Derivatives of Exponential and Logarithm Functions

The next set of functions that we want to take a look at are exponential and logarithm functions. The most common exponential and logarithm functions in a calculus course are the natural exponential function, $\mathbf{e}^x$, and the natural logarithm function, $\ln(x)$. We will take a more general approach however and look at the general exponential and logarithm function.

Exponential Functions

We'll start off by looking at the exponential function,

$$f(x) = a^x$$

We want to differentiate this. The power rule that we looked at a couple of sections ago won't work as that required the exponent to be a fixed number and the base to be a variable. That is exactly the opposite from what we've got with this function. So, we're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x a^h-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h} \end{aligned}$$

Now, the a^x is not affected by the limit since it doesn't have any *h*'s in it and so is a constant as far as the limit is concerned. We can therefore factor this out of the limit. This gives,

$$f'(x) = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

Now let's notice that the limit we've got above is exactly the definition of the derivative at of $f(x) = a^x$ at $x = 0$, *i.e.* $f'(0)$. Therefore, the derivative becomes,

$$f'(x) = f'(0)\,a^x$$

So, we are kind of stuck we need to know the derivative in order to get the derivative!

There is one value of *a* that we can deal with at this point. Back in the Exponential Functions section of the Review chapter we stated that $\mathbf{e} = 2.71828182845905\ldots$ What we didn't do however do actually define where **e** comes from. There are in fact a variety of ways to define **e**. Here are a three of them.

Some Definitions of e.

1. $\mathbf{e} = \lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$

2. $\mathbf{e}$ is the unique positive number for which $\lim\limits_{h \to 0} \dfrac{\mathbf{e}^h - 1}{h} = 1$

3. $\mathbf{e} = \sum\limits_{n=0}^{\infty} \dfrac{1}{n!}$

The second one is the important one for us because that limit is exactly the limit that we're working with above. So, this definition leads to the following fact,

Fact 1

For the natural exponential function, $f(x) = \mathbf{e}^x$ we have $f'(0) = \lim\limits_{h \to 0} \dfrac{\mathbf{e}^h - 1}{h} = 1$.

So, provided we are using the natural exponential function we get the following.

$$f(x) = \mathbf{e}^x \qquad \Rightarrow \qquad f'(x) = \mathbf{e}^x$$

At this point we're missing some knowledge that will allow us to easily get the derivative for a general function. Eventually we will be able to show that for a general exponential function we have,

$$f(x) = a^x \qquad \Rightarrow \qquad f'(x) = a^x \ln(a)$$

Logarithm Functions

Let's now briefly get the derivatives for logarithms. In this case we will need to start with the following fact about functions that are inverses of each other.

Fact 2

If *f(x)* and *g(x)* are inverses of each other then,

$$g'(x) = \frac{1}{f'(g(x))}$$

So, how is this fact useful to us? Well recall that the natural exponential function and the natural logarithm function are inverses of each other and we know what the derivative of the natural exponential function is!

So, if we have $f(x) = \mathbf{e}^x$ and $g(x) = \ln x$ then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\mathbf{e}^{g(x)}} = \frac{1}{\mathbf{e}^{\ln x}} = \frac{1}{x}$$

The last step just uses the fact that the two functions are inverses of each other.

Putting this all together gives,

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad\qquad x > 0$$

Note that we need to require that $x > 0$ since this is required for the logarithm and so must also be required for its derivative. In can also be shown that,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad\qquad x \neq 0$$

Using this all we need to avoid is $x = 0$.

In this case, unlike the exponential function case, we can actually find the derivative of the general logarithm function. All that we need is the derivative of the natural logarithm, which we just found, and the change of base formula. Using the change of base formula we can write a general logarithm as,

$$\log_a x = \frac{\ln x}{\ln a}$$

Differentiation is then fairly simple.

$$\begin{aligned}\frac{d}{dx}(\log_a x) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\frac{d}{dx}(\ln x)\\ &= \frac{1}{x\ln a}\end{aligned}$$

We took advantage of the fact that *a* was a constant and so $\ln a$ is also a constant and can be factored out of the derivative. Putting all this together gives,

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

Here is a summary of the derivatives in this section.

$$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x \qquad\qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad\qquad \frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

Okay, now that we have the derivations of the formulas out of the way let's compute a couple of derivatives.

Example 1 Differentiate each of the following functions.

