This may seem kind of silly, but it is needed to compute the derivative. Now, using this we can write the function as,

$$\begin{aligned} f(x) &= a^x \\ &= (a)^x \\ &= \left(\mathbf{e}^{\ln a}\right)^x \\ &= \mathbf{e}^{(\ln a)x} \\ &= \mathbf{e}^{x \ln a} \end{aligned}$$

Okay, now that we've gotten that taken care of all we need to remember is that *a* is a constant and so $\ln a$ is also a constant. Now, differentiating the final version of this function is a (hopefully) fairly simple Chain Rule problem.

$$f'(x) = \mathbf{e}^{x \ln a}(\ln a)$$

Now, all we need to do is rewrite the first term back as a^x to get,

$$f'(x) = a^x \ln(a)$$

So, not too bad if you can see the trick to rewrite *a* and with the Chain Rule.

Implicit Differentiation

To this point we've done quite a few derivatives, but they have all been derivatives of functions of the form $y = f(x)$. Unfortunately not all the functions that we're going to look at will fall into this form.

Let's take a look at an example of a function like this.

Example 1 Find y' for $xy = 1$.

Solution
There are actually two solution methods for this problem.

<u>*Solution 1 :*</u>
This is the simple way of doing the problem. Just solve for *y* to get the function in the form that we're used to dealing with and then differentiate.

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

So, that's easy enough to do. However, there are some functions for which this can't be done. That's where the second solution technique comes into play.

<u>*Solution 2 :*</u>
In this case we're going to leave the function in the form that we were given and work with it in that form. However, let's recall from the first part of this solution that if we could solve for *y* then we will get *y* as a function of *x*. In other words, if we could solve for *y* (as we could in this case, but won't always be able to do) we get $y = y(x)$. Let's rewrite the equation to note this.

$$xy = x\,y(x) = 1$$

Be careful here and note that when we write $y(x)$ we don't mean *y* time *x*. What we are noting here is that *y* is some (probably unknown) function of *x*. This is important to recall when doing this solution technique.

The next step in this solution is to differentiate both sides with respect to *x* as follows,

$$\frac{d}{dx}\left(x\,y(x)\right) = \frac{d}{dx}(1)$$

The right side is easy. It's just the derivative of a constant. The left side is also easy, but we've got to recognize that we've actually got a product here, the *x* and the $y(x)$. So to do the derivative of the left side we'll need to do the product rul. Doing this gives,

$$(1)\,y(x) + x\frac{d}{dx}\left(y(x)\right) = 0$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}\left(y\left(x\right)\right) = \frac{dy}{dx} = y'$$

Using this we get the follow,

$$y + xy' = 0$$

Note that we dropped the $\left(x\right)$ on the *y* as it was only there to remind us that the *y* was a function of *x* and now that we've taken the derivative it's no longer really needed. We just wanted it in the equation to recognize the product rule when we took the derivative.

So, let's now recall just what were we after. We were after the derivative, y', and notice that there is now a y' in the equation. So, to get the derivative all that we need to do is solve the equation for y'.

$$y' = -\frac{y}{x}$$

There it is. Using the second solution technique this is our answer. This is not what we got from the first solution however. Or at least it doesn't look like the same derivative that we got from the first solution. Recall however, that we really do know what *y* is in terms of *x* and if we plug that in we will get,

$$y' = -\frac{1/x}{x} = -\frac{1}{x^2}$$

which is what we got from the first solution. Regardless of the solution technique used we should get the same derivative.

The process that we used in the second solution to the previous example is called **implicit differentiation** and that is the subject of this section. In the previous example we were able to just solve for *y* and avoid implicit differentiation. However, in the remainder of the examples in this section we either won't be able to solve for *y* or, as we'll see in one of the examples below, the answer will not be in a form that we can deal with.

In the second solution above we replaced the *y* with $y\left(x\right)$ and then did the derivative. Recall that we did this to remind us that *y* is in fact a function of *x*. We'll be doing this quite a bit in these problems, although we rarely actually write $y\left(x\right)$. So, before we actually work anymore implicit differentiation problems let's do a quick set of "simple" derivatives that will hopefully help us with doing derivatives of functions that also contain a $y\left(x\right)$.

Example 2 Differentiate each of the following.

(a) $\left(5x^3-7x+1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$ [Solution]

(b) $\sin(3-6x)$, $\sin\left(y(x)\right)$ [Solution]

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$ [Solution]

Solution

These are written a little differently from what we're used to seeing here. This is because we want to match up these problems with what we'll be doing in this section. Also, each of these parts has several functions to differentiate starting with a specific function followed by a general function. This again, is to help us with some specific parts of the implicit differentiation process that we'll be doing.

(a) $\left(5x^3-7x+1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

With the first function here we're being asked to do the following,

$$\frac{d}{dx}\left[\left(5x^3-7x+1\right)^5\right]=5\left(5x^3-7x+1\right)^4\left(15x^2-7\right)$$

and this is just the chain rule. We differentiated the outside function (the exponent of 5) and then multiplied that by the derivative of the inside function (the stuff inside the parenthesis).

For the section function we're going to do basically the same thing. We're going to need to use the chain rule. The outside function is still the exponent of 5 while the inside function this time is simply $f(x)$. We don't have a specific function here, but that doesn't mean that we can't at least write down the chain rule for this function. Here is the derivative for this function,

$$\frac{d}{dx}\left[f(x)\right]^5=5\left[f(x)\right]^4 f'(x)$$

We don't actually know what $f(x)$ is so when we do the derivative of the inside function all we can do is write down notation for the derivative, *i.e.* $f'(x)$.

With the final function here we simply replaced the f in the second function with a y since most of our work in this section will involve y's instead of f's. Outside of that this function is identical to the second. So, the derivative is,

$$\frac{d}{dx}\left[y(x)\right]^5=5\left[y(x)\right]^4 y'(x)$$

[Return to Problems]

(b) $\sin(3-6x)$, $\sin\left(y(x)\right)$

The first function to differentiate here is just a quick chain rule problem again so here is it's derivative,

$$\frac{d}{dx}\left[\sin\left(3-6x\right)\right]=-6\cos\left(3-6x\right)$$

For the second function we didn't bother this time with using $f\left(x\right)$ and just jumped straight to $y\left(x\right)$ for the general version. This is still just a general version of what we did for the first function. The outside function is still the sine and the inside is give by $y\left(x\right)$ and while we don't have a formula for $y\left(x\right)$ and so we can't actually take its derivative we do have a notation for its derivative. Here is the derivative for this function,

$$\frac{d}{dx}\left[\sin\left(y\left(x\right)\right)\right]=y'\left(x\right)\cos\left(y\left(x\right)\right)$$

[Return to Problems]

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

In this part we'll just give the answers for each and leave out the explanation that we had in the first two parts.

$$\frac{d}{dx}\left(\mathbf{e}^{x^2-9x}\right)=\left(2x-9\right)\mathbf{e}^{x^2-9x} \qquad \frac{d}{dx}\left(\mathbf{e}^{y(x)}\right)=y'\left(x\right)\mathbf{e}^{y(x)}$$

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So, in this set of examples we were just doing some chain rule problems where the inside function was $y\left(x\right)$ instead of a specific function. This kind of derivative shows up all the time in doing implicit differentiation so we need to make sure that we can do them. Also note that we only did this for three kinds of functions but there are many more kinds of functions that we could have used here.

