Derivatives

Introduction

In this chapter we will start looking at the next major topic in a calculus class. We will be looking at derivatives in this chapter (as well as the next chapter). This chapter is devoted almost exclusively to finding derivatives. We will be looking at one application of them in this chapter. We will be leaving most of the applications of derivatives to the next chapter.

Here is a listing of the topics covered in this chapter.

The Definition of the Derivative – In this section we will be looking at the definition of the derivative.

Interpretation of the Derivative – Here we will take a quick look at some interpretations of the derivative.

Differentiation Formulas – Here we will start introducing some of the differentiation formulas used in a calculus course.

Product and Quotient Rule – In this section we will took at differentiating products and quotients of functions.

Derivatives of Trig Functions – We'll give the derivatives of the trig functions in this section.

Derivatives of Exponential and Logarithm Functions – In this section we will get the derivatives of the exponential and logarithm functions.

Derivatives of Inverse Trig Functions – Here we will look at the derivatives of inverse trig functions.

Derivatives of Hyperbolic Functions – Here we will look at the derivatives of hyperbolic functions.

Chain Rule – The Chain Rule is one of the more important differentiation rules and will allow us to differentiate a wider variety of functions. In this section we will take a look at it.

Implicit Differentiation – In this section we will be looking at implicit differentiation. Without this we won't be able to work some of the applications of derivatives.

Related Rates – In this section we will look at the lone application to derivatives in this chapter. This topic is here rather than the next chapter because it will help to cement in our minds one of the more important concepts about derivatives and because it requires implicit differentiation.

Higher Order Derivatives – Here we will introduce the idea of higher order derivatives.

Logarithmic Differentiation – The topic of logarithmic differentiation is not always presented in a standard calculus course. It is presented here for those how are interested in seeing how it is done and the types of functions on which it can be used.

The Definition of the Derivative

In the first section of the last chapter we saw that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x = a$ all required us to compute the following limit.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (3)$$

This is such an important limit and it arises in so many places that we give it a name. We call it a **derivative**. Here is the official definition of the derivative.

Definition

The **derivative of** $f(x)$ **with respect to** ***x*** is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (4)$$

Note that we replaced all the *a*'s in (1) with *x*'s to acknowledge the fact that the derivative is really a function as well. We often "read" $f'(x)$ as "*f* prime of *x*".

Let's compute a couple of derivatives using the definition.

Example 1 Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution

So, all we really need to do is to plug this function into the definition of the derivative, (1), and do some algebra. While, admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra so don't get excited about the fact that we're now computing derivatives.

First plug the function into the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \end{aligned}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous chapter that we can't just plug in $h = 0$ since this will give us a division by zero error. So we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h} \end{aligned}$$

Notice that every term in the numerator that didn't have an *h* in it canceled out and we can now factor an *h* out of the numerator which will cancel against the *h* in the denominator. After that we can compute the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{h(4x + 2h - 16)}{h} \\ &= \lim_{h\to 0} 4x + 2h - 16 \\ &= 4x - 16 \end{aligned}$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Example 2 Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution

This one is going to be a little messier as far as the algebra goes. However, outside of that it will work in exactly the same manner as the previous examples. First, we plug the function into the definition of the derivative,

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right) \end{aligned}$$

Note that we changed all the letters in the definition to match up with the given function. Also note that we wrote the fraction a much more compact manner to help us with the work.

As with the first problem we can't just plug in $h = 0$. So we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

$$
\begin{aligned}
g'(t) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{(t+h)(t+1)-t(t+h+1)}{(t+h+1)(t+1)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{t^2+t+th+h-\left(t^2+th+t\right)}{(t+h+1)(t+1)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right)
\end{aligned}
$$

Before finishing this let's note a couple of things. First, we didn't multiply out the denominator. Multiplying out the denominator will just overly complicate things so let's keep it simple. Next, as with the first example, after the simplification we only have terms with *h*'s in them left in the numerator and so we can now cancel an *h* out.