(a) $R(w) = 4^w - 5\log_9 w$

(b) $f(x) = 3\mathbf{e}^x + 10x^3 \ln x$

(c) $y = \dfrac{5\mathbf{e}^x}{3\mathbf{e}^x + 1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w) = 4^w \ln 4 - \frac{5}{w \ln 9}$$

(b) Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned} f'(x) &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^3\left(\frac{1}{x}\right) \\ &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^2 \end{aligned}$$

(c) We'll need to use the quotient rule on this one.

$$\begin{aligned} y &= \frac{5\mathbf{e}^x\left(3\mathbf{e}^x + 1\right) - \left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x + 1\right)^2} \\ &= \frac{15\mathbf{e}^{2x} + 5\mathbf{e}^x - 15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x + 1\right)^2} \\ &= \frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x + 1\right)^2} \end{aligned}$$

There's really not a lot to differentiating natural logarithms and natural exponential functions at this point as long as you remember the formulas. In later sections as we get more formulas under our belt they will become more complicated.

Next, we need to do our obligatory application/interpretation problem so we don't forget about them.

Example 2 Suppose that the position of an object is given by

$$s(t) = t\mathbf{e}^t$$

Does the object ever stop moving?

Solution

First we will need the derivative. We need this to determine if the object ever stops moving since at that point (provided there is one) the velocity will be zero and recall that the derivative of the position function is the velocity of the object.

The derivative is,

$$s'(t) = \mathbf{e}^t + t\mathbf{e}^t = (1+t)\mathbf{e}^t$$

So, we need to determine if the derivative is ever zero. To do this we will need to solve,

$$(1+t)\mathbf{e}^t = 0$$

Now, we know that exponential functions are never zero and so this will only be zero at $t = -1$. So, if we are going to allow negative values of *t* then the object will stop moving once at $t = -1$. If we aren't going to allow negative values of *t* then the object will never stop moving.

Before moving on to the next section we need to go back over a couple of derivatives to make sure that we don't confuse the two. The two derivatives are,

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$	Power Rule
$\dfrac{d}{dx}\left(a^x\right) = a^x \ln a$	Derivative of an exponential function

It is important to note that with the Power rule the exponent MUST be a constant and the base MUST be a variable while we need exactly the opposite for the derivative of an exponential function. For an exponential function the exponent MUST be a variable and the base MUST be a constant.

It is easy to get locked into one of these formulas and just use it for both of these. We also haven't even talked about what to do if both the exponent and the base involve variables. We'll see this situation in a later section.

Derivatives of Inverse Trig Functions

In this section we are going to look at the derivatives of the inverse trig functions. In order to derive the derivatives of inverse trig functions we'll need the formula from the last section relating the derivatives of inverse functions. If *f(x)* and *g(x)* are inverse functions then,

$$g'(x) = \frac{1}{f'(g(x))}$$

Recall as well that two functions are inverses if $f(g(x)) = x$ and $g(f(x)) = x$.

We'll go through inverse sine, inverse cosine and inverse tangent in detail here and leave the other three to you to derive if you'd like to.

Inverse Sine

Let's start with inverse sine. Here is the definition of the inverse sine.

$$y = \sin^{-1} x \qquad \Leftrightarrow \qquad \sin y = x \qquad \text{for} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

So, evaluating an inverse trig function is the same as asking what angle (*i.e. y*) did we plug into the sine function to get *x*. The restrictions on *y* given above are there to make sure that we get a consistent answer out of the inverse sine. We know that there are in fact an infinite number of angles that will work and we want a consistent value when we work with inverse sine. When using the range of angles above gives all possible values of the sine function exactly once. If you're not sure of that sketch out a unit circle and you'll see that that range of angles (the *y*'s) will cover all possible values of sine.

Note as well that since $-1 \le \sin(y) \le 1$ we also have $-1 \le x \le 1$.

Let's work a quick example.

Example 1 Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

Solution

So we are really asking what angle *y* solves the following equation.

$$\sin(y) = \frac{1}{2}$$

and we are restricted to the values of *y* above.

From a unit circle we can quickly see that $y = \frac{\pi}{6}$.

We have the following relationship between the inverse sine function and the sine function.

$$\sin\left(\sin^{-1} x\right) = x \qquad\qquad \sin^{-1}\left(\sin x\right) = x$$

In other words they are inverses of each other. This means that we can use the fact above to find the derivative of inverse sine. Let's start with,

$$f(x) = \sin x \qquad\qquad g(x) = \sin^{-1} x$$

Then,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{\cos\left(\sin^{-1} x\right)}$$

This is not a very useful formula. Let's see if we can get a better formula. Let's start by recalling the definition of the inverse sine function.

$$y = \sin^{-1}(x) \qquad \Rightarrow \qquad x = \sin(y)$$

Using the first part of this definition the denominator in the derivative becomes,

$$\cos\left(\sin^{-1} x\right) = \cos(y)$$

Now, recall that

$$\cos^2 y + \sin^2 y = 1 \qquad \Rightarrow \qquad \cos y = \sqrt{1 - \sin^2 y}$$