So, it's now time to do our first problem where implicit differentiation is required, unlike the first example where we could actually avoid implicit differentiation by solving for *y*.

Example 3 Find y' for the following function.

$$x^2+y^2=9$$

Solution

How, this is just a circle and we can solve for *y* which would give,

$$y=\pm\sqrt{9-x^2}$$

Prior to starting this problem we stated that we had to do implicit differentiation here because we couldn't just solve for *y* and yet that's what we just did. So, why can't we use implicit differentiation here? The problem is the "$\pm$". With this in the "solution" for *y* we see that *y* is in fact two different functions. Which should we use? Should we use both? We only want a single function for the derivative and at best we have two functions here.

So, in this example we really are going to need to do implicit differentiation so we can avoid this. In this example ee'll do the same thing we did in the first example and remind ourselves that *y* is really a function of *x* and write *y* as $y(x)$. Once we've done this all we need to do is differentiate each term with respect to *x*.

$$\frac{d}{dx}\left(x^2+\left[y(x)\right]^2\right)=\frac{d}{dx}(9)$$

As with the first example the right side is easy. The left side is also pretty easy since all we need to do is take the derivative of each term and note that the second term will be similar the part (a) of the second example. All we need to do for the second term is use the chain rule.

After taking the derivative we have,

$$2x+2\left[y(x)\right]^1 y'(x)=0$$

At this point we can drop the (x) part as it was only in the problem to help with the differentiation process. The final step is to simply solve the resulting equation for y'.

$$2x+2yy'=0$$
$$y'=-\frac{x}{y}$$

Unlike the first example we can't just plug in for *y* since we wouldn't know which of the two functions to use. Most answers from implicit differentiation will involve both *x* and *y* so don't get excited about that when it happens.

As always, we can't forget our interpretations of derivatives.

Example 4 Find the equation of the tangent line to

$$x^2+y^2=9$$

at the point $\left(2,\sqrt{5}\right)$.

Solution

First note that unlike all the other tangent line problems we've done in previous sections we need to be given both the *x* and the *y* values of the point. Notice as well that this point does lie on the graph of the circle (you can check by plugging the points into the equation) and so it's okay to talk about the tangent line at this point.

Recall that to write down the tangent line we need is slope of the tangent line and this is nothing more than the derivative evaluated at the given point. We've got the derivative from the previous example so as we need to do is plug in the given point.

$$m = y'\big|_{x=2,\, y=\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

The tangent line is then.

$$y = \sqrt{5} - \frac{2}{\sqrt{5}}\left(x-2\right)$$

Now, let's work some more examples. In the remaining examples we will no longer write $y\left(x\right)$ for *y*. This is just something that we were doing to remind ourselves that *y* is really a function of *x* to help with the derivatives. Seeing the $y\left(x\right)$ reminded us that we needed to do the chain rule on that portion of the problem. From this point on we'll leave the *y*'s written as *y*'s and in our head we'll need to remember that they really are $y\left(x\right)$ and that we'll need to do the chain rule.

There is an easy way to remember how to do the chain rule in these problems. The chain rule really tells us to differentiate the function as we usually would, except we need to add on a derivative of the inside function. In implicit differentiation this means that every time we are differentiating a term with *y* in it the inside function is the *y* and we will need to add a y' onto the term since that will be the derivative of the inside function.

Let's see a couple of examples.

Example 5 Find y' for each of the following.

(a) $x^3 y^5 + 3x = 8y^3 + 1$ [Solution]

(b) $x^2 \tan\left(y\right) + y^{10}\sec\left(x\right) = 2x$ [Solution]

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln\left(xy^3\right)$ [Solution]

Solution

(a) $x^3 y^5 + 3x = 8y^3 + 1$

First differentiate both sides with respect to *x* and remember that each *y* is really $y\left(x\right)$ we just aren't going to write it that way anymore. This means that the first term on the left will be a product rule.

We differentiated these kinds of functions involving *y*'s to a power with the chain rule in the Example 2 above. Also, recall the discussion prior to the start of this problem. When doing this kind of chain rule problem all that we need to do is differentiate the *y*'s as normal and then add on a y', which is nothing more than the derivative of the "inside function".

Here is the differentiation of each side for this function.

$$3x^2 y^5 + 5x^3 y^4 y' + 3 = 24y^2 y'$$

Now all that we need to do is solve for the derivative, y'. This is just basic solving algebra that you are capable of doing. The main problem is that it's liable to be messier than what you're used to doing. All we need to do is get all the terms with y' in them on one side and all the terms without y' in them on the other. Then factor y' out of all the terms containing it and divide both side by the "coefficient" of the y'. Here is the solving work for this one,

$$\begin{aligned} 3x^2y^5+3 &= 24y^2y' - 5x^3y^4y' \\ 3x^2y^5+3 &= \left(24y^2 - 5x^3y^4\right)y' \\ y' &= \frac{3x^2y^5+3}{24y^2-5x^3y^4} \end{aligned}$$

The algebra in these problems can be quite messy so be careful with that.

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(b) $x^2\tan(y)+y^{10}\sec(x)=2x$

We've got two product rules to deal with this time. Here is the derivative of this function.

$$2x\tan(y)+x^2\sec^2(y)\,y'+10y^9y'\sec(x)+y^{10}\sec(x)\tan(x)=2$$

Notice the derivative tacked onto the secant! Again, this is just a chain rule problem similar to the second part of Example 2 above.

Now, solve for the derivative.

$$\begin{aligned} \left(x^2\sec^2(y)+10y^9\sec(x)\right)y' &= 2-y^{10}\sec(x)\tan(x)-2x\tan(y) \\ y' &= \frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2\sec^2(y)+10y^9\sec(x)} \end{aligned}$$

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(c) $\mathbf{e}^{2x+3y}=x^2-\ln\left(xy^3\right)$

We're going to need to be careful with this problem. We've got a couple chain rules that we're going to need to deal with here that are a little different from those that we've dealt with prior to this problem.

In both the exponential and the logarithm we've got a "standard" chain rule in that there is something other than just an x or y inside the exponential and logarithm. So, this means we'll do the chain rule as usual here and then when we do the derivative of the inside function for each term we'll have to deal with differentiating y's.

Here is the derivative of this equation.

$$\mathbf{e}^{2x+3y}\left(2+3y'\right)=2x-\frac{y^3+3xy^2y'}{xy^3}$$

In both of the chain rules note that the y' didn't get tacked on until we actually differentiated the *y*'s in that term.

