Differentiation Formulas

In the first section of this chapter we saw the definition of the derivative and we computed a couple of derivatives using the definition. As we saw in those examples there was a fair amount of work involved in computing the limits and the functions that we worked with were not terribly complicated.

For more complex functions using the definition of the derivative would be an almost impossible task. Luckily for us we won't have to use the definition terribly often. We will have to use it on occasion, however we have a large collection of formulas and properties that we can use to simplify our life considerably and will allow us to avoid using the definition whenever possible.

We will introduce most of these formulas over the course of the next several sections. We will start in this section with some of the basic properties and formulas. We will give the properties and formulas in this section in both "prime" notation and "fraction" notation.

Properties

1) $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$ **OR** $\dfrac{d}{dx}\left(f(x) \pm g(x)\right) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$

In other words, to differentiate a sum or difference all we need to do is differentiate the individual terms and then put them back together with the appropriate signs. Note as well that this property is not limited to two functions.

See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this property. It's a very simple proof using the definition of the derivative.

2) $\left(cf(x)\right)' = cf'(x)$ **OR** $\dfrac{d}{dx}\left(cf(x)\right) = c\dfrac{df}{dx}$, c is any number

In other words, we can "factor" a multiplicative constant out of a derivative if we need to. See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this property.

Note that we have not included formulas for the derivative of products or quotients of two functions here. The derivative of a product or quotient of two functions is not the product or quotient of the derivatives of the individual pieces. We will take a look at these in the next section.

Next, let's take a quick look at a couple of basic "computation" formulas that will allow us to actually compute some derivatives.

Formulas

1) If $f(x) = c$ then $f'(x) = 0$ **OR** $\frac{d}{dx}(c) = 0$

The derivative of a constant is zero. See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this formula.

2) If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ **OR** $\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number.

This formula is sometimes called the **power rule**. All we are doing here is bringing the original exponent down in front and multiplying and then subtracting one from the original exponent.

Note as well that in order to use this formula n must be a number, it can't be a variable. Also note that the base, the x, must be a variable, it can't be a number. It will be tempting in some later sections to misuse the Power Rule when we run in some functions where the exponent isn't a number and/or the base isn't a variable.

See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this formula. There are actually three different proofs in this section. The first two restrict the formula to n being an integer because at this point that is all that we can do at this point. The third proof is for the general rule, but does suppose that you've read most of this chapter.

These are the only properties and formulas that we'll give in this section. Let's do compute some derivatives using these properties.

Example 1 Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$ [Solution]

(b) $g(t) = 2t^6 + 7t^{-6}$ [Solution]

(c) $y = 8z^3 - \frac{1}{3z^5} + z - 23$ [Solution]

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$ [Solution]

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$ [Solution]

Solution

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

In this case we have the sum and difference of four terms and so we will differentiate each of the terms using the first property from above and then put them back together with the proper sign. Also, for each term with a multiplicative constant remember that all we need to do is "factor" the constant out (using the second property) and then do the derivative.

$$f'(x) = 15(100)x^{99} - 3(12)x^{11} + 5(1)x^{0} - 0$$
$$= 1500x^{99} - 36x^{11} + 5$$

Notice that in the third term the exponent was a one and so upon subtracting 1 from the original exponent we get a new exponent of zero. Now recall that $x^0 = 1$. Don't forget to do any basic arithmetic that needs to be done such as any multiplication and/or division in the coefficients.

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(b) $g(t) = 2t^6 + 7t^{-6}$

The point of this problem is to make sure that you deal with negative exponents correctly. Here is the derivative.

$$g'(t) = 2(6)t^5 + 7(-6)t^{-7}$$
$$= 12t^5 - 42t^{-7}$$

Make sure that you correctly deal with the exponents in these cases, especially the negative exponents. It is an easy mistake to "go the other way" when subtracting one off from a negative exponent and get $-6t^{-5}$ instead of the correct $-6t^{-7}$.

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(c) $y = 8z^3 - \dfrac{1}{3z^5} + z - 23$

Now in this function the second term is not correctly set up for us to use the power rule. The power rule requires that the term be a variable to a power only and the term must be in the numerator. So, prior to differentiating we first need to rewrite the second term into a form that we can deal with.

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

Note that we left the 3 in the denominator and only moved the variable up to the numerator. Remember that the only thing that gets an exponent is the term that is immediately to the left of the exponent. If we'd wanted the three to come up as well we'd have written,

$$\frac{1}{(3z)^5}$$

so be careful with this! It's a very common mistake to bring the 3 up into the numerator as well at this stage.