Using this, the denominator is now,

$$\cos\left(\sin^{-1} x\right) = \cos(y) = \sqrt{1 - \sin^2 y}$$

Now, use the second part of the definition of the inverse sine function. The denominator is then,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Putting all of this together gives the following derivative.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

Inverse Cosine

Now let's take a look at the inverse cosine. Here is the definition for the inverse cosine.

$$y = \cos^{-1} x \qquad \Leftrightarrow \qquad \cos y = x \quad \text{for} \quad 0 \le y \le \pi$$

As with the inverse since we've got a restriction on the angles, *y*, that we get out of the inverse cosine function. Again, if you'd like to verify this a quick sketch of a unit circle should convince you that this range will cover all possible values of cosine exactly once. Also, we also have $-1 \le x \le 1$ because $-1 \le \cos(y) \le 1$.

Example 2 Evaluate $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution
As with the inverse sine we are really just asking the following.

$$\cos y = -\frac{\sqrt{2}}{2}$$

where y must meet the requirements given above. From a unit circle we can see that we must have $y = \frac{3\pi}{4}$.

The inverse cosine and cosine functions are also inverses of each other and so we have,

$$\cos\left(\cos^{-1} x\right) = x \qquad\qquad \cos^{-1}\left(\cos x\right) = x$$

To find the derivative we'll do the same kind of work that we did with the inverse sine above. If we start with

$$f\left(x\right) = \cos x \qquad\qquad g\left(x\right) = \cos^{-1} x$$

then,

$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{-\sin\left(\cos^{-1} x\right)}$$

Simplifying the denominator here is almost identical to the work we did for the inverse sine and so isn't shown here. Upon simplifying we get the following derivative.

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

So, the derivative of the inverse cosine is nearly identical to the derivative of the inverse sine. The only difference is the negative sign.

Inverse Tangent
Here is the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad \Leftrightarrow \qquad \tan y = x \qquad \text{for} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Again, we have a restriction on y, but notice that we can't let y be either of the two endpoints in the restriction above since tangent isn't even defined at those two points. To convince yourself that this range will cover all possible values of tangent do a quick sketch of the tangent function and we can see that in this range we do indeed cover all possible values of tangent. Also, in this case there are no restrictions on x because tangent can take on all possible values.

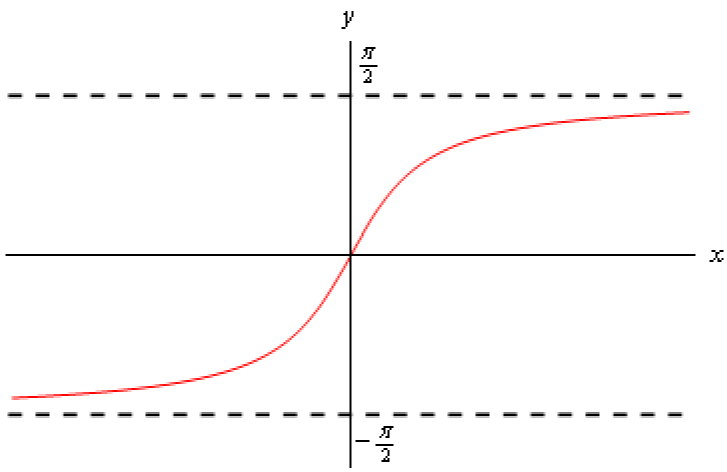
Example 3 Evaluate $\tan^{-1} 1$

Solution
Here we are asking,

$$\tan y = 1$$

where y satisfies the restrictions given above. From a unit circle we can see that $y = \frac{\pi}{4}$.

Because there is no restriction on x we can ask for the limits of the inverse tangent function as x goes to plus or minus infinity. To do this we'll need the graph of the inverse tangent function. This is shown below.



From this graph we can see that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \qquad\qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

The tangent and inverse tangent functions are inverse functions so,

$$\tan\left(\tan^{-1} x\right) = x \qquad\qquad \tan^{-1}\left(\tan x\right) = x$$

Therefore to find the derivative of the inverse tangent function we can start with

$$f\left(x\right) = \tan x \qquad\qquad g\left(x\right) = \tan^{-1} x$$

We then have,

$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{\sec^2\left(\tan^{-1} x\right)}$$

Simplifying the denominator is similar to the inverse sine, but different enough to warrant showing the details. We'll start with the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad\qquad \Rightarrow \qquad\qquad \tan y = x$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y$$

Now, if we start with the fact that

$$\cos^2 y + \sin^2 y = 1$$

and divide every term by $\cos^2 y$ we will get,

$$1 + \tan^2 y = \sec^2 y$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y = 1 + \tan^2 y$$

Finally using the second portion of the definition of the inverse tangent function gives us,

$$\sec^2\left(\tan^{-1} x\right) = 1 + \tan^2 y = 1 + x^2$$

The derivative of the inverse tangent is then,

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

There are three more inverse trig functions but the three shown here the most common ones. Formulas for the remaining three could be derived by a similar process as we did those above. Here are the derivatives of all six inverse trig functions.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

We should probably now do a couple of quick derivatives here before moving on to the next section.