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$\begin{aligned}
2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y}&=2x-\frac{y^3}{xy^3}-\frac{3xy^2y'}{xy^3}\\
2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y}&=2x-\frac{1}{x}-\frac{3y'}{y}\\
\left(3\mathbf{e}^{2x+3y}+3y^{-1}\right)y'&=2x-x^{-1}-2\mathbf{e}^{2x+3y}\\
y'&=\frac{2x-x^{-1}-2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y}+3y^{-1}}
\end{aligned}$$

Note that to make the derivative at least look a little nicer we converted all the fractions to negative exponents.

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Okay, we've seen one application of implicit differentiation in the tangent line example above. However, there is another application that we will be seeing in every problem in the next section.

In some cases we will have two (or more) functions all of which are functions of a third variable. So, we might have $x(t)$ and $y(t)$, for example and in these cases we will be differentiating with respect to *t*. This is just implicit differentiation like we did in the previous examples, but there is a difference however.

In the previous examples we have functions involving *x*'s and *y*'s and thinking of *y* as $y(x)$. In these problems we differentiated with respect to *x* and so when faced with *x*'s in the function we differentiated as normal and when faced with *y*'s we differentiated as normal except we then added a y' onto that term because we were really doing a chain rule.

In the new example we want to look at we're assume that $x=x(t)$ and that $y=y(t)$ and differentiating with respect to *t*. This means that every time we are faced with an *x* or a *y* we'll be doing the chain rule. This in turn means that when we differentiate an *x* we will need to add on an x' and whenever we differentiate a *y* we will add on a y'.

These new types of problems are really the same kind of problem we've been doing in this section. They are just expanded out a little to include more than one function that will require a chain rule.

Let's take a look at an example of this kind of problem.

Example 6 Assume that $x = x(t)$ and $y = y(t)$ and differentiate the following equation with respect to *t*.

$$x^3 y^6 + \mathbf{e}^{1-x} - \cos(5y) = y^2$$

Solution

So, just differentiate as normal and add on an appropriate derivative at each step. Note as well that the first term will be a product rule since both *x* and *y* are functions of *t*.

$$3x^2 x' y^6 + 6x^3 y^5 y' - x' \mathbf{e}^{1-x} + 5y' \sin(5y) = 2yy'$$

There really isn't all that much to this problem. Since there are two derivatives in the problem we won't be bothering to solve for one of them. When we do this kind of problem in the next section the problem will imply which one we need to solve for.

At this point there doesn't seem be any real reason for doing this kind of problem, but as we'll see in the next section every problem that we'll be doing there will involve this kind of implicit differentiation.

Related Rates

In this section we are going to look at an application of implicit differentiation. Most of the applications of derivatives are in the next chapter however there are a couple of reasons for placing it in this chapter as opposed to putting it into the next chapter with the other applications. The first reason is that it's an application of implicit differentiation and so putting right after that section means that we won't have forgotten how to do implicit differentiation. The other reason is simply that after doing all these derivatives we need to be reminded that there really are actual applications to derivatives. Sometimes it is easy to forget there really is a reason that we're spending all this time on derivatives.

For these related rates problems it's usually best to just jump right into some problems and see how they work.

Example 1 Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution

The first thing that we'll need to do here is to identify what information that we've been given and what we want to find. Before we do that let's notice that both the volume of the balloon and the radius of the balloon will vary with time and so are really functions of time, *i.e.* $V(t)$ and $r(t)$.

We know that air is being pumped into the balloon at a rate of 5 cm^3/min. This is the rate at which the volume is increasing. Recall that rates of change are nothing more than derivatives and so we know that,

$$V'(t) = 5$$

We want to determine the rate at which the radius is changing. Again, rates are derivatives and so it looks like we want to determine,

$$r'(t) = ? \qquad \text{when} \quad r(t) = \frac{d}{2} = 10\,\text{cm}$$

Note that we needed to convert the diameter to a radius.

Now that we've identified what we have been given and what we want to find we need to relate these two quantities to each other. In this case we can relate the volume and the radius with the formula for the volume of a sphere.

$$V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$$

As in the previous section when we looked at implicit differentiation, we will typically not use the (t) part of things in the formulas, but since this is the first time through one of these we will do that to remind ourselves that they are really functions of *t*.

Now we don't really want a relationship between the volume and the radius. What we really want is a relationship between their derivatives. We can do this by differentiating both sides with respect to *t*. In other words, we will need to do implicit differentiation on the above formula. Doing this gives,

$$V' = 4\pi r^2 r'$$

Note that at this point we went ahead and dropped the (t) from each of the terms. Now all that we need to do is plug in what we know and solve for what we want to find.

$$5 = 4\pi\left(10^2\right)r' \qquad \Rightarrow \qquad r' = \frac{1}{80\pi} \text{ cm/min}$$

We can get the units of the derivative be recalling that,

$$r' = \frac{dr}{dt}$$

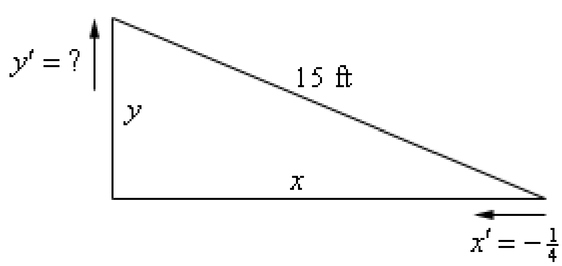
The units of the derivative will be the units of the numerator (cm in the previous example) divided by the units of the denominator (min in the previous example).

Let's work some more examples.

Example 2 A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?

Solution

The first thing to do in this case is to sketch picture that shows us what is going on.



We've defined the distance of the bottom of the latter from the wall to be *x* and the distance of the top of the ladder from the floor to be *y*. Note as well that these are changing with time and so we really should write $x(t)$ and $y(t)$. However, as is often the case with related rates/implicit differentiation problems we don't write the (t) part just try to remember this in our heads as we proceed with the problem.

Next we need to identify what we know and what we want to find. We know that the rate at which the bottom of the ladder is moving towards the wall. This is,

$$x' = -\frac{1}{4}$$

Note as well that the rate is negative since the distance from the wall, *x*, is decreasing. We always need to be careful with signs with these problems.

We want to find the rate at which the top of the ladder is moving away from the floor. This is y'. Note as well that this quantity should be positive since *y* will be increasing.

As with the first example we first need a relationship between *x* and *y*. We can get this using Pythagorean theorem.

$$x^2 + y^2 = (15)^2 = 225$$

All that we need to do at this point is to differentiate both sides with respect to *t*, remembering that *x* and *y* are really functions of *t* and so we'll need to do implicit differentiation. Doing this gives an equation that shows the relationship between the derivatives.

$$2xx' + 2yy' = 0 \qquad (1)$$

Next, let's see which of the various parts of this equation that we know and what we need to find. We know x' and are being asked to determine y' so it's okay that we don't know that. However, we still need to determine *x* and *y*.

