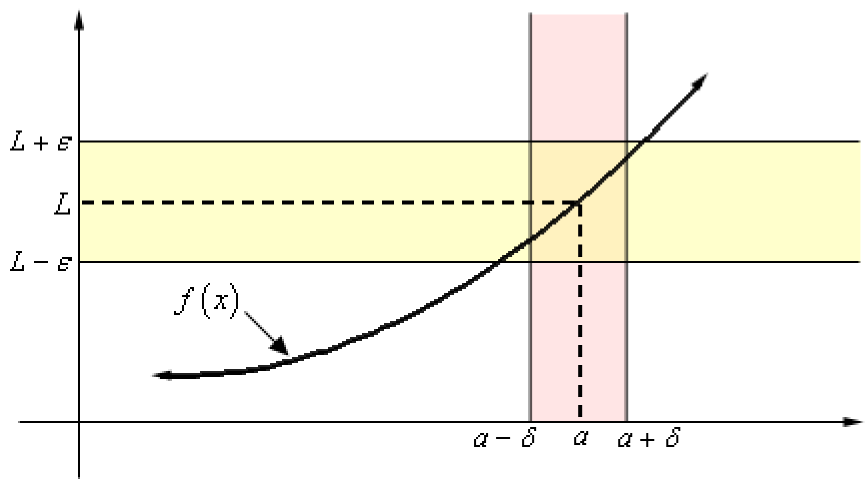
$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$$\left| f(x) - L \right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta$$

Wow. That's a mouth full. Now that it's written down, just what does this mean?

Let's take a look at the following graph and let's also assume that the limit does exist.



What the definition is telling us is that for **any** number $\varepsilon > 0$ that we pick we can go to our graph and sketch two horizontal lines at $L + \varepsilon$ and $L - \varepsilon$ as shown on the graph above. Then somewhere out there in the world is another number $\delta > 0$, which we will need to determine, that will allow us to add in two vertical lines to our graph at $a + \delta$ and $a - \delta$.

Now, if we take any *x* in the pink region, *i.e.* between $a + \delta$ and $a - \delta$, then this *x* will be closer to *a* than either of $a + \delta$ and $a - \delta$. Or,

$$\left| x - a \right| < \delta$$

If we now identify the point on the graph that our choice of *x* gives then this point on the graph **will** lie in the intersection of the pink and yellow region. This means that this function value *f(x)* will be closer to *L* than either of $L+\varepsilon$ and $L-\varepsilon$. Or,

$$\left|f\left(x\right)-L\right|<\varepsilon$$

So, if we take any value of *x* in the pink region then the graph for those values of *x* will lie in the yellow region.

Notice that there are actually an infinite number of possible δ's that we can choose. In fact, if we go back and look at the graph above it looks like we could have taken a slightly larger δ and still gotten the graph from that pink region to be completely contained in the yellow region.

Also, notice that as the definition points out we only need to make sure that the function is defined in some interval around $x=a$ but we don't really care if it is defined at $x=a$. Remember that limits do not care what is happening at the point, they only care what is happening around the point in question.

Okay, now that we've gotten the definition out of the way and made an attempt to understand it let's see how it's actually used in practice.

These are a little tricky sometimes and it can take a lot of practice to get good at these so don't feel too bad if you don't pick up on this stuff right away. We're going to be looking a couple of examples that work out fairly easily.

Example 1 Use the definition of the limit to prove the following limit.

$$\lim_{x\to 0} x^2 = 0$$

Solution

In this case both *L* and *a* are zero. So, let $\varepsilon>0$ be any number. Don't worry about what the number is, ε is just some arbitrary number. Now according to the definition of the limit, if this limit is to be true we will need to find some other number $\delta>0$ so that the following will be true.

$$\left|x^2-0\right|<\varepsilon \qquad \text{whenever} \qquad 0<\left|x-0\right|<\delta$$

Or upon simplifying things we need,

$$\left|x^2\right|<\varepsilon \qquad \text{whenever} \qquad 0<\left|x\right|<\delta$$

Often the way to go through these is to start with the left inequality and do a little simplification and see if that suggests a choice for δ. We'll start by bringing the exponent out of the absolute value bars and then taking the square root of both sides.

$$|x|^2 < \varepsilon \qquad \Rightarrow \qquad |x| < \sqrt{\varepsilon}$$

Now, the results of this simplification looks an awful lot like $0 < |x| < \delta$ with the exception of the "$0 <$" part. Missing that however isn't a problem, it is just telling us that we can't take $x = 0$. So, it looks like if we choose $\delta = \sqrt{\varepsilon}$ we should get what we want.