Now that we've gotten the function rewritten into a proper form that allows us to use the Power Rule we can differentiate the function. Here is the derivative for this part.

$$y' = 24z^2 + \frac{5}{3}z^{-6} + 1$$

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(d) $T\left(x\right)=\sqrt{x}+9\sqrt[3]{x^7}-\dfrac{2}{\sqrt[5]{x^2}}$

All of the terms in this function have roots in them. In order to use the power rule we need to first convert all the roots to fractional exponents. Again, remember that the Power Rule requires us to have a variable to a number and that it must be in the numerator of the term. Here is the function written in "proper" form.

$$\begin{aligned}T\left(x\right)&=x^{\frac{1}{2}}+9\left(x^7\right)^{\frac{1}{3}}-\frac{2}{\left(x^2\right)^{\frac{1}{5}}}\\&=x^{\frac{1}{2}}+9x^{\frac{7}{3}}-\frac{2}{x^{\frac{2}{5}}}\\&=x^{\frac{1}{2}}+9x^{\frac{7}{3}}-2x^{-\frac{2}{5}}\end{aligned}$$

In the last two terms we combined the exponents. You should always do this with this kind of term. In a later section we will learn of a technique that would allow us to differentiate this term without combining exponents, however it will take significantly more work to do. Also don't forget to move the term in the denominator of the third term up to the numerator. We can now differentiate the function.

$$\begin{aligned}T'\left(x\right)&=\frac{1}{2}x^{-\frac{1}{2}}+9\left(\frac{7}{3}\right)x^{\frac{4}{3}}-2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}}\\&=\frac{1}{2}x^{-\frac{1}{2}}+\frac{63}{3}x^{\frac{4}{3}}+\frac{4}{5}x^{-\frac{7}{5}}\end{aligned}$$

Make sure that you can deal with fractional exponents. You will see a lot of them in this class.

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(e) $h\left(x\right)=x^{\pi}-x^{\sqrt{2}}$

In all of the previous examples the exponents have been nice integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number so don't get excited about problems like this one. They work exactly the same.

$$h'\left(x\right)=\pi x^{\pi-1}-\sqrt{2}x^{\sqrt{2}-1}$$

The answer is a little messy and we won't reduce the exponents down to decimals. However, this problem is not terribly difficult it just looks that way initially.

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There is a general rule about derivatives in this class that you will need to get into the habit of using. When you see radicals you should always first convert the radical to a fractional exponent and then simplify exponents as much as possible. Following this rule will save you a lot of grief in the future.

Back when we first put down the properties we noted that we hadn't included a property for products and quotients. That doesn't mean that we can't differentiate any product or quotient at this point. There are some that we can do.

Example 2 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$ [Solution]

(b) $h\left(t\right) = \dfrac{2t^5 + t^2 - 5}{t^2}$ [Solution]

Solution

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

In this function we can't just differentiate the first term, differentiate the second term and then multiply the two back together. That just won't work. We will discuss this in detail in the next section so if you're not sure you believe that hold on for a bit and we'll be looking at that soon as well as showing you an example of what it won't work.

It is still possible to do this derivative however. All that we need to do is convert the radical to fractional exponents (as we should anyway) and then multiply this through the parenthesis.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

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(b) $h\left(t\right) = \dfrac{2t^5 + t^2 - 5}{t^2}$

As with the first part we can't just differentiate the numerator and the denominator and the put it back together as a fraction. Again, if you're not sure you believe this hold on until the next section and we'll take a more detailed look at this.

We can simplify this rational expression however as follows.

$$h\left(t\right) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'\left(t\right) = 6t^2 + 10t^{-3}$$

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So, as we saw in this example there are a few products and quotients that we can differentiate. If we can first do some simplification the functions will sometimes simplify into a form that can be differentiated using the properties and formulas in this section.

Before moving on to the next section let's work a couple of examples to remind us once again of some of the interpretations of the derivative.

Example 3 Is $f(x) = 2x^3 + \frac{300}{x^3} + 4$ increasing, decreasing or not changing at $x = -2$?

Solution

We know that the rate of change of a function is given by the functions derivative so all we need to do is it rewrite the function (to deal with the second term) and then take the derivative.

$$f(x) = 2x^3 + 300x^{-3} + 4 \qquad \Rightarrow \qquad f'(x) = 6x^2 - 900x^{-4} = 6x^2 - \frac{900}{x^4}$$

Note that we rewrote the last term in the derivative back as a fraction. This is not something we've done to this point and is only being done here to help with the evaluation in the next step. It's often easier to do the evaluation with positive exponents.

So, upon evaluating the derivative we get

$$f'(-2) = 6(4) - \frac{900}{32} = -\frac{129}{4} = -32.25$$

So, at $x = -2$ the derivative is negative and so the function is decreasing at $x = -2$.

Example 4 Find the equation of the tangent line to $f(x) = 4x - 8\sqrt{x}$ at $x = 16$.