Determining *x* and *y* is actually fairly simple. We know that initially $x = 10$ and the end is being pushed in towards the wall at a rate of $\frac{1}{4}$ ft/sec and that we are interested in what has happened after 12 seconds. We know that,

$$\begin{aligned} \text{distance} &= \text{rate} \times \text{time} \\ &= \left(\frac{1}{4}\right)(12) = 3 \end{aligned}$$

So, the end of the ladder has been pushed in 3 feet and so after 12 seconds we must have $x = 7$. Note that we could have computed this in one step as follows,

$$x = 10 - \frac{1}{4}(12) = 7$$

To find *y* (after 12 seconds) all that we need to do is reuse the Pythagorean Theorem with the values of *x* that we just found above.

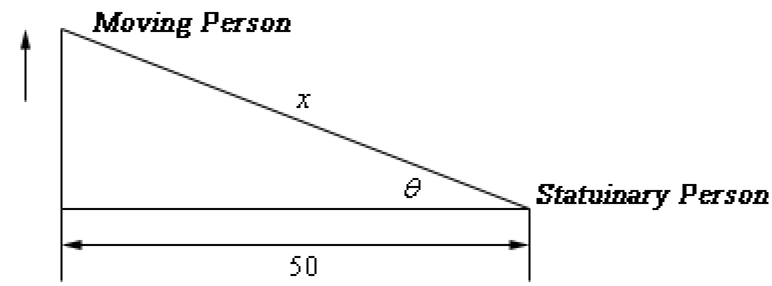
$$y = \sqrt{225 - x^2} = \sqrt{225 - 49} = \sqrt{176}$$

Now all that we need to do is plug into (1) and solve for y'.

$$2(7)\left(-\frac{1}{4}\right)+2\left(\sqrt{176}\right)y'=0 \qquad \Rightarrow \qquad y'=\frac{7/4}{\sqrt{176}}=\frac{7}{4\sqrt{176}}=0.1319 \text{ ft/sec}$$

Notice that we got the correct sign for y'. If we'd gotten a negative then we'd have known that we had made a mistake and we could go back and look for it.

Example 3 Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01 rad/min. At what rate is distance between the two people changing when $\theta = 0.5$ radians?



Solution

This example is not as tricky as it might at first appear. Let's call the distance between them at any point in time *x* as noted above. We can then relate all the known quantities by one of two trig formulas.

$$\cos\theta = \frac{50}{x} \qquad\qquad \sec\theta = \frac{x}{50}$$

We want to find x' and we could find *x* if we wanted to at the point in question using cosine since we also know the angle at that point in time. However, if we use the second formula we won't need to know *x* as you'll see. So, let's differentiate that formula.

$$\sec\theta\tan\theta\ \theta' = \frac{x'}{50}$$

As noted, there are no *x*'s in this formula. We want to determine x' and we know that $\theta = 0.5$ and $\theta' = 0.01$ (do you agree with it being positive?). So, just plug in and solve.

$$(50)(0.01)\sec(0.5)\tan(0.5) = x' \qquad \Rightarrow \qquad x' = 0.311254 \text{ ft / min}$$

So far we we've seen three related rates problems. While each one was worked in a very different manner the process was essentially the same in each. In each problem we identified what we were given and what we wanted to find. We next wrote down a relationship between all the various quantities and used implicit differentiation to arrive at a relationship between the various derivatives in the problem. Finally, we plugged into the equation to find the value we were after.

So, in a general sense each problem was worked in pretty much the same manner. The only real difference between them was coming up with the relationship between the known and unknown quantities. This is often the hardest part of the problem. In many problems the best way to come up with the relationship is to sketch a diagram that shows the situation. This often seems like a silly step, but can make all the difference in whether we can find the relationship or not.

Let's work another problem that uses some different ideas and shows some of the different kinds of things that can show up in related rates problems.

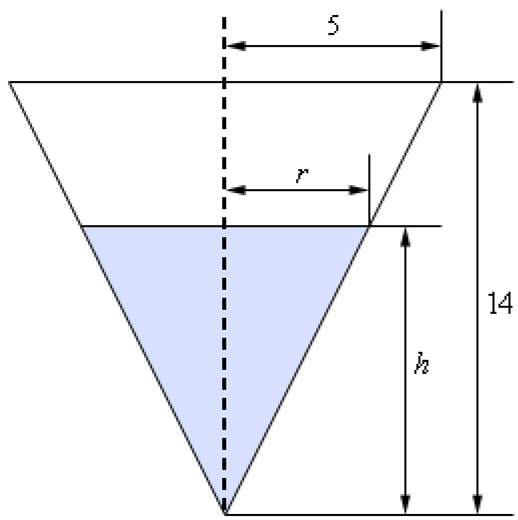
Example 4 A tank of water in the shape of a cone is leaking water at a constant rate of $2\,\text{ft}^3/\text{hour}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

Solution

Okay, we should probably start off with a quick sketch (probably not to scale) of what is going on here.



As we can see, the water in the tank actually forms a smaller cone with the same central angle as the tank itself. The radius of the "water" cone at any time is given by *r* and the height of the "water" cone at any time is given by *h*. The volume of water in the tank at any time *t* is given by,

$$V = \frac{1}{3}\pi r^2 h$$

and we've been given that $V' = -2$.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

For this part we need to determine h' when $h = 6$ and now we have a problem. The only formula that we've got that will relate the volume to the height also includes the radius and so if we were to differentiate this with respect to *t* we would get,

$$V' = \frac{2}{3}\pi r r' h + \frac{1}{3}\pi r^2 h'$$

So, in this equation we know V' and *h* and want to find h', but we don't know *r* and r'. As we'll see finding *r* isn't too bad, but we just don't have enough information, at this point, that will allow us to find r' and h' simultaneously.

To fix this we'll need to eliminate the *r* from the volume formula in some way. This is actually easier than it might at first look. If we go back to our sketch above and look at just the right half of the tank we see that we have two similar triangles and when we say similar we mean similar in the geometric sense. Recall that two triangles are called similar if their angles are identical, which in the case here. When we have two similar triangles then ratios of any two sides will be equal. For our set this means that we have,

$$\frac{r}{h} = \frac{5}{14} \qquad \Rightarrow \qquad r = \frac{5}{14}h$$

If we take this and plug into our volume formula we have,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 h = \frac{25}{588}\pi h^3$$

This gives us a volume formula that only involved the volume and the height of the water. Note however that this volume formula is only valid for our cone, so don't be tempted to use it for other cones! If we now differentiate this we have,

$$V' = \frac{25}{196}\pi h^2 h'$$

At this point all we need to do is plug in what we know and solve for h'.

$$-2 = \frac{25}{196}\pi\left(6^2\right)h' \qquad \Rightarrow \qquad h' = \frac{-98}{255\pi} = -0.1223$$

So, it looks like the height is decreasing at a rate of 0.04413 ft/hr.

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

In this case we are asking for r' and there is an easy way to do this part and a difficult (well, more difficult than the easy way anyway….) way to do it. The "difficult" way is to redo the work part (a) above only this time use,

$$\frac{h}{r} = \frac{14}{5} \qquad \Rightarrow \qquad h = \frac{14}{5}r$$

to get the volume in terms of V and r and then proceed as before.