We'll next need to verify that our choice of δ will give us what we want, *i.e.*,

$$|x^2| < \varepsilon \qquad \text{whenever} \qquad 0 < |x| < \sqrt{\varepsilon}$$

Verification is in fact pretty much the same work that we did to get our guess. First, let's again let $\varepsilon > 0$ be any number and then choose $\delta = \sqrt{\varepsilon}$. Now, assume that $0 < |x| < \sqrt{\varepsilon}$. We need to show that by choosing *x* to satisfy this we will get,

$$|x^2| < \varepsilon$$

To start the verification process we'll start with $|x^2|$ and then first strip out the exponent from the absolute values. Once this is done we'll use our assumption on *x*, namely that $|x| < \sqrt{\varepsilon}$. Doing all this gives,

$$\begin{aligned}
|x^2| &= |x|^2 && \text{strip exponent out of absolute value bars} \\
&< \left(\sqrt{\varepsilon}\right)^2 && \text{use the assumption that } |x| < \sqrt{\varepsilon} \\
&= \varepsilon && \text{simplify}
\end{aligned}$$

Or, upon taking the middle terms out, if we assume that $0 < |x| < \sqrt{\varepsilon}$ then we will get,

$$|x^2| < \varepsilon$$

and this is exactly what we needed to show.

So, just what have we done? We've shown that if we choose $\varepsilon > 0$ then we can find a $\delta > 0$ so that we have,

$$|x^2 - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 0| < \sqrt{\varepsilon}$$

and according to our definition this means that,

$$\lim_{x \to 0} x^2 = 0$$

These can be a little tricky the first couple times through. Especially when it seems like we've got to do the work twice. In the previous example we did some simplification on the left hand inequality to get our guess for δ and then seemingly went through exactly the same work to then prove that our guess was correct. This is often who these work, although we will see an example here in a bit where things don't work out quite so nicely.

So, having said that let's take a look at a slightly more complicated limit, although this one will still be fairly similar to the first example.

Example 2 Use the definition of the limit to prove the following limit.

$$\lim_{x\to 2} 5x-4=6$$

Solution

We'll start this one out the same way that we did the first one. We won't be putting in quite the same amount of explanation however.

Let's start off by letting $\varepsilon > 0$ be any number then we need to find a number $\delta > 0$ so that the following will be true.

$$\left|(5x-4)-6\right|<\varepsilon \qquad \text{whenever} \qquad 0<\left|x-2\right|<\delta$$

We'll start by simplifying the left inequality in an attempt to get a guess for δ. Doing this gives,

$$\left|(5x-4)-6\right|=\left|5x-10\right|=5\left|x-2\right|<\varepsilon \qquad \Rightarrow \qquad \left|x-2\right|<\frac{\varepsilon}{5}$$

So, as with the first example it looks like if we do enough simplification on the left inequality we get something that looks an awful lot like the right inequality and this leads us to choose $\delta=\frac{\varepsilon}{5}$.

Let's now verify this guess. So, again let $\varepsilon > 0$ be any number and then choose $\delta=\frac{\varepsilon}{5}$. Next, assume that $0<\left|x-2\right|<\delta=\frac{\varepsilon}{5}$ and we get the following,

$$\begin{aligned}
\left|(5x-4)-6\right| &= \left|5x-10\right| && \text{simplify things a little}\\
&= 5\left|x-2\right| && \text{more simplification....}\\
&< 5\left(\frac{\varepsilon}{5}\right) && \text{use the assumption } \delta=\frac{\varepsilon}{5}\\
&= \varepsilon && \text{and some more simplification}
\end{aligned}$$

So, we've shown that

$$\left|(5x-4)-6\right|<\varepsilon \qquad \text{whenever} \qquad 0<\left|x-2\right|<\frac{\varepsilon}{5}$$

and so by our definition we have,

$$\lim_{x\to 2} 5x-4=6$$

Okay, so again the process seems to suggest that we have to essentially redo all our work twice, once to make the guess for δ and then another time to prove our guess. Let's do an example that doesn't work out quite so nicely.