Example 4 Differentiate the following functions.

(a) $f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$

(b) $y = \sqrt{z}\sin^{-1}(z)$

Solution

(a) Not much to do with this one other than differentiate each term.

$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

(b) Don't forget to convert the radical to fractional exponents before using the product rule.

$$y' = \frac{1}{2} z^{-\frac{1}{2}} \sin^{-1}\left(z\right) + \frac{\sqrt{z}}{\sqrt{1-z^2}}$$

Alternate Notation

There is some alternate notation that is used on occasion to denote the inverse trig functions. This notation is,

$$\sin^{-1} x = \arcsin x \qquad \cos^{-1} x = \arccos x$$

$$\tan^{-1} x = \arctan x \qquad \cot^{-1} x = \operatorname{arccot} x$$

$$\sec^{-1} x = \operatorname{arcsec} x \qquad \csc^{-1} x = \operatorname{arccsc} x$$

Derivatives of Hyperbolic Functions

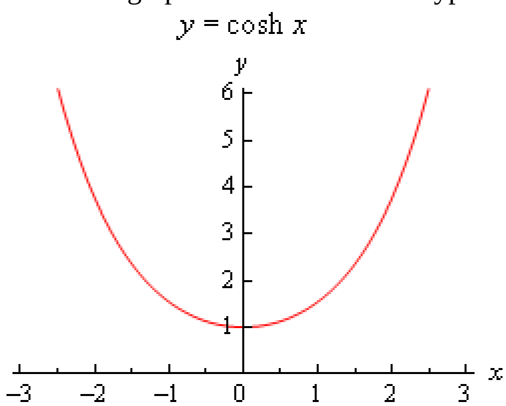
The last set of functions that we're going to be looking in this chapter at are the hyperbolic functions. In many physical situations combinations of $\mathbf{e}^{x}$ and $\mathbf{e}^{-x}$ arise fairly often. Because of this these combinations are given names. There are six hyperbolic functions and they are defined as follows.

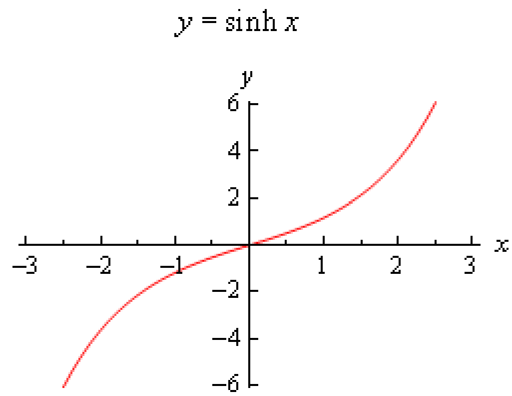
$$\sinh x = \frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2} \qquad \cosh x = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2}$$

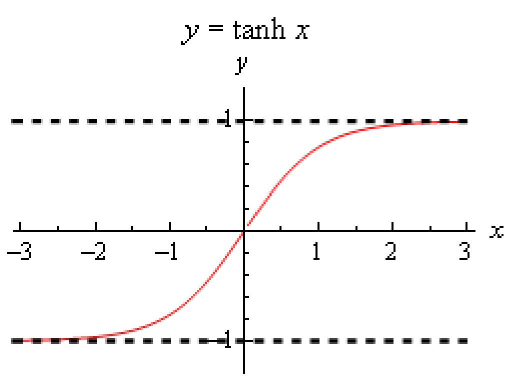
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Here are the graphs of the three main hyperbolic functions.







We also have the following facts about the hyperbolic functions.

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad 1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

You'll note that these are similar, but not quite the same, to some of the more common trig identities so be careful to not confuse the identities here with those of the standard trig functions.

Because the hyperbolic functions are defined in terms of exponential functions finding their derivatives is fairly simple provided you've already read through the next section. We haven't however so we'll need the following formula that can be easily proved after we've covered the next section.

$$\frac{d}{dx}\left(\mathbf{e}^{-x}\right) = -\mathbf{e}^{-x}$$

With this formula we'll do the derivative for hyperbolic sine and leave the rest to you as an exercise.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2}\right) = \frac{\mathbf{e}^{x} - \left(-\mathbf{e}^{-x}\right)}{2} = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2} = \cosh x$$

For the rest we can either use the definition of the hyperbolic function and/or the quotient rule. Here are all six derivatives.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Here are a couple of quick derivatives using hyperbolic functions.