That's not terribly difficult, but it is more work that we need to so. Recall from the first part that we have,

$$r = \frac{5}{14}h \qquad \Rightarrow \qquad r' = \frac{5}{14}h'$$

So, as we can see if we take the relationship that relates r and h that we used in the first part and differentiate it we get a relationship between r' and h'. At this point all we need to do here is use the result from the first part to get,

$$r' = \frac{5}{14}\left(\frac{-98}{255\pi}\right) = -\frac{7}{51\pi} = -0.4369$$

Much easier that redoing all of the first part. Note however, that we were only able to do this the "easier" way because it was asking for r' at exactly the same time that we asked for h' in the first part. If we hadn't been using the same time then we would have had no choice but to do this the "difficult" way.

In the second part of the previous problem we saw an important idea in dealing with related rates. In order to find the asked for rate all we need is an equations that relates the rate we're looking for to a rate that we already know. Sometimes there are multiple equations that we can use and sometimes one will be easier than another.

Also, this problem showed us that we will often have an equation that contains more variables that we have information about and so, in these cases, we will need to eliminate one (or more) of the variables. In this problem we eliminated the extra variable using the idea of similar triangles. This will not always be how we do this, but many of these problems do use similar triangles so make sure you can use that idea.

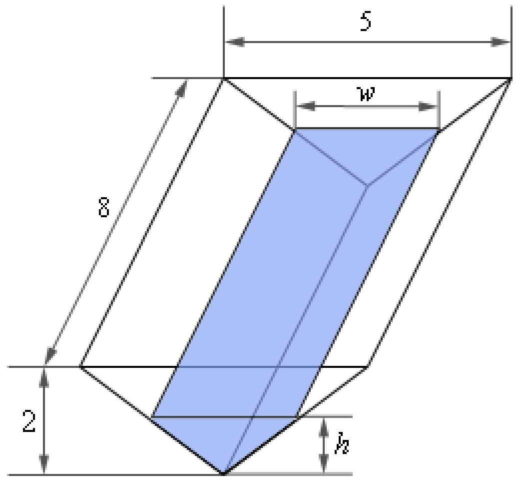
Let's work some more problems.

Example 5 A trough of water is 8 meters deep and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of $6 \text{ m}^3/\text{sec}$. At what rate is the height of the water changing when the water has a height of 120 cm?

Solution

Note that an isosceles triangle is just a triangle in which two of the sides are the same length. In our case sides of the tank have the same length.

We definitely need a sketch of this situation to get us going here so here. A sketch of the trough is shown below.



Now, in this problem we know that $V' = 6 \text{ m}^3/\text{sec}$ and we want to determine h' when $h = 1.2\,\text{m}$. Note that because V' is in terms of meters we need to convert *h* into meters as well. So, we need an equation that will relate these two quantities and the volume of the tank will do it.

The volume of this kind of tank is simple to compute. The volume is the area of the end times the depth. For our case the volume of the water in the tank is,

$$\begin{aligned} V &= (\text{Area of End})(\text{depth}) \\ &= \left(\tfrac{1}{2}\text{base} \times \text{height}\right)(\text{depth}) \\ &= \tfrac{1}{2}hw(8) \\ &= 4hw \end{aligned}$$

As with the previous example we've got an extra quantity here, *w*, that is also changing with time and so we need to get eliminate it from the problem. To do this we'll again make use of the idea of similar triangles. If we look at the end of the tank we'll see that we again have two similar triangles. One for the tank itself and on formed by the water in the tank. Again, remember that with similar triangles that ratios of sides must be equal. In our case we'll use,

$$\frac{w}{5} = \frac{h}{2} \qquad \Rightarrow \qquad w = \frac{5}{2}h$$

Plugging this into the volume gives a formula for the volume (and only for this tank) that only involved the height of the water.

$$V = 4hw = 4h\left(\frac{5}{2}h\right) = 10h^2$$

We can now differentiate this to get,

$$V' = 20hh'$$

Finally, all we need to do is plug in and solve for h'.

$$6 = 20(1.2)h' \qquad \Rightarrow \qquad h' = 0.25 \text{ m/sec}$$

So, the height of the water is raising at a rate of 0.25 m/sec.

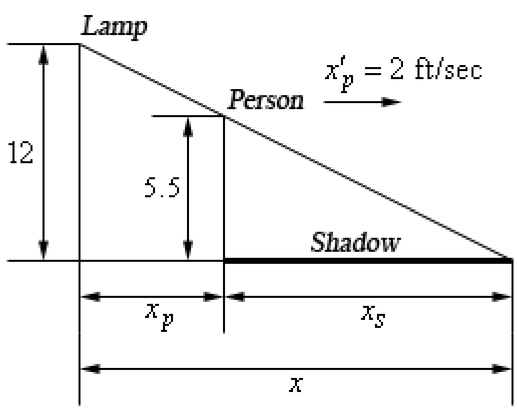
Example 6 A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

Solution

We'll definitely need a sketch of this situation to get us started here. The tip of the shadow is defined by the rays of light just getting past the person and so we can form the following set of similar triangles.



Here *x* is the distance of the tip of the shadow from the pole, x_p is the distance of the person from the pole and x_s is the length of the shadow. Also note that we converted the persons height over to 5.5 feet since all the other measurements are in feet.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

In this case we want to determine x' when $x_p = 25$ given that $x_p' = 2$.

The equation we'll need here is,

$$x = x_p + x_s$$

but we'll need to eliminate x_s from the equation in order to get an answer. To do this we can again make use of the fact that the two triangles are similar to get,

$$\frac{5.5}{12} = \frac{x_s}{x} = \frac{x_s}{x_p + x_s} \qquad\qquad \text{Note : } \frac{5.5}{12} = \frac{\frac{11}{2}}{12} = \frac{11}{24}$$

We'll need to solve this for x_s.

$$\frac{11}{24}\left(x_p + x_s\right) = x_s$$
$$\frac{11}{24}x_p = \frac{13}{24}x_s$$
$$\frac{11}{13}x_p = x_s$$

Our equation then becomes,

$$x = x_p + \frac{11}{13}x_p = \frac{24}{13}x_p$$

Now all that we need to do is differentiate this, plug in and solve for x'.

$$x' = \frac{24}{13}x'_p \qquad \Rightarrow \qquad x' = \frac{24}{13}(2) = 3.6923 \text{ ft/sec}$$

The tip of the shadow is then moving away from the pole at a rate of 3.6923 ft/sec. Notice as well that we never actually had to use the fact that $x_p = 25$ for this problem. That will happen on rare occasions.

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

This part is actually quite simple if we have the answer from (a) in hand, which we do of course. In this case we know that x_s represents the length of the shadow, or the distance of the tip of the shadow from the person so it looks like we want to determine x'_s when $x_p = 25$.

Again, we can use $x = x_p + x_s$, however unlike the first part we now know that $x'_p = 2$ and $x' = 3.6923 \text{ ft/sec}$ so in this case all we need to do is differentiate the equation and plug in for all the known quantities.