So, upon canceling the *h* we can evaluate the limit and get the derivative.

$$
\begin{aligned}
g'(t) &= \lim_{h\to 0}\frac{1}{(t+h+1)(t+1)}\\
&= \frac{1}{(t+1)(t+1)}\\
&= \frac{1}{(t+1)^2}
\end{aligned}
$$

The derivative is then,

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3 Find the derivative of the following function using the derivative.

$$R(z) = \sqrt{5z-8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$
\begin{aligned}
R'(z) &= \lim_{h\to 0}\frac{R(z+h)-R(z)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{5(z+h)-8}-\sqrt{5z-8}}{h}
\end{aligned}
$$

In this problem we're going to have to rationalize the numerator. You do remember rationalization from an Algebra class right? In an Algebra class you probably only rationalized the denominator, but you can also rationalize numerators. Remember that in rationalizing the numerator (in this case) we multiply both the numerator and denominator by the numerator except we change the sign between the two terms. Here's the rationalizing work for this problem,

$$R'(z)=\lim_{h\to 0}\frac{\left(\sqrt{5(z+h)-8}-\sqrt{5z-8}\right)}{h}\frac{\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}$$
$$=\lim_{h\to 0}\frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}$$
$$=\lim_{h\to 0}\frac{5h}{h\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}$$

Again, after the simplification we have only *h*'s left in the numerator. So, cancel the *h* and evaluate the limit.

$$R'(z)=\lim_{h\to 0}\frac{5}{\sqrt{5(z+h)-8}+\sqrt{5z-8}}$$
$$=\frac{5}{\sqrt{5z-8}+\sqrt{5z-8}}$$
$$=\frac{5}{2\sqrt{5z-8}}$$

And so we get a derivative of,

$$R'(z)=\frac{5}{2\sqrt{5z-8}}$$

Let's work one more example. This one will be a little different, but it's got a point that needs to be made.

Example 4 Determine $f'(0)$ for $f(x)=|x|$

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{|0+h|-|0|}{h}$$
$$=\lim_{h\to 0}\frac{|h|}{h}$$

We saw a situation like this back when we were looking at limits at infinity. As in that section we can't just cancel the h's. We will have to look at the two one sided limits and recall that

$$|h| = \begin{cases} h & \text{if } h \ge 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\begin{aligned} \lim_{h\to 0^-} \frac{|h|}{h} &= \lim_{h\to 0^-} \frac{-h}{h} && \text{because } h<0 \text{ in a left-hand limit.} \\ &= \lim_{h\to 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^+} \frac{|h|}{h} &= \lim_{h\to 0^+} \frac{h}{h} && \text{because } h>0 \text{ in a right-hand limit.} \\ &= \lim_{h\to 0^+} 1 \\ &= 1 \end{aligned}$$

The two one-sided limits are different and so

$$\lim_{h\to 0} \frac{|h|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

In this example we have finally seen a function for which the derivative doesn't exist at a point. This is a fact of life that we've got to be aware of. Derivatives will not always exist. Note as well that this doesn't say anything about whether or not the derivative exists anywhere else. In fact, the derivative of the absolute value function exists at every point except the one we just looked at, $x = 0$.

The preceding discussion leads to the following definition.

Definition

A function $f(x)$ is called **differentiable** at $x = a$ if $f'(x)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

The next theorem shows us a very nice relationship between functions that are continuous and those that are differentiable.

Theorem

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this theorem.

Note that this theorem does not work in reverse. Consider $f(x) = |x|$ and take a look at,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

So, $f(x) = |x|$ is continuous at $x = 0$ but we've just shown above in Example 4 that $f(x) = |x|$ is not differentiable at $x = 0$.

Alternate Notation

Next we need to discuss some alternate notation for the derivative. The typical derivative notation is the "prime" notation. However, there is another notation that is used on occasion so let's cover that.

Given a function $y = f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to *x*.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}(y)$$

Because we also need to evaluate derivatives on occasion we also need a notation for evaluating derivatives when using the fractional notation. So if we want to evaluate the derivative at *x*=*a* all of the following are equivalent.