Example 1 Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \dfrac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$

Chain Rule

We've taken a lot of derivatives over the course of the last few sections. However, if you look back they have all been functions similar to the following kinds of functions.

$$R(z)=\sqrt{z} \qquad f(t)=t^{50} \qquad y=\tan(x) \qquad h(w)=\mathbf{e}^{w} \qquad g(x)=\ln x$$

These are all fairly simple functions in that wherever the variable appears it is by itself. What about functions like the following,

$$R(z)=\sqrt{5z-8} \qquad f(t)=\left(2t^3+\cos(t)\right)^{50} \qquad y=\tan\left(\sqrt[3]{3x^2}+\tan(5x)\right)$$

$$h(w)=\mathbf{e}^{w^4-3w^2+9} \qquad g(x)=\ln\left(x^{-4}+x^4\right)$$

None of our rules will work on these functions and yet some of these functions are closer to the derivatives that we're liable to run into than the functions in the first set.

Let's take the first one for example. Back in the section on the definition of the derivative we actually used the definition to compute this derivative. In that section we found that,

$$R'(z)=\frac{5}{2\sqrt{5z-8}}$$

If we were to just use the power rule on this we would get,

$$\frac{1}{2}(5z-8)^{-\frac{1}{2}}=\frac{1}{2\sqrt{5z-8}}$$

which is not the derivative that we computed using the definition. It is close, but it's not the same. So, the power rule alone simply won't work to get the derivative here.

Let's keep looking at this function and note that if we define,

$$f(z)=\sqrt{z} \qquad g(z)=5z-8$$

then we can write the function as a composition.

$$R(z)=(f\circ g)(z)=f\left(g(z)\right)=\sqrt{5z-8}$$

and it turns out that it's actually fairly simple to differentiate a function composition using the **Chain Rule**. There are two forms of the chain rule. Here they are.

Chain Rule

Suppose that we have two functions *f(x)* and *g(x)* and they are both differentiable.

1. If we define $F(x)=(f\circ g)(x)$ then the derivative of *F(x)* is,

$$F'(x)=f'\left(g(x)\right)\,g'(x)$$

2. If we have $y=f(u)$ and $u=g(x)$ then the derivative of *y* is,

$$\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dx}$$

Each of these forms have their uses, however we will work mostly with the first form in this class. To see the proof of the Chain Rule see the Proof of Various Derivative Formulas section of the Extras chapter.

Now, let's go back and use the Chain Rule on the function that we used when we opened this section.

Example 1 Use the Chain Rule to differentiate $R(z) = \sqrt{5z-8}$.

Solution

We've already identified the two functions that we needed for the composition, but let's write them back down anyway and take their derivatives.

$$f(z) = \sqrt{z} \qquad\qquad g(z) = 5z - 8$$

$$f'(z) = \frac{1}{2\sqrt{z}} \qquad\qquad g'(z) = 5$$

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z))\, g'(z) \\ &= f'(5z-8)\, g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}(5) \\ &= \frac{1}{2\sqrt{5z-8}}(5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

And this is what we got using the definition of the derivative.

In general we don't really do all the composition stuff in using the Chain Rule. That can get a little complicated and in fact obscures the fact that there is a quick and easy way of remembering the chain rule that doesn't require us to think in terms of function composition.

Let's take the function from the previous example and rewrite it slightly.

$$R(z) = \underbrace{(5z-8)}_{\text{inside function}}{}^{\overbrace{\frac{1}{2}}^{}}\underbrace{}_{\substack{\text{outside}\\\text{function}}}$$

This function has an "inside function" and an "outside function". The outside function is the square root or the exponent of $\frac{1}{2}$ depending on how you want to think of it and the inside function is the stuff that we're taking the square root of or raising to the $\frac{1}{2}$, again depending on how you want to look at it.

The derivative is then,

$$R'(z) = \overbrace{\frac{1}{2}\underbrace{(5z-8)}_{\substack{\text{inside function}\\ \text{left alone}}}^{-\frac{1}{2}}}^{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(5)}_{\substack{\text{derivative of}\\ \text{inside function}}}$$

In general this is how we think of the chain rule. We identify the "inside function" and the "outside function". We then we differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(g(x))}_{\substack{\text{inside function}\\ \text{left alone}}} \underbrace{g'(x)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

We can always identify the "outside function" in the examples below by asking ourselves how we would evaluate the function. For instance in the *R(z)* case if we were to ask ourselves what *R(2)* is we would first evaluate the stuff under the radical and then finally take the square root of this result. The square root is the last operation that we perform in the evaluation and this is also the outside function. The outside function will always be the last operation you would perform if you were going to evaluate the function.

Let's take a look at some examples of the Chain Rule.

Example 2 Differentiate each of the following.