Solution

We know that the equation of a tangent line is given by,

$$y = f(a) + f'(a)(x - a)$$

So, we will need the derivative of the function (don't forget to get rid of the radical).

$$f(x) = 4x - 8x^{\frac{1}{2}} \qquad \Rightarrow \qquad f'(x) = 4 - 4x^{-\frac{1}{2}} = 4 - \frac{4}{x^{\frac{1}{2}}}$$

Again, notice that we eliminated the negative exponent in the derivative solely for the sake of the evaluation. All we need to do then is evaluate the function and the derivative at the point in question, $x = 16$.

$$f(16) = 64 - 8(4) = 32 \qquad\qquad f'(x) = 4 - \frac{4}{4} = 3$$

The tangent line is then,

$$y = 32 + 3(x - 16) = 3x - 16$$

Example 5 The position of an object at any time *t* (in hours) is given by,

$$s(t) = 2t^3 - 21t^2 + 60t - 10$$

Determine when the object is moving to the right and when the object is moving to the left.

Solution

The only way that we'll know for sure which direction the object is moving is to have the velocity in hand. Recall that if the velocity is positive the object is moving off to the right and if the velocity is negative then the object is moving to the left.

So, we need the derivative since the derivative is the velocity of the object. The derivative is,

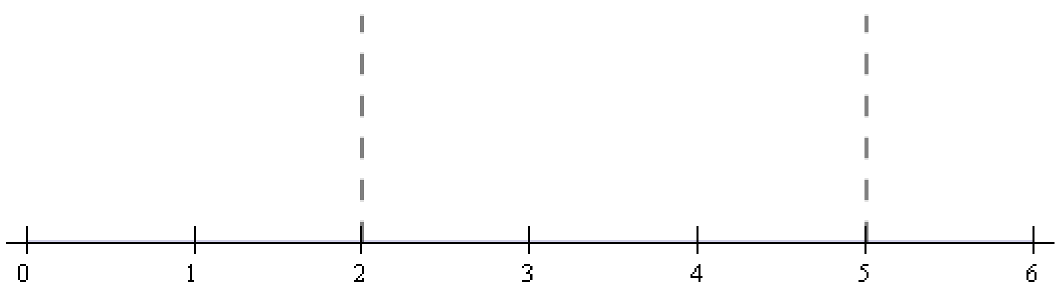
$$s'(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10) = 6(t-2)(t-5)$$

The reason for factoring the derivative will be apparent shortly.

Now, we need to determine where the derivative is positive and where the derivative is negative. There are several ways to do this. The method that I tend to prefer is the following.

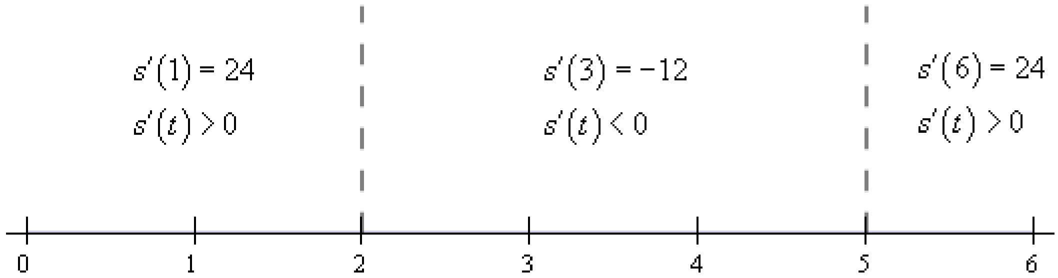
Since polynomials are continuous we know from the Intermediate Value Theorem that if the polynomial ever changes sign then it must have first gone through zero. So, if we knew where the derivative was zero we would know the only points where the derivative *might* change sign.

We can see from the factored form of the derivative that the derivative will be zero at $t = 2$ and $t = 5$. Let's graph these points on a number line.



Now, we can see that these two points divide the number line into three distinct regions. In reach of these regions we **know** that the derivative will be the same sign. Recall the derivative can only change sign at the two points that are used to divide the number line up into the regions.

Therefore, all that we need to do is to check the derivative at a test point in each region and the derivative in that region will have the same sign as the test point. Here is the number line with the test points and results shown.



Here are the intervals in which the derivative is positive and negative.

$$\text{positive}: \quad -\infty < t < 2 \quad \& \quad 5 < t < \infty$$
$$\text{negative}: \quad 2 < t < 5$$

We included negative *t*'s here because we could even though they may not make much sense for this problem. Once we know this we also can answer the question. The object is moving to the right and left in the following intervals.

$$\text{moving to the right}: \quad -\infty < t < 2 \quad \& \quad 5 < t < \infty$$
$$\text{moving to the left}: \quad 2 < t < 5$$

Make sure that you can do the kind of work that we just did in this example. You will be asked numerous times over the course of the next two chapters to determine where functions are positive and/or negative. If you need some review or want to practice these kinds of problems you should check out the Solving Inequalities section of my Algebra/Trig Review.

Product and Quotient Rule

In the previous section we noted that we had to be careful when differentiating products or quotients. It's now time to look at products and quotients and see why.