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

Note as well that on occasion we will drop the *(x)* part on the function to simplify the notation somewhat. In these cases the following are equivalent.

$$f'(x) = f'$$

As a final note in this section we'll acknowledge that computing most derivatives directly from the definition is a fairly complex (and sometimes painful) process filled with opportunities to make mistakes. In a couple of section we'll start developing formulas and/or properties that will help us to take the derivative of many of the common functions so we won't need to resort to the definition of the derivative too often.

This does not mean however that it isn't important to know the definition of the derivative! It is an important definition that we should always know and keep in the back of our minds. It is just something that we're not going to be working with all that much.

Interpretations of the Derivative

Before moving on to the section where we learn how to compute derivatives by avoiding the limits we were evaluating in the previous section we need to take a quick look at some of the interpretations of the derivative. All of these interpretations arise from recalling how our definition of the derivative came about. The definition came about by noticing that all the problems that we worked in the first section in the chapter on limits required us to evaluate the same limit.

Rate of Change

The first interpretation of a derivative is rate of change. This was not the first problem that we looked at in the limit chapter, but it is the most important interpretation of the derivative. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

Example 1 Suppose that the amount of water in a holding tank at *t* minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute? [Solution]

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes? [Solution]

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes? [Solution]

(d) Is the volume of water in the tank ever not changing? If so, when? [Solution]

Solution

In the solution to this example we will use both notations for the derivative just to get you familiar with the different notations.

We are going to need the rate of change of the volume to answer these questions. This means that we will need the derivative of this function since that will give us a formula for the rate of change at any time *t*. Now, notice that the function giving the volume of water in the tank is the same function that we saw in Example 1 in the last section except the letters have changed. The change in letters between the function in this example versus the function in the example from the last section won't affect the work and so we can just use the answer from that example with an appropriate change in letters.

The derivative is.

$$V'(t) = 4t - 16 \qquad \text{OR} \qquad \frac{dV}{dt} = 4t - 16$$

Recall from our work in the first limits section that we determined that if the rate of change was positive then the quantity was increasing and if the rate of change was negative then the quantity was decreasing.

We can now work the problem.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

In this case all that we need is the rate of change of the volume at $t = 1$ or,

$$V'(1) = -12 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=1} = -12$$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

[Return to Problems]

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

Again, we will need the rate of change at $t = 5$.

$$V'(5) = 4 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at $t = 5$.

[Return to Problems]

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

To answer this question all that we look at is the size of the rate of change and we don't worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change. So, in this case the volume is changing faster at $t = 1$ than at $t = 5$.

[Return to Problems]

(d) Is the volume of water in the tank ever not changing? If so, when?

The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$V'(t) = 0 \qquad \text{OR} \qquad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \qquad \Rightarrow \qquad t = 4$$

So at $t = 4$ the volume isn't changing. Note that all this is saying is that for a brief instant the volume isn't changing. It doesn't say that at this point the volume will quit changing permanently.

If we go back to our answers from parts (a) and (b) we can get an idea about what is going on. At $t=1$ the volume is decreasing and at $t=5$ the volume is increasing. So at some point in time the volume needs to switch from decreasing to increasing. That time is $t=4$.

This is the time in which the volume goes from decreasing to increasing and so for the briefest instant in time the volume will quit changing as it changes from decreasing to increasing.

[Return to Problems]

Note that one of the more common mistakes that students make in these kinds of problems is to try and determine increasing/decreasing from the function values rather than the derivatives. In this case if we took the function values at $t=0$, $t=1$ and $t=5$ we would get,

$$V(0)=35 \qquad V(1)=21 \qquad V(5)=5$$

Clearly as we go from $t=0$ to $t=1$ the volume has decreased. This might lead us to decide that AT $t=1$ the volume is decreasing. However, we just can't say that. All we can say is that between $t=0$ and $t=1$ the volume has decreased at some point in time. The only way to know what is happening right at $t=1$ is to compute $V'(1)$ and look at its sign to determine increasing/decreasing. In this case $V'(1)$ is negative and so the volume really is decreasing at $t=1$.

Now, if we'd plugged into the function rather than the derivative we would have been gotten the correct answer for $t=1$ even though our reasoning would have been wrong. It's important to not let this give you the idea that this will always be the case. It just happened to work out in the case of $t=1$.