Example 3 Use the definition of the limit to prove the following limit.

$$\lim_{x \to 4} x^2 + x - 11 = 9$$

Solution

So, let's get started. Let $\varepsilon > 0$ be any number then we need to find a number $\delta > 0$ so that the following will be true.

$$\left|\left(x^2 + x - 11\right) - 9\right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left|x - 4\right| < \delta$$

We'll start the guess process in the same manner as the previous two examples.

$$\left|\left(x^2 + x - 11\right) - 9\right| = \left|x^2 + x - 20\right| = \left|\left(x+5\right)\left(x-4\right)\right| = \left|x+5\right|\left|x-4\right| < \varepsilon$$

Okay, we've managed to show that $\left|\left(x^2 + x - 11\right) - 9\right| < \varepsilon$ is equivalent to $\left|x+5\right|\left|x-4\right| < \varepsilon$. However, unlike the previous two examples, we've got an extra term in here that doesn't show up in the right inequality above. If we have any hope of proceeding here we're going need to find some way to get rid of the $\left|x+5\right|$.

To do this let's just note that if, by some chance, we can show that $\left|x+5\right| < K$ for some number K then, we'll have the following,

$$\left|x+5\right|\left|x-4\right| < K\left|x-4\right|$$

If we now assume that what we really want to show is $K\left|x-4\right| < \varepsilon$ instead of $\left|x+5\right|\left|x-4\right| < \varepsilon$ we get the following,

$$\left|x-4\right| < \frac{\varepsilon}{K}$$

This is starting to seem familiar isn't it?

All this work however, is based on the assumption that we can show that $\left|x+5\right| < K$ for some K. Without this assumption we can't do anything so let's see if we can do this.

Let's first remember that we are working on a limit here and let's also remember that limits are only really concerned with what is happening around the point in question, $x = 4$ in this case. So, it is safe to assume that whatever x is, it must be close to $x = 4$. This means we can safely assume that whatever x is, it is within a distance of, say one of $x = 4$. Or in terms of an inequality, we can assume that,

$$|x-4|<1$$

Why choose 1 here? There is no reason other than it's a nice number to work with. We could just have easily chosen 2, or 5, or $\frac{1}{3}$. The only difference our choice will make is on the actual value of *K* that we end up with. You might want to go through this process with another choice of *K* and see if you can do it.

So, let's start with $|x-4|<1$ and get rid of the absolute value bars and this solve the resulting inequality for *x* as follows,

$$-1<x-4<1 \qquad \Rightarrow \qquad 3<x<5$$

If we now add 5 to all parts of this inequality we get,

$$8<x+5<10$$

Now, since $x+5>8>0$ (the positive part is important here) we can say that, provided $|x-4|<1$ we know that $x+5=|x+5|$. Or, if take the double inequality above we have,

$$8<|x+5|<10 \qquad \Rightarrow \qquad |x+5|<10 \qquad \Rightarrow \qquad K=10$$

So, provided $|x-4|<1$ we can see that $|x+5|<10$ which in turn gives us,

$$|x-4|<\frac{\varepsilon}{K}=\frac{\varepsilon}{10}$$

So, to this point we make two assumptions about $|x-4|$ We've assumed that,

$$|x-4|<\frac{\varepsilon}{10} \qquad \text{AND} \qquad |x-4|<1$$

It may not seem like it, but we're now ready to chose a δ. In the previous examples we had only a single assumption and we used that to give us δ. In this case we've got two and they BOTH need to be true. So, we'll let δ be the smaller of the two assumptions, 1 and $\frac{\varepsilon}{10}$.
Mathematically, this is written as,

$$\delta=\min\left\{1,\frac{3}{10}\right\}$$

By doing this we can guarantee that,

$$\delta\le\frac{\varepsilon}{10} \qquad \text{AND} \qquad \delta\le 1$$

Now that we've made our choice for δ we need to verify it. So, $\varepsilon>0$ be any number and then choose $\delta = \min\left\{1, \frac{3}{10}\right\}$. Assume that $0 < |x-4| < \delta = \min\left\{1, \frac{\varepsilon}{10}\right\}$. First, we get that,

$$0 < |x-4| < \delta \le \frac{\varepsilon}{10} \qquad \Rightarrow \qquad |x-4| < \frac{\varepsilon}{10}$$