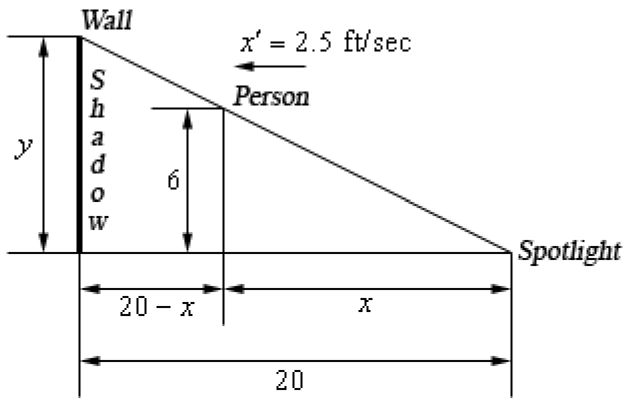
$$x' = x'_p + x'_s$$
$$3.6923 = 2 + x'_s \qquad\qquad x'_s = 1.6923 \text{ ft/sec}$$

The tip of the shadow is then moving away from the person at a rate of 1.6923 ft/sec.

Example 7 A spot light is on the ground 20 ft away from a wall and a 6 ft tall person is walking towards the wall at a rate of 2.5 ft/sec. How fast the height of the shadow changing when the person is 8 feet from the wall? Is the shadow increasing or decreasing in height at this time?

Solution

Let's start off with a sketch of this situation and the sketch here will be similar to that of the previous problem. The top of the shadow will be defined by the light rays going over the head of the person and so we again get yet another set of similar triangles.



In this case we want to determine y' when the person is 8 ft from wall or $x = 12$ ft. Also, if the person is moving towards the wall at 2.5 ft/sec then the person must be moving away from the spotlight at 2.5 ft/sec and so we also know that $x' = 2.5$.

In all the previous problems that used similar triangles we used the similar triangles to eliminate one of the variables from the equation we were working with. In this case however, we can get the equation that relates *x* and *y* directly from the two similar triangles. In this case the equation we're going to work with is,

$$\frac{y}{6} = \frac{20}{x} \qquad \Rightarrow \qquad y = \frac{120}{x}$$

Now all that we need to do is differentiate and plug values into solve to get y'.

$$y' = -\frac{120}{x^2}x' \qquad \Rightarrow \qquad y' = -\frac{120}{12^2}(2.5) = -2.0833 \text{ ft/sec}$$

The height of the shadow is then decreasing at a rate of 2.0833 ft/sec.

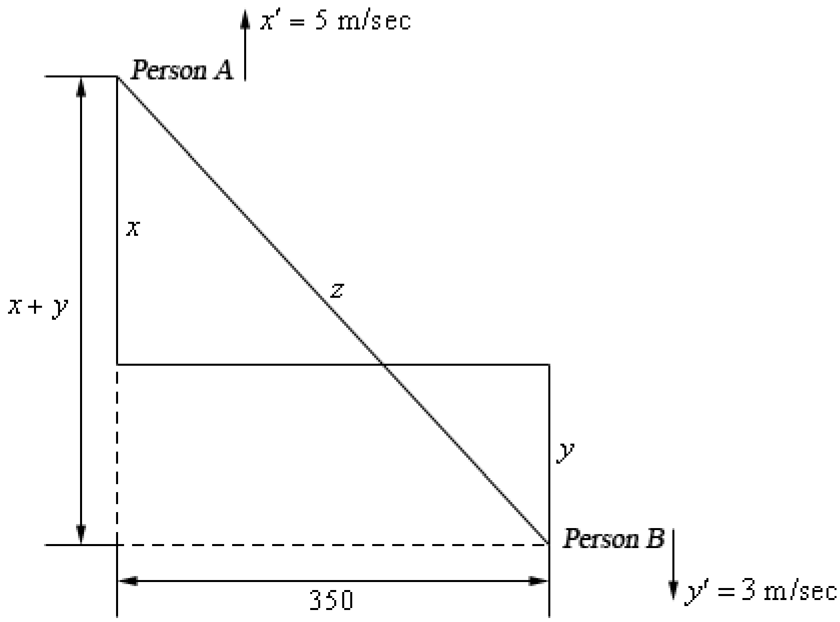
Okay, we've worked quite a few problems now that involved similar triangles in one form or another so make sure you can do these kinds of problems.

It's now time to do a problem that while similar to some of the problems we've done to this point is also sufficiently different that it can cause problems until you've seen how to do it.

Example 8 Two people on bikes are separated by 350 meters. Person A starts riding north at a rate of 5 m/sec and 7 minutes later Person B starts riding south at 3 m/sec. At what rate is the distance separating the two people changing 25 minutes after Person A starts riding?

Solution

There is a lot to digest here with this problem. Let's start off with a sketch of the situation.



Now we are after z' and we know that $x' = 5$ and $y' = 3$. We want to know z' after Person A had been riding for 25 minutes and Person B has been riding for $25 - 7 = 18$ minutes. After converting these times to seconds (because our rates are all in m/sec) this means that at the time we're interested in each of the bike riders has rode,

$$x = 5(25 \times 60) = 7500 \text{ m} \qquad\qquad y = 3(18 \times 60) = 3240 \text{ m}$$

Next, the Pythagorean theorem tells us that,

$$z^2 = (x + y)^2 + 350^2 \tag{2}$$

Therefore, 25 minutes after Person A starts riding the two bike riders are

$$z = \sqrt{(x+y)^2 + 350^2} = \sqrt{(7500 + 3240)^2 + 350^2} = 10745.7015 \text{ m}$$

apart.

To determine the rate at which the two riders are moving apart all we need to do then is differentiate (2) and plug in all the quantities that we know to find z'.

$$2zz' = 2(x+y)(x'+y')$$
$$2(10745.7015)z' = 2(7500+3240)(5+3)$$
$$z' = 7.9958 \text{ m/sec}$$

So, the two riders are moving apart at a rate of 7.9958 m/sec.

Every problem that we've worked to this point has come down to needing a geometric formula and we should probably work a quick problem that is not geometric in nature.

Example 9 Suppose that we have two resistors connected in parallel with resistances R_1 and R_2 measured in ohms (Ω). The total resistance, *R*, is then given by,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that R_1 is increasing at a rate of 0.4 Ω/min and R_2 is decreasing at a rate of 0.7Ω/min. At what rate is *R* changing when $R_1 = 80\,\Omega$ and $R_2 = 105\,\Omega$?

Solution

Okay, unlike the previous problems there really isn't a whole lot to do here. First, let's note that we're looking for R' and that we know $R_1' = 0.4$ and $R_2' = -0.7$. Be careful with the signs here.

Also, since we'll eventually need it let's determine *R* at the time we're interested in.