(a) $f(x) = \sin(3x^2 + x)$ [Solution]

(b) $f(t) = (2t^3 + \cos(t))^{50}$ [Solution]

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$ [Solution]

(d) $g(x) = \ln(x^{-4} + x^4)$ [Solution]

(e) $y = \sec(1 - 5x)$ [Solution]

(f) $P(t) = \cos^4(t) + \cos(t^4)$ [Solution]

Solution

(a) $f(x) = \sin(3x^2 + x)$

It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(3x^2 + x)}_{\substack{\text{leave inside}\\ \text{function alone}}} \underbrace{(6x+1)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

Or with a little rewriting,

$$f'(x) = (6x+1)\cos\left(3x^2+x\right)$$

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(b) $f(t) = \left(2t^3 + \cos(t)\right)^{50}$

In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\begin{aligned} f'(t) &= 50\left(2t^3+\cos(t)\right)^{49}\left(6t^2-\sin(t)\right) \\ &= 50\left(6t^2-\sin(t)\right)\left(2t^3+\cos(t)\right)^{49} \end{aligned}$$

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(c) $h(w) = \mathbf{e}^{w^4-3w^2+9}$

Identifying the outside function in the previous two was fairly simple since it really was the "outside" function in some sense. In this case we need to be a little careful. Recall that the outside function is the last operation that we would perform in an evaluation. In this case if we were to evaluate this function the last operation would be the exponential. Therefore the outside function is the exponential function and the inside function is its exponent.

Here's the derivative.

$$\begin{aligned} h'(w) &= \mathbf{e}^{w^4-3w^2+9}\left(4w^3-6w\right) \\ &= \left(4w^3-6w\right)\mathbf{e}^{w^4-3w^2+9} \end{aligned}$$

Remember, we leave the inside function alone when we differentiate the outside function. So, the derivative of the exponential function (with the inside left alone) is just the original function.

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(d) $g(x) = \ln\left(x^{-4}+x^4\right)$

Here the outside function is the natural logarithm and the inside function is stuff on the inside of the logarithm.

$$g'(x) = \frac{1}{x^{-4}+x^4}\left(-4x^{-5}+4x^3\right) = \frac{-4x^{-5}+4x^3}{x^{-4}+x^4}$$

Again remember to leave the inside function along when differentiating the outside function. So, upon differentiating the logarithm we end up not with 1/*x* but instead with 1/(inside function).

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(e) $y = \sec(1-5x)$

In this case the outside function is the secant and the inside is the $1-5x$.

$$y' = \sec(1-5x)\tan(1-5x)(-5)$$
$$= -5\sec(1-5x)\tan(1-5x)$$

In this case the derivative of the outside function is $\sec(x)\tan(x)$. However, since we leave the inside function alone we don't get *x*'s in both. Instead we get $1-5x$ in both.

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(f) $P(t) = \cos^4(t) + \cos(t^4)$

There are two points to this problem. First, there are two terms and each will require a different application of the chain rule. That will often be the case so don't expect just a single chain rule when doing these problems. Second, we need to be very careful in choosing the outside and inside function for each term.

Recall that the first term can actually be written as,

$$\cos^4(t) = (\cos(t))^4$$

So, in the first term the outside function is the exponent of 4 and the inside function is the cosine. In the second term it's exactly the opposite. In the second term the outside function is the cosine and the inside function is t^4. Here's the derivative for this function.

$$P'(t) = 4\cos^3(t)(-\sin(t)) - \sin(t^4)(4t^3)$$
$$= -4\sin(t)\cos^3(t) - 4t^3\sin(t^4)$$

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There are a couple of general formulas that we can get for some special cases of the chain rule. Let's take a quick look at those.

Example 3 Differentiate each of the following.

(a) $f(x) = [g(x)]^n$

(b) $f(x) = \mathbf{e}^{g(x)}$

(c) $f(x) = \ln(g(x))$

Solution

(a) The outside function is the exponent and the inside is *g(x)*.

$$f'(x) = n[g(x)]^{n-1}g'(x)$$

(b) The outside function is the exponential function and the inside is *g(x)*.

$$f'(x) = g'(x)\mathbf{e}^{g(x)}$$

(c) The outside function is the logarithm and the inside is *g(x)*.

$$f'(x) = \frac{1}{g(x)} g'(x) = \frac{g'(x)}{g(x)}$$

The formulas in this example are really just special cases of the Chain Rule but may be useful to remember in order to quickly do some of these derivatives.

Now, let's also not forget the other rules that we've got for doing derivatives. For the most part we'll not be explicitly identifying the inside and outside functions for the remainder of the problems in this section. We will be assuming that you can see our choices based on the previous examples and the work that we have shown.

Example 4 Differentiate each of the following.