We also get,

$$0 < |x-4| < \delta \le 1 \qquad \Rightarrow \qquad |x-4| < 1 \qquad \Rightarrow \qquad |x+5| < 10$$

Finally, all we need to do is,

$$\begin{aligned}
\left|\left(x^2+x-11\right)-9\right| &= \left|x^2+x-20\right| && \text{simplify things a little} \\
&= |x+5||x-4| && \text{factor} \\
&< 10|x-4| && \text{use the assumption that } |x+5|<10 \\
&< 10\left(\frac{\varepsilon}{10}\right) && \text{use the assumption that } |x-4| < \frac{\varepsilon}{10} \\
&< \varepsilon && \text{a little final simplification}
\end{aligned}$$

We've now managed to show that,

$$\left|\left(x^2+x-11\right)-9\right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-4| < \min\left\{1, \frac{\varepsilon}{10}\right\}$$

and so by our definition we have,

$$\lim_{x\to 4} x^2+x-11 = 9$$

Okay, that was a lot more work that the first two examples and unfortunately, it wasn't all that difficult of a problem. Well, maybe we should say that in comparison to some of the other limits we could have tried to prove it wasn't all that difficult. When first faced with these kinds of proofs using the precise definition of a limit they can all seem pretty difficult.

Do not feel bad if you don't get this stuff right away. It's very common to not understand this right away and to have to struggle a little to fully start to understand how these kinds of limit definition proofs work.

Next, let's give the precise definitions for the right- and left-handed limits.

Definition 2 For the right-hand limit we say that,

$$\lim_{x\to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$$|f(x)-L| < \varepsilon \qquad \text{whenever} \qquad 0 < x-a < \delta \quad (\text{or } a < x < a+\delta)$$

Definition 3 For the left-hand limit we say that,

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$$\left| f(x) - L \right| < \varepsilon \qquad \text{whenever} \qquad -\delta < x - a < 0 \qquad (\text{or } a - \delta < x < a)$$

Note that with both of these definitions there are two ways to deal with the restriction on *x* and the one in parenthesis is probably the easier to use, although the main one given more closely matches the definition of the normal limit above.

Let's work a quick example of one of these, although as you'll see they work in much the same manner as the normal limit problems do.

Example 4 Use the definition of the limit to prove the following limit.

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Solution

Let $\varepsilon > 0$ be any number then we need to find a number $\delta > 0$ so that the following will be true.

$$\left| \sqrt{x} - 0 \right| < \varepsilon \qquad \text{whenever} \qquad 0 < x - 0 < \delta$$

Or upon a little simplification we need to show,

$$\sqrt{x} < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta$$

As with the previous problems let's start with the left hand inequality and see if we can't use that to get a guess for δ. The only simplification that we really need to do here is to square both sides.

$$\sqrt{x} < \varepsilon \qquad \Rightarrow \qquad x < \varepsilon^2$$

So, it looks like we can chose $\delta = \varepsilon^2$.

Let's verify this. Let $\varepsilon > 0$ be any number and chose $\delta = \varepsilon^2$. Next assume that $0 < x < \varepsilon^2$. This gives,

$$\begin{aligned} \left| \sqrt{x} - 0 \right| &= \sqrt{x} && \text{some quick simplification} \\ &< \sqrt{\varepsilon^2} && \text{use the assumption that } x < \varepsilon^2 \\ &< \varepsilon && \text{one final simplification} \end{aligned}$$

We now shown that,

$$\left| \sqrt{x} - 0 \right| < \varepsilon \qquad \text{whenever} \qquad 0 < x - 0 < \varepsilon^2$$

and so by the definition of the right-hand limit we have,

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Let's now move onto the definition of infinite limits. Here are the two definitions that we need to cover both possibilities, limits that are positive infinity and limits that are negative infinity.