First let's take a look at why we have to be careful with products and quotients. Suppose that we have the two functions $f\left(x\right)=x^3$ and $g\left(x\right)=x^6$. Let's start by computing the derivative of the product of these two functions. This is easy enough to do directly.

$$\left(f\,g\right)'=\left(x^3x^6\right)'=\left(x^9\right)'=9x^8$$

Remember that on occasion we will drop the *(x)* part on the functions to simplify notation somewhat. We've done that in the work above.

Now, let's try the following.

$$f'\left(x\right)g'\left(x\right)=\left(3x^2\right)\left(6x^5\right)=18x^7$$

So, we can very quickly see that.

$$\left(f\,g\right)'\neq f'\,g'$$

In other words, the derivative of a product is not the product of the derivatives.

Using the same functions we can do the same thing for quotients.

$$\left(\frac{f}{g}\right)'=\left(\frac{x^3}{x^6}\right)'=\left(\frac{1}{x^3}\right)'=\left(x^{-3}\right)'=-3x^{-4}=-\frac{3}{x^4}$$

$$\frac{f'\left(x\right)}{g'\left(x\right)}=\frac{3x^2}{6x^5}=\frac{1}{2x^3}$$

So, again we can see that,

$$\left(\frac{f}{g}\right)'\neq\frac{f'}{g'}$$

To differentiate products and quotients we have the **Product Rule** and the **Quotient Rule**.

Product Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$\left(f\,g\right)'=f'\,g+f\,g'$$

The proof of the Product Rule is shown in the Proof of Various Derivative Formulas section of the Extras chapter.

Quotient Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$$

Note that the numerator of the quotient rule is very similar to the product rule so be careful to not mix the two up!

The proof of the Product Rule is shown in the Proof of Various Derivative Formulas section of the Extras chapter.

Let's do a couple of examples of the product rule.

Example 1 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$ [Solution]

(b) $f\left(x\right) = \left(6x^3 - x\right)\left(10 - 20x\right)$ [Solution]

Solution

At this point there really aren't a lot of reasons to use the product rule. As we noted in the previous section all we would need to do for either of these is to just multiply out the product and then differentiate.

With that said we will use the product rule on these so we can see an example or two. As we add more functions to our repertoire and as the functions become more complicated the product rule will become more useful and in many cases required.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

Note that we took the derivative of this function in the previous section and didn't use the product rule at that point. We should however get the same result here as we did then.

Now let's do the problem here. There's not really a lot to do here other than use the product rule. However, before doing that we should convert the radical to a fractional exponent as always.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

Now let's take the derivative. So we take the derivative of the first function times the second then add on to that the first function times the derivative of the second function.

$$y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}\left(2 - 2x\right)$$

This is NOT what we got in the previous section for this derivative. However, with some simplification we can arrive at the same answer.

$$y' = \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

This is what we got for an answer in the previous section so that is a good check of the product rule.

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(b) $f(x) = (6x^3 - x)(10 - 20x)$

This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'(x) &= (18x^2 - 1)(10 - 20x) + (6x^3 - x)(-20) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Since it was easy to do we went ahead and simplified the results a little.

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Let's now work an example or two with the quotient rule. In this case, unlike the product rule examples, a couple of these functions will require the quotient rule in order to get the derivative. The last two however, we can avoid the quotient rule if we'd like to as we'll see.

Example 2 Differentiate each of the following functions.

(a) $W(z) = \dfrac{3z + 9}{2 - z}$ [Solution]

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$ [Solution]

(c) $f(x) = \dfrac{4}{x^6}$ [Solution]

(d) $y = \dfrac{w^6}{5}$ [Solution]

Solution

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'(z) &= \frac{3(2 - z) - (3z + 9)(-1)}{(2 - z)^2} \\ &= \frac{15}{(2 - z)^2} \end{aligned}$$

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(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$\begin{aligned} h'(x) &= \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\left(x^2-2\right) - 4x^{\frac{1}{2}}\left(2x\right)}{\left(x^2-2\right)^2} \\ &= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{\left(x^2-2\right)^2} \\ &= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{\left(x^2-2\right)^2} \end{aligned}$$

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(c) $f(x) = \dfrac{4}{x^6}$

It seems strange to have this one here rather than being the first part of this example given that it definitely appears to be easier than any of the previous two. In fact, it is easier. There is a point to doing it here rather than first. In this case there are two ways to do compute this derivative. There is an easy way and a hard way and in this case the hard way is the quotient rule. That's the point of this example.

Let's do the quotient rule and see what we get.

$$f'(x) = \frac{(0)\left(x^6\right) - 4\left(6x^5\right)}{\left(x^6\right)^2} = \frac{-24x^5}{x^{12}} = -\frac{24}{x^7}$$

Now, that was the "hard" way. So, what was so hard about it? Well actually it wasn't that hard, there is just an easier way to do it that's all. However, having said that, a common mistake here is to do the derivative of the numerator (a constant) incorrectly. For some reason many people will give the derivative of the numerator in these kinds of problems as a 1 instead of 0! Also, there is some simplification that needs to be done in these kinds of problems if you do the quotient rule.