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{105} = \frac{37}{1680} \qquad \Rightarrow \qquad R = \frac{1680}{37} = 45.4054\,\Omega$$

Next we need to differentiate the equation given in the problem statement.

$$-\frac{1}{R^2}R' = -\frac{1}{(R_1)^2}R_1' - \frac{1}{(R_2)^2}R_2'$$
$$R' = R^2\left(\frac{1}{(R_1)^2}R_1' + \frac{1}{(R_2)^2}R_2'\right)$$

Finally, all we need to do is plug into this and do some quick computations.

$$R' = (45.4054)^2\left(\frac{1}{80^2}(0.4) + \frac{1}{105^2}(-0.7)\right) = -0.002045$$

So, it looks like *R* is decreasing at a rate of 0.002045Ω/min.

We've seen quite a few related rates problems in this section that cover a wide variety of possible problems. There are still many more different kinds of related rates problems out there in the world, but the ones that we've worked here should give you a pretty good idea on how to at least start most of the problems that you're liable to run into.

Higher Order Derivatives

Let's start this section with the following function.

$$f(x) = 5x^3 - 3x^2 + 10x - 5$$

By this point we should be able to differentiate this function without any problems. Doing this we get,

$$f'(x) = 15x^2 - 6x + 10$$

Now, this is a function and so it can be differentiated. Here is the notation that we'll use for that, as well as the derivative.

$$f''(x) = \left(f'(x)\right)' = 30x - 6$$

This is called the **second derivative** and $f'(x)$ is now called the **first derivative**.

Again, this is a function as so we can differentiate it again. This will be called the **third derivative**. Here is that derivative as well as the notation for the third derivative.

$$f'''(x) = \left(f''(x)\right)' = 30$$

Continuing, we can differentiate again. This is called, oddly enough, the **fourth derivative**. We're also going to be changing notation at this point. We can keep adding on primes, but that will get cumbersome after awhile.

$$f^{(4)}(x) = \left(f'''(x)\right)' = 0$$

This process can continue but notice that we will get zero for all derivatives after this point. This set of derivatives leads us to the following fact about the differentiation of polynomials.

Fact

If *p(x)* is a polynomial of degree *n* (*i.e.* the largest exponent in the polynomial) then,

$$p^{(k)}(x) = 0 \qquad\qquad \text{for } k \geq n+1$$

We will need to be careful with the "non-prime" notation for derivatives. Consider each of the following.

$$f^{(2)}(x) = f''(x)$$

$$f^{2}(x) = \left[f(x)\right]^2$$

The presence of parenthesis in the exponent denotes differentiation while the absence of parenthesis denotes exponentiation.

Collectively the second, third, fourth, *etc.* derivatives are called **higher order derivatives**.

Let's take a look at some examples of higher order derivatives.

Example 1 Find the first four derivatives for each of the following.

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$ [Solution]

(b) $y = \cos x$ [Solution]

(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$ [Solution]

Solution

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

There really isn't a lot to do here other than do the derivatives.

$$R'(t) = 6t + 4t^{-\frac{1}{2}} + \mathbf{e}^t$$

$$R''(t) = 6 - 2t^{-\frac{3}{2}} + \mathbf{e}^t$$

$$R'''(t) = 3t^{-\frac{5}{2}} + \mathbf{e}^t$$

$$R^{(4)}(t) = -\frac{15}{2}t^{-\frac{7}{2}} + \mathbf{e}^t$$

Notice that differentiating an exponential function is very simple. It doesn't change with each differentiation.

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(b) $y = \cos x$

Again, let's just do some derivatives.

$$\begin{aligned} y &= \cos x \\ y' &= -\sin x \\ y'' &= -\cos x \\ y''' &= \sin x \\ y^{(4)} &= \cos x \end{aligned}$$

Note that cosine (and sine) will repeat every four derivatives. The other four trig functions will not exhibit this behavior. You might want to take a few derivatives to convince yourself of this.

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(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$

In the previous two examples we saw some patterns in the differentiation of exponential functions, cosines and sines. We need to be careful however since they only work if there is just a t or an x in argument. This is the point of this example. In this example we will need to use the chain rule on each derivative.

$$f'(y) = 3\cos(3y) - 2\mathbf{e}^{-2y} + \frac{1}{y} = 3\cos(3y) - 2\mathbf{e}^{-2y} + y^{-1}$$
$$f''(y) = -9\sin(3y) + 4\mathbf{e}^{-2y} - y^{-2}$$
$$f'''(y) = -27\cos(3y) - 8\mathbf{e}^{-2y} + 2y^{-3}$$
$$f^{(4)}(y) = 81\sin(3y) + 16\mathbf{e}^{-2y} - 6y^{-4}$$

So, we can see with slightly more complicated arguments the patterns that we saw for exponential functions, sines and cosines no longer completely hold.

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Let's do a couple more examples to make a couple of points.

Example 2 Find the second derivative for each of the following functions.

(a) $Q(x) = \sec(5t)$ [Solution]

(b) $g(w) = \mathbf{e}^{1-2w^3}$ [Solution]

(c) $f(t) = \ln(1+t^2)$ [Solution]

Solution

(a) $Q(x) = \sec(5t)$

Here's the first derivative.

$$Q'(x) = 5\sec(5t)\tan(5t)$$

Notice that the second derivative will now require the product rule.

$$\begin{aligned} Q''(x) &= 25\sec(5t)\tan(5t)\tan(5t) + 25\sec(5t)\sec^2(5t) \\ &= 25\sec(5t)\tan^2(5t) + 25\sec^3(5t) \end{aligned}$$

Notice that each successive derivative will require a product and/or chain rule and that as noted above this will not end up returning back to just a secant after four (or another other number for that matter) derivatives as sine and cosine will.

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(b) $g(w) = \mathbf{e}^{1-2w^3}$

Again, let's start with the first derivative.

$$g'(w) = -6w^2\mathbf{e}^{1-2w^3}$$

As with the first example we will need the product rule for the second derivative.

$$\begin{aligned} g''(w) &= -12w\mathbf{e}^{1-2w^3} - 6w^2\left(-6w^2\right)\mathbf{e}^{1-2w^3} \\ &= -12w\mathbf{e}^{1-2w^3} + 36w^4\mathbf{e}^{1-2w^3} \end{aligned}$$

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(c) $f(t) = \ln\left(1+t^2\right)$

Same thing here.

$$f'(t) = \frac{2t}{1+t^2}$$

The second derivative this time will require the quotient rule.

$$\begin{aligned} f''(t) &= \frac{2\left(1+t^2\right) - (2t)(2t)}{\left(1+t^2\right)^2} \\ &= \frac{2-2t^2}{\left(1+t^2\right)^2} \end{aligned}$$

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As we saw in this last set of examples we will often need to use the product or quotient rule for the higher order derivatives, even when the first derivative didn't require these rules.

Let's work one more example that will illustrate how to use implicit differentiation to find higher order derivatives.