Definition 4 Let *f(x)* be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

$$\lim_{x \to a} f(x) = \infty$$

if for every number $M > 0$ there is some number $\delta > 0$ such that

$$f(x) > M \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

Definition 5 Let *f(x)* be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

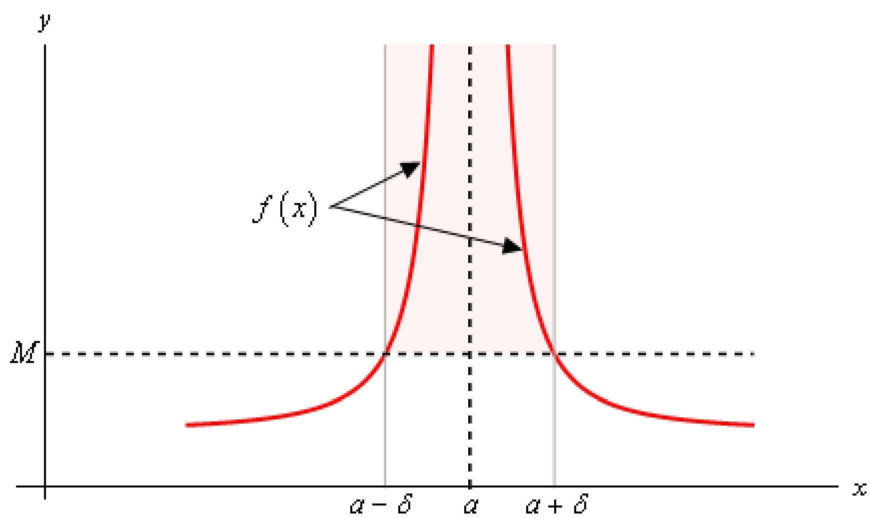
$$\lim_{x \to a} f(x) = -\infty$$

if for every number $N < 0$ there is some number $\delta > 0$ such that

$$f(x) < N \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

In these two definitions note that *M* must be a positive number and that *N* must be a negative number. That's an easy distinction to miss if you aren't paying close attention.
Also note that we could also write down definitions for one-sided limits that are infinity if we wanted to. We'll leave that to you to do if you'd like to.

Here is a quick sketch illustrating Definition 4.



What Definition 4 is telling us is that no matter how large we choose *M* to be we can always find an interval around $x = a$, given by $0 < |x - a| < \delta$ for some number δ, so that as long as we stay within that interval the graph of the function will be above the line $y = M$ as shown in the graph. Also note that we don't need the function to actually exist at $x = a$ in order for the definition to hold. This is also illustrated in the sketch above.

Note as well that the larger *M* is the smaller we're probably going to need to make δ.

To see an illustration of Definition 5 reflect the above graph about the *x*-axis and you'll see a sketch of Definition 5.

Let's work a quick example of one of these to see how these differ from the previous examples.

Example 5 Use the definition of the limit to prove the following limit.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Solution

These work in pretty much the same manner as the previous set of examples do. The main difference is that we're working with an *M* now instead of an ε. So, let's get going.

Let $M > 0$ be any number and we'll need to choose a $\delta > 0$ so that,

$$\frac{1}{x^2} > M \qquad \text{whenever} \qquad 0 < |x - 0| = |x| < \delta$$

As with the all the previous problems we'll start with the left inequality and try to get something in the end that looks like the right inequality. To do this we'll basically solve the left inequality for *x* and we'll need to recall that $\sqrt{x^2} = |x|$. So, here's that work.

$$\frac{1}{x^2} > M \qquad \Rightarrow \qquad x^2 < \frac{1}{M} \qquad \Rightarrow \qquad |x| < \frac{1}{\sqrt{M}}$$

So, it looks like we can chose $\delta = \frac{1}{\sqrt{M}}$. All we need to do now is verify this guess.

Let $M > 0$ be any number, choose $\delta = \frac{1}{\sqrt{M}}$ and assume that $0 < |x| < \frac{1}{\sqrt{M}}$.

In the previous examples we tried to show that our assumptions satisfied the left inequality by working with it directly. However, in this, the function and our assumption on *x* that we've got actually will make this easier to start with the assumption on *x* and show that we can get the left inequality out of that. Note that this is being done this way mostly because of the function that we're working with and not because of the type of limit that we've got.