The easy way is to do what we did in the previous section.

$$f'(x) = 4x^{-6} = -24x^{-7} = -\frac{24}{x^7}$$

Either way will work, but I'd rather take the easier route if I had the choice.

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(d) $y = \frac{w^6}{5}$

This problem also seems a little out of place. However, it is here again to make a point. Do not confuse this with a quotient rule problem. While you can do the quotient rule on this function there is no reason to use the quotient rule on this. Simply rewrite the function as

$$y = \frac{1}{5} w^6$$

and differentiate as always.

$$y' = \frac{6}{5} w^5$$

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Finally, let's not forget about our applications of derivatives.

Example 3 Suppose that the amount of air in a balloon at any time *t* is given by

$$V(t) = \frac{6\sqrt[3]{t}}{4t+1}$$

Determine if the balloon is being filled with air or being drained of air at $t = 8$.

Solution

If the balloon is being filled with air then the volume is increasing and if it's being drained of air then the volume will be decreasing. In other words, we need to get the derivative so that we can determine the rate of change of the volume at $t = 8$.

This will require the quotient rule.

$$\begin{aligned} V'(t) &= \frac{2t^{-\frac{2}{3}}(4t+1) - 6t^{\frac{1}{3}}(4)}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + \frac{2}{t^{\frac{2}{3}}}}{(4t+1)^2} \end{aligned}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate.

The rate of change of the volume at $t = 8$ is then,

$$V'(8) = \frac{-16(2) + \frac{2}{4}}{(33)^2} \qquad (8)^{\frac{1}{3}} = 2 \qquad (8)^{\frac{2}{3}} = \left((8)^{\frac{1}{3}}\right)^2 = (2)^2 = 4$$

$$= -\frac{63}{2178}$$

So, the rate of change of the volume at $t = 8$ is negative and so the volume must be decreasing. Therefore air is being drained out of the balloon at $t = 8$.

As a final topic let's note that the product rule can be extended to more than two functions, for instance.

$$(f\,g\,h)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$

$$(f\,g\,h\,w)' = f'\,g\,h\,w + f\,g'\,h\,w + f\,g\,h'\,w + f\,g\,h\,w'$$

With this section and the previous section we are now able to differentiate powers of *x* as well as sums, differences, products and quotients of these kinds of functions. However, there are many more functions out there in the world that are not in this form. The next few sections give many of these functions as well as give their derivatives.

Derivatives of Trig Functions

With this section we're going to start looking at the derivatives of functions other than polynomials or roots of polynomials. We'll start this process off by taking a look at the derivatives of the six trig functions. Two of the derivatives will be derived. The remaining four are left to the reader and will follow similar proofs for the two given here.

Before we actually get into the derivatives of the trig functions we need to give a couple of limits that will show up in the derivation of two of the derivatives.

Fact

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad\qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

See the Proof of Trig Limits section of the Extras chapter to see the proof of these two limits.

Before we start differentiating trig functions let's work a quick set of limit problems that this fact now allows us to do.

Example 1 Evaluate each of the following limits.

(a) $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{6\theta}$ [Solution]

(b) $\displaystyle\lim_{x \to 0} \frac{\sin(6x)}{x}$ [Solution]

(c) $\displaystyle\lim_{x \to 0} \frac{x}{\sin(7x)}$ [Solution]

(d) $\displaystyle\lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)}$ [Solution]

(e) $\displaystyle\lim_{x \to 4} \frac{\sin(x-4)}{x-4}$ [Solution]

(f) $\displaystyle\lim_{z \to 0} \frac{\cos(2z)-1}{z}$ [Solution]

Solution

(a) $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{6\theta}$

There really isn't a whole lot to this limit. In fact, it's only here to contrast with the next example so you can see the difference in how these work. In this case since there is only a 6 in the denominator we'll just factor this out and then use the fact.

$$\lim_{\theta \to 0} \frac{\sin\theta}{6\theta} = \frac{1}{6}\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{1}{6}(1) = 1$$

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(b) $\lim\limits_{x \to 0} \dfrac{\sin(6x)}{x}$

Now, in this case we can't factor the 6 out of the sine so we're stuck with it there and we'll need to figure out a way to deal with it. To do this problem we need to notice that in the fact the argument of the sine is the same as the denominator (*i.e.* both θ's). So we need to get both of the argument of the sine and the denominator to be the same. We can do this by multiplying the numerator and the denominator by 6 as follows.

$$\lim_{x \to 0} \frac{\sin(6x)}{x} = \lim_{x \to 0} \frac{6\sin(6x)}{6x} = 6\lim_{x \to 0} \frac{\sin(6x)}{6x}$$

Note that we factored the 6 in the numerator out of the limit. At this point, while it may not look like it, we can use the fact above to finish the limit.