(a) $T(x) = \tan^{-1}(2x)\sqrt[3]{1-3x^2}$ [Solution]

(b) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$ [Solution]

Solution

(a) $T(x) = \tan^{-1}(2x)\sqrt[3]{1-3x^2}$

This requires the product rule and each derivative in the product rule will require a chain rule application as well.

$$\begin{aligned} T'(x) &= \frac{1}{1+(2x)^2}(2)\left(1-3x^2\right)^{\frac{1}{3}} + \tan^{-1}(2x)\left(\frac{1}{3}\right)\left(1-3x^2\right)^{-\frac{2}{3}}(-6x) \\ &= \frac{2\left(1-3x^2\right)^{\frac{1}{3}}}{1+(2x)^2} - 2x\left(1-3x^2\right)^{-\frac{2}{3}}\tan^{-1}(2x) \end{aligned}$$

In this part be careful with the inverse tangent. We know that,

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

When doing the chain rule with this we remember that we've got to leave the inside function alone. That means that where we have the x^2 in the derivative of $\tan^{-1}x$ we will need to have $(\text{inside function})^2$.

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(b) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

In this case we will be using the chain rule in concert with the quotient rule.

$$y' = \frac{5\left(x^3+4\right)^4\left(3x^2\right)\left(1-2x^2\right)^3 - \left(x^3+4\right)^5\left(3\right)\left(1-2x^2\right)^2\left(-4x\right)}{\left(\left(1-2x^2\right)^3\right)^2}$$

These tend to be a little messy. Notice that when we go to simplify that we'll be able to a fair amount of factoring in the numerator and this will often greatly simplify the derivative.

$$\begin{aligned} y' &= \frac{\left(x^3+4\right)^4\left(1-2x^2\right)^2\left(5\left(3x^2\right)\left(1-2x^2\right) - \left(x^3+4\right)\left(3\right)\left(-4x\right)\right)}{\left(1-2x^2\right)^6} \\ &= \frac{3x\left(x^3+4\right)^4\left(5x-6x^3+16\right)}{\left(1-2x^2\right)^4} \end{aligned}$$

After factoring we were able to cancel some of the terms in the numerator against the denominator. So even though the initial chain rule was fairly messy the final answer is significantly simpler because of the factoring.

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The point of this last example is to not forget the other derivative rules that we've got. Most of the examples in this section won't involve the product rule or the quotient rule to make the problems a little shorter. However, in practice they will often be in the same problem.

Now, let's take a look at some more complicated examples.

Example 5 Differentiate each of the following.

(a) $h(z) = \dfrac{2}{\left(4z+\mathbf{e}^{-9z}\right)^{10}}$ [Solution]

(b) $f(y) = \sqrt{2y+\left(3y+4y^2\right)^3}$ [Solution]

(c) $y = \tan\left(\sqrt[3]{3x^2}+\ln\left(5x^4\right)\right)$ [Solution]

(d) $g(t) = \sin^3\left(\mathbf{e}^{1-t}+3\sin\left(6t\right)\right)$ [Solution]

Solution

We're going to be a little more careful in these problems than we were in the previous ones. The reason will be quickly apparent.

(a) $h(z) = \dfrac{2}{\left(4z + \mathbf{e}^{-9z}\right)^{10}}$

In this case let's first rewrite the function in a form that will be a little easier to deal with.

$$h(z) = 2\left(4z + \mathbf{e}^{-9z}\right)^{-10}$$

Now, let's start the derivative.

$$h'(z) = -20\left(4z + \mathbf{e}^{-9z}\right)^{-11} \frac{d}{dz}\left(4z + \mathbf{e}^{-9z}\right)$$

Notice that we didn't actually do the derivative of the inside function yet. This is to allow us to notice that when we do differentiate the second term we will require the chain rule again. Notice as well that we will only need the chain rule on the exponential and not the first term. In many functions we will be using the chain rule more than once so don't get excited about this when it happens.

Let's go ahead and finish this example out.

$$h'(z) = -20\left(4z + \mathbf{e}^{-9z}\right)^{-11}\left(4 - 9\mathbf{e}^{-9z}\right)$$

Be careful with the second application of the chain rule. Only the exponential gets multiplied by the "-9" since that's the derivative of the inside function for that term only. One of the more common mistakes in these kinds of problems is to multiply the whole thing by the "-9" and not just the second term.

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(b) $f(y) = \sqrt{2y + \left(3y + 4y^2\right)^3}$

We'll not put as many words into this example, but we're still going to be careful with this derivative so make sure you can follow each of the steps here.

$$\begin{aligned} f'(y) &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}} \frac{d}{dy}\left(2y + \left(3y + 4y^2\right)^3\right) \\ &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}}\left(2 + 3\left(3y + 4y^2\right)^2(3 + 8y)\right) \\ &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}}\left(2 + (9 + 24y)\left(3y + 4y^2\right)^2\right) \end{aligned}$$

As with the first example the second term of the inside function required the chain rule to differentiate it. Also note that again we need to be careful when multiplying by the derivative of the inside function when doing the chain rule on the second term.