Doing this work gives,

$$|x| < \frac{1}{\sqrt{M}}$$

$$|x|^2 < \frac{1}{M} \qquad \text{square both sides}$$

$$x^2 < \frac{1}{M} \qquad \text{acknowledge that } |x|^2 = x^2$$

$$\frac{1}{x^2} > M \qquad \text{solve for } x^2$$

So, we've managed to show that,

$$\frac{1}{x^2} > M \qquad \text{whenever} \qquad 0 < |x - 0| < \frac{1}{\sqrt{M}}$$

and so by the definition of the limit we have,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

For our next set of limit definitions let's take a look at the two definitions for limits at infinity. Again, we need one for a limit at plus infinity and another for negative infinity.

Definition 6 Let *f(x)* be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

$$\lim_{x \to \infty} f(x) = L$$

if for every number $\varepsilon > 0$ there is some number $M > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x > M$$

Definition 7 Let *f(x)* be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

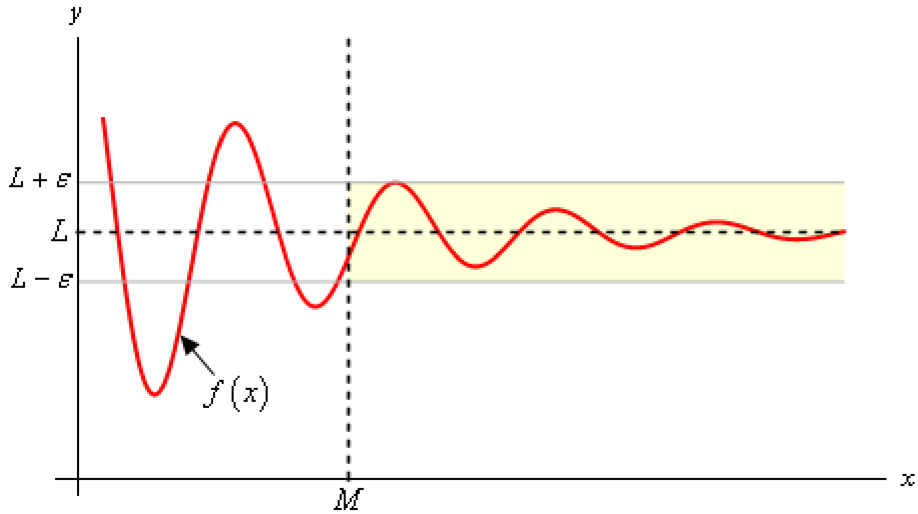
$$\lim_{x \to -\infty} f(x) = L$$

if for every number $\varepsilon > 0$ there is some number $N < 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x < N$$

To see what these definitions are telling us here is a quick sketch illustrating Definition 6. Definition 6 tells us is that no matter how close to *L* we want to get, mathematically this is given by $|f(x) - L| < \varepsilon$ for any chosen ε, we can find another number *M* such that provided we take any *x* bigger than *M*, then the graph of the function for that *x* will be closer to *L* than either $L - \varepsilon$ and $L + \varepsilon$. Or, in other words, the graph will be in the shaded region as shown in the sketch below.

Finally, note that the smaller we make ε the larger we'll probably need to make *M*.



Here's a quick example of one of these limits.

Example 6 Use the definition of the limit to prove the following limit.

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Solution

Let $\varepsilon > 0$ be any number and we'll need to choose a $N < 0$ so that,

$$\left| \frac{1}{x} - 0 \right| = \frac{1}{|x|} < \varepsilon \qquad \text{whenever} \qquad x < N$$

Getting our guess for *N* isn't too bad here.

$$\frac{1}{|x|} < \varepsilon \qquad \Rightarrow \qquad |x| > \frac{1}{\varepsilon}$$

Since we're heading out towards negative infinity it looks like we can choose $N = -\frac{1}{\varepsilon}$. Note that we need the "-" to make sure that *N* is negative (recall that $\varepsilon > 0$).