To see that we can use the fact on this limit let's do a **change of variables**. A change of variables is really just a renaming of portions of the problem to make something look more like something we know how to deal with. They can't always be done, but sometimes, such as this case, they can simplify the problem. The change of variables here is to let $\theta = 6x$ and then notice that as $x \to 0$ we also have $\theta \to 0$. When doing a change of variables in a limit we need to change all the *x*'s into θ's and that includes the one in the limit.

Doing the change of variables on this limit gives,

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(6x)}{x} &= 6\lim_{x \to 0} \frac{\sin(6x)}{6x} \qquad\qquad \text{let } \theta = 6x \\ &= 6\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \\ &= 6(1) \\ &= 6 \end{aligned}$$

And there we are. Note that we didn't really need to do a change of variables here. All we really need to notice is that the argument of the sine is the same as the denominator and then we can use the fact. A change of variables, in this case, is really only needed to make it clear that the fact does work.

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(c) $\lim\limits_{x \to 0} \dfrac{x}{\sin(7x)}$

In this case we appear to have a small problem in that the function we're taking the limit of here is upside down compared to that in the fact. This is not the problem it appears to be once we notice that,

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$

and then all we need to do is recall a nice property of limits that allows us to do ,

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(7x)} &= \lim_{x\to 0}\frac{1}{\frac{\sin(7x)}{x}}\\ &= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\frac{\sin(7x)}{x}}\\ &= \frac{1}{\lim_{x\to 0}\frac{\sin(7x)}{x}}\end{aligned}$$

With a little rewriting we can see that we do in fact end up needing to do a limit like the one we did in the previous part. So, let's do the limit here and this time we won't bother with a change of variable to help us out. All we need to do is multiply the numerator and denominator of the fraction in the denominator by 7 to get things set up to use the fact. Here is the work for this limit.

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(7x)} &= \frac{1}{\lim_{x\to 0}\frac{7\sin(7x)}{7x}}\\ &= \frac{1}{7\lim_{x\to 0}\frac{\sin(7x)}{7x}}\\ &= \frac{1}{(7)(1)}\\ &= \frac{1}{7}\end{aligned}$$

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(d) $\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}$

This limit looks nothing like the limit in the fact, however it can be thought of as a combination of the previous two parts by doing a little rewriting. First, we'll split the fraction up as follows,

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} = \lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}$$

Now, the fact wants a *t* in the denominator of the first and in the numerator of the second. This is easy enough to do if we multiply the whole thing by $\frac{t}{t}$ (which is just one after all and so won't change the problem) and then do a little rearranging as follows,

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}\frac{t}{t}\\ &= \lim_{t\to 0}\frac{\sin(3t)}{t}\frac{t}{\sin(8t)}\\ &= \left(\lim_{t\to 0}\frac{\sin(3t)}{t}\right)\left(\lim_{t\to 0}\frac{t}{\sin(8t)}\right)\end{aligned}$$

At this point we can see that this really is two limits that we've seen before. Here is the work for each of these and notice on the second limit that we're going to work it a little differently than we did in the previous part. This time we're going to notice that it doesn't really matter whether the sine is in the numerator or the denominator as long as the argument of the sine is the same as what's in the numerator the limit is still one.

Here is the work for this limit.

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \left(\lim_{t\to 0}\frac{3\sin(3t)}{3t}\right)\left(\lim_{t\to 0}\frac{8t}{8\sin(8t)}\right)\\ &= \left(3\lim_{t\to 0}\frac{\sin(3t)}{3t}\right)\left(\frac{1}{8}\lim_{t\to 0}\frac{8t}{\sin(8t)}\right)\\ &= (3)\left(\frac{1}{8}\right)\\ &= \frac{3}{8}\end{aligned}$$

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(e) $\displaystyle\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$

This limit almost looks the same as that in the fact in the sense that the argument of the sine is the same as what is in the denominator. However, notice that, in the limit, x is going to 4 and not 0 as the fact requires. However, with a change of variables we can see that this limit is in fact set to use the fact above regardless.

So, let $\theta = x-4$ and then notice that as $x\to 4$ we have $\theta\to 0$. Therefore, after doing the change of variable the limit becomes,

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

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(f) $\lim_{z\to 0}\frac{\cos(2z)-1}{z}$

The previous parts of this example all used the sine portion of the fact. However, we could just have easily used the cosine portion so here is a quick example using the cosine portion to illustrate this. We'll not put in much explanation here as this really does work in the same manner as the sine portion.

$$\begin{aligned}\lim_{z\to 0}\frac{\cos(2z)-1}{z} &= \lim_{z\to 0}\frac{2\left(\cos(2z)-1\right)}{2z}\\ &= 2\lim_{z\to 0}\frac{\cos(2z)-1}{2z}\\ &= 2(0)\\ &0\end{aligned}$$

All that is required to use the fact is that the argument of the cosine is the same as the denominator.