To see that this won't always work let's now look at $t=5$. If we plug $t=1$ and $t=5$ into the volume we can see that again as we go from $t=1$ to $t=5$ the volume has decreases. Again, however all this says is that the volume HAS decreased somewhere between $t=1$ and $t=5$. It does NOT say that the volume is decreasing at $t=5$. The only way to know what is going on right at $t=5$ is to compute $V'(5)$ and in this case $V'(5)$ is positive and so the volume is actually increasing at $t=5$.

So, be careful. When asked to determine if a function is increasing or decreasing at a point make sure and look at the derivative. It is the only sure way to get the correct answer. We are not looking to determine is the function has increased/decreased by the time we reach a particular point. We are looking to determine if the function is increasing/decreasing at that point in question.

Slope of Tangent Line

This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x = a$ is $f'(a)$. The tangent line then is given by,

$$y = f(a) + f'(a)(x - a)$$

Example 2 Find the tangent line to the following function at $z = 3$.

$$R(z) = \sqrt{5z - 8}$$

Solution

We first need the derivative of the function and we found that in Example 3 in the last section. The derivative is,

$$R'(z) = \frac{5}{2\sqrt{5z - 8}}$$

Now all that we need is the function value and derivative (for the slope) at $z = 3$.

$$R(3) = \sqrt{7} \qquad\qquad m = R'(3) = \frac{5}{2\sqrt{7}}$$

The tangent line is then,

$$y = \sqrt{7} + \frac{5}{2\sqrt{7}}(z - 3)$$

Velocity

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after *t* units of time the velocity of the object at $t = a$ is given by $f'(a)$.

Example 3 Suppose that the position of an object after *t* hours is given by,

$$g(t) = \frac{t}{t + 1}$$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at $t = 10$ hours? [Solution]
(b) Does the object ever stop moving? [Solution]

Solution

Once again we need the derivative and we found that in Example 2 in the last section. The derivative is,

$$g'(t) = \frac{1}{(t + 1)^2}$$

(a) Is the object moving to the right or the left at $t = 10$ hours?

To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at $t = 10$.

$$g'(10) = \frac{1}{121}$$

So the velocity at $t = 10$ is positive and so the object is moving to the right at $t = 10$.

[Return to Problems]

(b) Does the object ever stop moving?
The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can't be zero.

Therefore, the velocity will never stop moving.

In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

[Return to Problems]

We've seen three major interpretations of the derivative here. You will need to remember these, especially the rate of change, as they will show up continually throughout this course.

Differentiation Formulas

In the first section of this chapter we saw the definition of the derivative and we computed a couple of derivatives using the definition. As we saw in those examples there was a fair amount of work involved in computing the limits and the functions that we worked with were not terribly complicated.

For more complex functions using the definition of the derivative would be an almost impossible task. Luckily for us we won't have to use the definition terribly often. We will have to use it on occasion, however we have a large collection of formulas and properties that we can use to simplify our life considerably and will allow us to avoid using the definition whenever possible.

We will introduce most of these formulas over the course of the next several sections. We will start in this section with some of the basic properties and formulas. We will give the properties and formulas in this section in both "prime" notation and "fraction" notation.

Properties

1) $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$ **OR** $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$

In other words, to differentiate a sum or difference all we need to do is differentiate the individual terms and then put them back together with the appropriate signs. Note as well that this property is not limited to two functions.

See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this property. It's a very simple proof using the definition of the derivative.

2) $\left(cf(x)\right)' = cf'(x)$ **OR** $\frac{d}{dx}\left(cf(x)\right) = c\frac{df}{dx}$, c is any number

In other words, we can "factor" a multiplicative constant out of a derivative if we need to. See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this property.

Note that we have not included formulas for the derivative of products or quotients of two functions here. The derivative of a product or quotient of two functions is not the product or quotient of the derivatives of the individual pieces. We will take a look at these in the next section.

Next, let's take a quick look at a couple of basic "computation" formulas that will allow us to actually compute some derivatives.