Let's verify that our guess will work. Let $\varepsilon > 0$ and choose $N = -\frac{1}{\varepsilon}$ and assume that $x < -\frac{1}{\varepsilon}$. As with the previous example the function that we're working with here suggests that it will be easier to start with this assumption and show that we can get the left inequality out of that.

$$x < -\frac{1}{\varepsilon}$$

$$|x| > \left|-\frac{1}{\varepsilon}\right| \qquad \text{take the absolute value}$$

$$|x| > \frac{1}{\varepsilon} \qquad \text{do a little simplification}$$

$$\frac{1}{|x|} < \varepsilon \qquad \text{solve for } |x|$$

$$\left|\frac{1}{x} - 0\right| < \varepsilon \qquad \text{rewrite things a little}$$

Note that when we took the absolute value of both sides we changed both sides from negative numbers to positive numbers and so also had to change the direction of the inequality.

So, we've shown that,

$$\left|\frac{1}{x} - 0\right| = \frac{1}{|x|} < \varepsilon \qquad \text{whenever} \qquad x < -\frac{1}{\varepsilon}$$

and so by the definition of the limit we have,

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

For our final limit definition let's look at limits at infinity that are also infinite in value. There are four possible limits to define here. We'll do one of them and leave the other three to you to write down if you'd like to.

Definition 8 Let *f(x)* be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

$$\lim_{x \to \infty} f(x) = \infty$$

if for every number $N > 0$ there is some number $M > 0$ such that

$$f(x) > N \qquad \text{whenever} \qquad x > M$$

The other three definitions are almost identical. The only differences are the signs of *M* and/or *N* and the corresponding inequality directions.

As a final definition in this section let's recall that we previously said that a function was continuous if,

$$\lim_{x \to a} f(x) = f(a)$$

So, since continuity, as we previously defined it, is defined in terms of a limit we can also now give a more precise definition of continuity. Here it is,

Definition 9 Let *f(x)* be a function defined on an interval that contains $x = a$. Then we say that *f(x)* is continuous at $x = a$ if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$$\left| f(x) - f(a) \right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta$$

This definition is very similar to the first definition in this section and of course that should make some sense since that is exactly the kind of limit that we're doing to show that a function is continuous. The only real difference is that here we need to make sure that the function is actually defined at $x = a$, while we didn't need to worry about that for the first definition since limits don't really care what is happening at the point.

We won't do any examples of proving a function is continuous at a point here mostly because we've already done some examples. Go back and look at the first three examples. In each of these examples the value of the limit was the value of the function evaluated at $x = a$ and so in each of these examples not only did we prove the value of the limit we also managed to prove that each of these functions are continuous at the point in question.

Derivatives

Introduction

In this chapter we will start looking at the next major topic in a calculus class. We will be looking at derivatives in this chapter (as well as the next chapter). This chapter is devoted almost exclusively to finding derivatives. We will be looking at one application of them in this chapter. We will be leaving most of the applications of derivatives to the next chapter.

Here is a listing of the topics covered in this chapter.

The Definition of the Derivative – In this section we will be looking at the definition of the derivative.

Interpretation of the Derivative – Here we will take a quick look at some interpretations of the derivative.

Differentiation Formulas – Here we will start introducing some of the differentiation formulas used in a calculus course.

Product and Quotient Rule – In this section we will took at differentiating products and quotients of functions.

Derivatives of Trig Functions – We'll give the derivatives of the trig functions in this section.

Derivatives of Exponential and Logarithm Functions – In this section we will get the derivatives of the exponential and logarithm functions.

Derivatives of Inverse Trig Functions – Here we will look at the derivatives of inverse trig functions.

Derivatives of Hyperbolic Functions – Here we will look at the derivatives of hyperbolic functions.

Chain Rule – The Chain Rule is one of the more important differentiation rules and will allow us to differentiate a wider variety of functions. In this section we will take a look at it.

Implicit Differentiation – In this section we will be looking at implicit differentiation. Without this we won't be able to work some of the applications of derivatives.

Related Rates – In this section we will look at the lone application to derivatives in this chapter. This topic is here rather than the next chapter because it will help to cement in our minds one of the more important concepts about derivatives and because it requires implicit differentiation.

Higher Order Derivatives – Here we will introduce the idea of higher order derivatives.