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Okay, now that we've gotten this set of limit examples out of the way let's get back to the main point of this section, differentiating trig functions.

We'll start with finding the derivative of the sine function. To do this we will need to use the definition of the derivative. It's been a while since we've had to use this, but sometimes there just isn't anything we can do about it. Here is the definition of the derivative for the sine function.

$$\frac{d}{dx}\left(\sin(x)\right)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

Since we can't just plug in $h=0$ to evaluate the limit we will need to use the following trig formula on the first sine in the numerator.

$$\sin(x+h)=\sin(x)\cos(h)+\cos(x)\sin(h)$$

Doing this gives us,

$$\begin{aligned}\frac{d}{dx}\left(\sin(x)\right) &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin(x)\left(\cos(h)-1\right)+\cos(x)\sin(h)}{h}\\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h}+\lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}\end{aligned}$$

As you can see upon using the trig formula we can combine the first and third term and then factor a sine out of that. We can then break up the fraction into two pieces, both of which can be dealt with separately.

Now, both of the limits here are limits as *h* approaches zero. In the first limit we have a sin*(x)* and in the second limit we have a cos*(x)*. Both of these are only functions of *x* only and as *h* moves in towards zero this has no affect on the value of *x*. Therefore, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this gives,

$$\frac{d}{dx}\left(\sin\left(x\right)\right)=\sin\left(x\right)\lim_{h\to 0}\frac{\cos\left(h\right)-1}{h}+\cos\left(x\right)\lim_{h\to 0}\frac{\sin\left(h\right)}{h}$$

At this point all we need to do is use the limits in the fact above to finish out this problem.

$$\frac{d}{dx}\left(\sin\left(x\right)\right)=\sin\left(x\right)\left(0\right)+\cos\left(x\right)\left(1\right)=\cos\left(x\right)$$

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

$$\frac{d}{dx}\left(\cos\left(x\right)\right)=-\sin\left(x\right)$$

With these two out of the way the remaining four are fairly simple to get. All the remaining four trig functions can be defined in terms of sine and cosine and these definitions, along with appropriate derivative rules, can be used to get their derivatives.

Let's take a look at tangent. Tangent is defined as,

$$\tan\left(x\right)=\frac{\sin\left(x\right)}{\cos\left(x\right)}$$

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let's do that.

$$\begin{aligned}\frac{d}{dx}\left(\tan\left(x\right)\right)&=\frac{d}{dx}\left(\frac{\sin\left(x\right)}{\cos\left(x\right)}\right)\\&=\frac{\cos\left(x\right)\cos\left(x\right)-\sin\left(x\right)\left(-\sin\left(x\right)\right)}{\left(\cos\left(x\right)\right)^2}\\&=\frac{\cos^2\left(x\right)+\sin^2\left(x\right)}{\cos^2\left(x\right)}\end{aligned}$$

Now, recall that $\cos^2(x) + \sin^2(x) = 1$ and if we also recall the definition of secant in terms of cosine we arrive at,

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x)\end{aligned}$$

The remaining three trig functions are also quotients involving sine and/or cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

Derivatives of the six trig functions

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$$

At this point we should work some examples.

Example 2 Differentiate each of the following functions.

(a) $g(x) = 3\sec(x) - 10\cot(x)$ [Solution]

(b) $h(w) = 3w^{-4} - w^2\tan(w)$ [Solution]

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$ [Solution]

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$ [Solution]

Solution

(a) $g(x) = 3\sec(x) - 10\cot(x)$

There really isn't a whole lot to this problem. We'll just differentiate each term using the formulas from above.

$$\begin{aligned}g'(x) &= 3\sec(x)\tan(x) - 10\left(-\csc^2(x)\right) \\ &= 3\sec(x)\tan(x) + 10\csc^2(x)\end{aligned}$$

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(b) $h(w) = 3w^{-4} - w^2\tan(w)$

In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike what we saw when we first looked at the product rule. When we first looked at the product rule the only functions we knew how to differentiate were polynomials and in those cases all we really needed to do was multiply them out and we could take the derivative without the product rule. We are now getting into the point where we will be forced to do the product rule at times regardless of whether or not we want to.

We will also need to be careful with the minus sign in front of the second term and make sure that it gets dealt with properly. There are two ways to deal with this. One way it to make sure that you use a set of parenthesis as follows,

$$\begin{aligned} h'(w) &= -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right) \\ &= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w) \end{aligned}$$

Because the second term is being subtracted off of the first term then the whole derivative of the second term must also be subtracted off of the derivative of the first term. The parenthesis make this idea clear.

A potentially easier way to do this is to think of the minus sign as part of the first function in the product. Or, in other words the two functions in the product, using this idea, are $-w^2$ and $\tan(w)$. Doing this gives,

$$h'(w) = -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

So, regardless of how you approach this problem you will get the same derivative.