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(c) $y = \tan\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)$

Let's jump right into this one.

$$y' = \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)\frac{d}{dx}\left(\left(3x^2\right)^{\frac{1}{3}} + \ln\left(5x^4\right)\right)$$
$$= \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)\left(\frac{1}{3}\left(3x^2\right)^{-\frac{2}{3}}\left(6x\right) + \frac{20x^3}{5x^4}\right)$$
$$= \left(2x\left(3x^2\right)^{-\frac{2}{3}} + \frac{4}{x}\right)\sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)$$

In this example both of the terms in the inside function required a separate application of the chain rule.

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(d) $g\left(t\right) = \sin^3\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$

We'll need to be a little careful with this one.

$$g'\left(t\right) = 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\frac{d}{dt}\sin\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$$
$$= 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\frac{d}{dt}\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$$
$$= 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\left(\mathbf{e}^{1-t}\left(-1\right) + 3\cos\left(6t\right)\left(6\right)\right)$$
$$= 3\left(-\mathbf{e}^{1-t} + 18\cos\left(6t\right)\right)\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$$

This problem required a total of 4 chain rules to complete.

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Sometimes these can get quite unpleasant and require many applications of the chain rule. Initially, in these cases it's usually best to be careful as we did in this previous set of examples and write out a couple of extra steps rather than trying to do it all in one step in your head. Once you get better at the chain rule you'll find that you can do these fairly quickly in your head.

Finally, before we move onto the next section there is one more issue that we need to address. In the Derivatives of Exponential and Logarithm Functions section we claimed that,

$$f\left(x\right) = a^x \qquad \Rightarrow \qquad f'\left(x\right) = a^x \ln\left(a\right)$$

but at the time we didn't have the knowledge to do this. We now do. What we needed was the Chain Rule.

First, notice that using a property of logarithms we can write a as,

$$a = \mathbf{e}^{\ln a}$$

This may seem kind of silly, but it is needed to compute the derivative. Now, using this we can write the function as,

$$\begin{aligned} f(x) &= a^x \\ &= (a)^x \\ &= \left(\mathbf{e}^{\ln a}\right)^x \\ &= \mathbf{e}^{(\ln a)x} \\ &= \mathbf{e}^{x \ln a} \end{aligned}$$

Okay, now that we've gotten that taken care of all we need to remember is that *a* is a constant and so $\ln a$ is also a constant. Now, differentiating the final version of this function is a (hopefully) fairly simple Chain Rule problem.

$$f'(x) = \mathbf{e}^{x \ln a}(\ln a)$$

Now, all we need to do is rewrite the first term back as a^x to get,

$$f'(x) = a^x \ln(a)$$

So, not too bad if you can see the trick to rewrite *a* and with the Chain Rule.

Implicit Differentiation

To this point we've done quite a few derivatives, but they have all been derivatives of functions of the form $y = f(x)$. Unfortunately not all the functions that we're going to look at will fall into this form.

Let's take a look at an example of a function like this.

Example 1 Find y' for $xy = 1$.

Solution
There are actually two solution methods for this problem.

<u>*Solution 1 :*</u>
This is the simple way of doing the problem. Just solve for *y* to get the function in the form that we're used to dealing with and then differentiate.

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

So, that's easy enough to do. However, there are some functions for which this can't be done. That's where the second solution technique comes into play.

<u>*Solution 2 :*</u>
In this case we're going to leave the function in the form that we were given and work with it in that form. However, let's recall from the first part of this solution that if we could solve for *y* then we will get *y* as a function of *x*. In other words, if we could solve for *y* (as we could in this case, but won't always be able to do) we get $y = y(x)$. Let's rewrite the equation to note this.

$$xy = x\,y(x) = 1$$

Be careful here and note that when we write $y(x)$ we don't mean *y* time *x*. What we are noting here is that *y* is some (probably unknown) function of *x*. This is important to recall when doing this solution technique.

The next step in this solution is to differentiate both sides with respect to *x* as follows,

$$\frac{d}{dx}\left(x\,y(x)\right) = \frac{d}{dx}(1)$$

The right side is easy. It's just the derivative of a constant. The left side is also easy, but we've got to recognize that we've actually got a product here, the *x* and the $y(x)$. So to do the derivative of the left side we'll need to do the product rul. Doing this gives,

$$(1)\,y(x) + x\frac{d}{dx}\left(y(x)\right) = 0$$