Logarithmic Differentiation – The topic of logarithmic differentiation is not always presented in a standard calculus course. It is presented here for those how are interested in seeing how it is done and the types of functions on which it can be used.

The Definition of the Derivative

In the first section of the last chapter we saw that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x = a$ all required us to compute the following limit.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{3}$$

This is such an important limit and it arises in so many places that we give it a name. We call it a **derivative**. Here is the official definition of the derivative.

Definition

The **derivative of** $f(x)$ **with respect to *x*** is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{4}$$

Note that we replaced all the *a*'s in (1) with *x*'s to acknowledge the fact that the derivative is really a function as well. We often "read" $f'(x)$ as "*f* prime of *x*".

Let's compute a couple of derivatives using the definition.

Example 1 Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution

So, all we really need to do is to plug this function into the definition of the derivative, (1), and do some algebra. While, admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra so don't get excited about the fact that we're now computing derivatives.

First plug the function into the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \end{aligned}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous chapter that we can't just plug in $h = 0$ since this will give us a division by zero error. So we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h} \end{aligned}$$

Notice that every term in the numerator that didn't have an *h* in it canceled out and we can now factor an *h* out of the numerator which will cancel against the *h* in the denominator. After that we can compute the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{h(4x + 2h - 16)}{h} \\ &= \lim_{h\to 0} 4x + 2h - 16 \\ &= 4x - 16 \end{aligned}$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Example 2 Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution

This one is going to be a little messier as far as the algebra goes. However, outside of that it will work in exactly the same manner as the previous examples. First, we plug the function into the definition of the derivative,

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right) \end{aligned}$$

Note that we changed all the letters in the definition to match up with the given function. Also note that we wrote the fraction a much more compact manner to help us with the work.

As with the first problem we can't just plug in $h = 0$. So we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

$$
\begin{aligned}
g'(t) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{(t+h)(t+1)-t(t+h+1)}{(t+h+1)(t+1)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{t^2+t+th+h-\left(t^2+th+t\right)}{(t+h+1)(t+1)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right)
\end{aligned}
$$

Before finishing this let's note a couple of things. First, we didn't multiply out the denominator. Multiplying out the denominator will just overly complicate things so let's keep it simple. Next, as with the first example, after the simplification we only have terms with *h*'s in them left in the numerator and so we can now cancel an *h* out.

So, upon canceling the *h* we can evaluate the limit and get the derivative.

$$
\begin{aligned}
g'(t) &= \lim_{h\to 0}\frac{1}{(t+h+1)(t+1)}\\
&= \frac{1}{(t+1)(t+1)}\\
&= \frac{1}{(t+1)^2}
\end{aligned}
$$

The derivative is then,

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3 Find the derivative of the following function using the derivative.

$$R(z) = \sqrt{5z-8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$
\begin{aligned}
R'(z) &= \lim_{h\to 0}\frac{R(z+h)-R(z)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{5(z+h)-8}-\sqrt{5z-8}}{h}
\end{aligned}
$$

In this problem we're going to have to rationalize the numerator. You do remember rationalization from an Algebra class right? In an Algebra class you probably only rationalized the denominator, but you can also rationalize numerators. Remember that in rationalizing the numerator (in this case) we multiply both the numerator and denominator by the numerator except we change the sign between the two terms. Here's the rationalizing work for this problem,

$$R'(z) = \lim_{h \to 0} \frac{\left(\sqrt{5(z+h)-8} - \sqrt{5z-8}\right)}{h} \frac{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$
$$= \lim_{h \to 0} \frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$
$$= \lim_{h \to 0} \frac{5h}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

Again, after the simplification we have only *h*'s left in the numerator. So, cancel the *h* and evaluate the limit.

$$R'(z) = \lim_{h \to 0} \frac{5}{\sqrt{5(z+h)-8} + \sqrt{5z-8}}$$
$$= \frac{5}{\sqrt{5z-8} + \sqrt{5z-8}}$$
$$= \frac{5}{2\sqrt{5z-8}}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Let's work one more example. This one will be a little different, but it's got a point that needs to be made.

Example 4 Determine $f'(0)$ for $f(x) = |x|$

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$
$$= \lim_{h \to 0} \frac{|h|}{h}$$