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(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

As with the previous part we'll need to use the product rule on the first term. We will also think of the 5 as part of the first function in the product to make sure we deal with it correctly. Alternatively, you could make use of a set of parenthesis to make sure the 5 gets dealt with properly. Either way will work, but we'll stick with thinking of the 5 as part of the first term in the product. Here's the derivative of this function.

$$\begin{aligned} y' &= 5\cos(x)\cos(x) + 5\sin(x)\left(-\sin(x)\right) - 4\csc(x)\cot(x) \\ &= 5\cos^2(x) - 5\sin^2(x) - 4\csc(x)\cot(x) \end{aligned}$$

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(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

In this part we'll need to use the quotient rule to take the derivative.

$$P'(t) = \frac{\cos(t)(3-2\cos(t)) - \sin(t)(2\sin(t))}{(3-2\cos(t))^2}$$
$$= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{(3-2\cos(t))^2}$$

Be careful with the signs when differentiating the denominator. The negative sign we get from differentiating the cosine will cancel against the negative sign that is already there.

This appears to be done, but there is actually a fair amount of simplification that can yet be done. To do this we need to factor out a "-2" from the last two terms in the numerator and the make use of the fact that $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$P'(t) = \frac{3\cos(t) - 2\left(\cos^2(t) + \sin^2(t)\right)}{(3-2\cos(t))^2}$$
$$= \frac{3\cos(t) - 2}{(3-2\cos(t))^2}$$

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As a final problem here let's not forget that we still have our standard interpretations to derivatives.

Example 3 Suppose that the amount of money in a bank account is given by

$$P(t) = 500 + 100\cos(t) - 150\sin(t)$$

where *t* is in years. During the first 10 years in which the account is open when is the amount of money in the account increasing?

Solution

To determine when the amount of money is increasing we need to determine when the rate of change is positive. Since we know that the rate of change is given by the derivative that is the first thing that we need to find.

$$P'(t) = -100\sin(t) - 150\cos(t)$$

Now, we need to determine where in the first 10 years this will be positive. This is equivalent to asking where in the interval [0, 10] is the derivative positive. Recall that both sine and cosine are continuous functions and so the derivative is also a continuous function. The Intermediate Value Theorem then tells us that the derivative can only change sign if it first goes through zero.

So, we need to solve the following equation.

$$-100\sin(t) - 150\cos(t) = 0$$
$$100\sin(t) = -150\cos(t)$$
$$\frac{\sin(t)}{\cos(t)} = -1.5$$
$$\tan(t) = -1.5$$

The solution to this equation is,

$$t = 2.1588 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$t = 5.3004 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

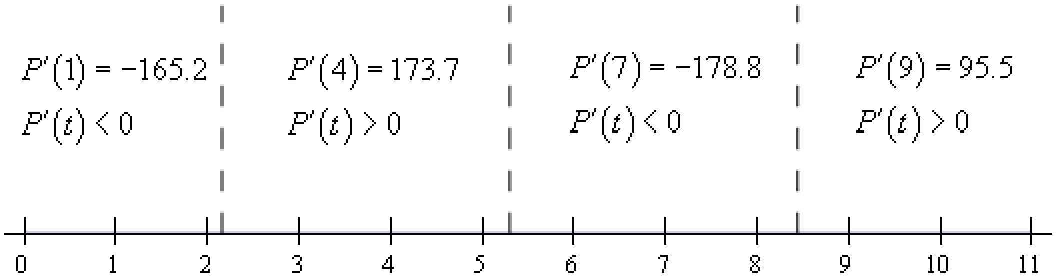
If you don't recall how to solve trig equations go back and take a look at the sections on solving trig equations in the Review chapter.

We are only interested in those solutions that fall in the range [0, 10]. Plugging in values of *n* into the solutions above we see that the values we need are,

$$t = 2.1588 \qquad t = 2.1588 + 2\pi = 8.4420$$
$$t = 5.3004$$

So, much like solving polynomial inequalities all that we need to do is sketch in a number line and add in these points. These points will divide the number line into regions in which the derivative must always be the same sign. All that we need to do then is choose a test point from each region to determine the sign of the derivative in that region.

Here is the number line with all the information on it.



So, it looks like the amount of money in the bank account will be increasing during the following intervals.

$$2.1588 < t < 5.3004 \qquad 8.4420 < t < 10$$

Note that we can't say anything about what is happening after $t = 10$ since we haven't done any work for *t*'s after that point.

In this section we saw how to differentiate trig functions. We also saw in the last example that our interpretations of the derivative are still valid so we can't forget those.

Also, it is important that we be able to solve trig equations as this is something that will arise off and on in this course. It is also important that we can do the kinds of number lines that we used in the last example to determine where a function is positive and where a function is negative. This is something that we will be doing on occasion in both this chapter and the next.