Example 3 Find y'' for

$$x^2 + y^4 = 10$$

Solution

Okay, we know that in order to get the second derivative we need the first derivative and in order to get that we'll need to do implicit differentiation. Here is the work for that.

$$\begin{aligned} 2x + 4y^3 y' &= 0 \\ y' &= -\frac{x}{2y^3} \end{aligned}$$

Now, this is the first derivative. We get the second derivative by differentiating this, which will require implicit differentiation again.

$$\begin{aligned} y'' &= \left(-\frac{x}{2y^3}\right)' \\ &= -\frac{2y^3 - x\left(6y^2 y'\right)}{\left(2y^3\right)^2} \\ &= -\frac{2y^3 - 6xy^2 y'}{4y^6} \\ &= -\frac{y - 3xy'}{2y^4} \end{aligned}$$

This is fine as far as it goes. However, we would like there to be no derivatives in the answer. We don't, generally, mind having *x*'s and/or *y*'s in the answer when doing implicit differentiation, but we really don't like derivatives in the answer. We can get rid of the derivative however by acknowledging that we know what the first derivative is and substituting this into the second derivative equation. Doing this gives,

$$\begin{aligned} y'' &= -\frac{y-3xy'}{2y^4} \\ &= -\frac{y-3x\left(-\frac{x}{2y^3}\right)}{2y^4} \\ &= -\frac{y+\frac{3}{2}x^2y^{-3}}{2y^4} \end{aligned}$$

Now that we've found some higher order derivatives we should probably talk about an interpretation of the second derivative.

If the position of an object is given by *s(t)* we know that the velocity is the first derivative of the position.

$$v(t) = s'(t)$$

The acceleration of the object is the first derivative of the velocity, but since this is the first derivative of the position function we can also think of the acceleration as the second derivative of the position function.

$$a(t) = v'(t) = s''(t)$$

Alternate Notation

There is some alternate notation for higher order derivatives as well. Recall that there was a fractional notation for the first derivative.

$$f'(x) = \frac{df}{dx}$$

We can extend this to higher order derivatives.

$$f''(x) = \frac{d^2y}{dx^2} \qquad\qquad f'''(x) = \frac{d^3y}{dx^3} \qquad\qquad etc.$$

Logarithmic Differentiation

There is one last topic to discuss in this section. Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called **logarithmic differentiation**.

It's easiest to see how this works in an example.

Example 1 Differentiate the function.

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. However, that would be a fairly messy process. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

Of course, this isn't really simpler. What we need to do is use the properties of logarithms to expand the right side as follows.

$$\ln y = \ln\left(x^5\right) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

This doesn't look all the simple. However, the differentiation process will be simpler. What we need to do at this point is differentiate both sides with respect to *x*. Note that this is really implicit differentiation.

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+2\right)^{-\frac{1}{2}}(2x)}{\left(x^2+2\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

To finish the problem all that we need to do is multiply both sides by *y* and the plug in for *y* since we do know what that is.

$$\begin{aligned} y' &= y\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right) \\ &= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right) \end{aligned}$$

Depending upon the person doing this would probably be slightly easier than doing both the product and quotient rule. The answer is almost definitely simpler that what we would have gotten using the product and quotient rule.

So, as the first example has shown we can use logarithmic differentiation to avoid using the product rule and/or quotient rule.

We can also use logarithmic differentiation to differentiation functions in the form.

$$y = \left(f\left(x\right)\right)^{g(x)}$$

Let's take a quick look at a simple example of this.

Example 2 Differentiate $y = x^x$

Solution

We've seen two functions similar to this at this point.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad\qquad \frac{d}{dx}\left(a^x\right) = a^x \ln a$$

Neither of these two will work here since both require either the base or the exponent to be a constant. In this case both the base and the exponent are variables and so we have no way to differentiate this function using only known rules from previous sections.

With logarithmic differentiation we can do this however. First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\begin{aligned} \ln y &= \ln x^x \\ \ln y &= x \ln x \end{aligned}$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

As with the first example multiply by y and substitute back in for y.

$$\begin{aligned} y' &= y\left(1 + \ln x\right) \\ &= x^x\left(1 + \ln x\right) \end{aligned}$$

Let's take a look at a more complicated example of this.

Example 3 Differentiate $y = (1-3x)^{\cos(x)}$

Solution

Now, this look much more complicated than the previous example, but is in fact only slightly more complicated. The process is pretty much identical so we first take the log of both sides and then simplify the right side.

$$\ln y = \ln\left[(1-3x)^{\cos(x)}\right] = \cos(x)\ln(1-3x)$$

Next, do some implicit differentiation.

$$\frac{y'}{y} = -\sin(x)\ln(1-3x) + \cos(x)\frac{-3}{1-3x} = -\sin(x)\ln(1-3x) - \cos(x)\frac{3}{1-3x}$$

Finally, solve for y' and substitute back in for *y*.

$$\begin{aligned} y' &= -y\left(\sin(x)\ln(1-3x) + \cos(x)\frac{3}{1-3x}\right) \\ &= -(1-3x)^{\cos(x)}\left(\sin(x)\ln(1-3x) + \cos(x)\frac{3}{1-3x}\right) \end{aligned}$$

A messy answer but there it is.

We'll close this section out with a quick recap of all the various ways we've seen of differentiating functions with exponents. It is important to not get all of these confused.

$\frac{d}{dx}\left(a^b\right) = 0$	This is a constant
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	Power Rule
$\frac{d}{dx}\left(a^x\right) = a^x \ln a$	Derivative of an exponential function
$\frac{d}{dx}\left(x^x\right) = x^x(1+\ln x)$	Logarithmic Differentiation

It is sometimes easy to get these various functions confused and use the wrong rule for differentiation. Always remember that each rule has very specific rules for where the variable and constants must be. For example, the Power Rule requires that the base be a variable and the exponent be a constant, while the exponential function requires exactly the opposite.

If you can keep straight all the rules you can't go wrong with these.

Applications of Derivatives

Introduction

In the previous chapter we focused almost exclusively on the computation of derivatives. In this chapter will focus on applications of derivatives. It is important to always remember that we didn't spend a whole chapter talking about computing derivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we'll be looking at in this chapter are using derivatives to determine information about graphs of functions and optimization problems. These will not be the only applications however. We will be revisiting limits and taking a look at an application of derivatives that will allow us to compute limits that we haven't been able to compute previously. We will also see how derivatives can be used to estimate solutions to equations.

Here is a listing of the topics in this section.

Rates of Change – The point of this section is to remind us of the application/interpretation of derivatives that we were dealing with in the previous chapter. Namely, rates of change.

Critical Points – In this section we will define critical points. Critical points will show up in many of the sections in this chapter so it will be important to understand them.

Minimum and Maximum Values – In this section we will take a look at some of the basic definitions and facts involving minimum and maximum values of functions.

Finding Absolute Extrema – Here is the first application of derivatives that we'll look at in this chapter. We will be determining the largest and smallest value of a function on an interval.

The Shape of a Graph, Part I – We will start looking at the information that the first derivatives can tell us about the graph of a function. We will be looking at increasing/decreasing functions as well as the First Derivative Test.

The Shape of a Graph, Part II – In this section we will look at the information about the graph of a function that the second derivatives can tell us. We will look at inflection points, concavity, and the Second Derivative Test.

The Mean Value Theorem – Here we will take a look that the Mean Value Theorem.

