Optimization

In this section we are going to look at optimization problems. In optimization problems we are looking for the largest value or the smallest value that a function can take. We saw how to one kind of optimization problem in the Absolute Extrema section where we found the largest and smallest value that a function would take on an interval.

In this section we are going to look at another type of optimization problem. Here we will be looking for the largest or smallest value of a function subject to some kind of constraint. The constraint will be some condition (that can usually be described by some equation) that must absolutely, positively be true no matter what our solution is. On occasion, the constraint will not be easily described by an equation, but in these problems it will be easy to deal with as we'll see.

This section is generally one of the more difficult for students taking a Calculus course. One of the main reasons for this is that a subtle change of wording can completely change the problem. There is also the problem of identifying the quantity that we'll be optimizing and the quantity that is the constraint and writing down equations for each.

The first step in all of these problems should be to very carefully read the problem. Once you've done that the next step is to identify the quantity to be optimized and the constraint.

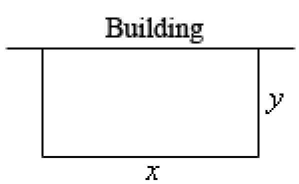
In identifying the constraint remember that the constraint is something that must true regardless of the solution. In almost every one of the problems we'll be looking at here one quantity will be clearly indicated as having a fixed value and so must be the constraint. Once you've got that identified the quantity to be optimized should be fairly simple to get. It is however easy to confuse the two if you just skim the problem so make sure you carefully read the problem first!

Let's start the section off with a simple problem to illustrate the kinds of issues will be dealing with here.

Example 1 We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Solution

In all of these problems we will have two functions. The first is the function that we are actually trying to optimize and the second will be the constraint. Sketching the situation will often help us to arrive at these equations so let's do that.



In this problem we want to maximize the area of a field and we know that will use 500 ft of fencing material. So, the area will be the function we are trying to optimize and the amount of fencing is the constraint. The two equations for these are,

$$\text{Maximize : } A = xy$$
$$\text{Contraint : } 500 = x + 2y$$

Okay, we know how to find the largest or smallest value of a function provided it's only got a single variable. The area function (as well as the constraint) has two variables in it and so what we know about finding absolute extrema won't work. However, if we solve the constraint for one of the two variables we can substitute this into the area and we will then have a function of a single variable.

So, let's solve the constraint for *x*. Note that we could have just as easily solved for *y* but that would have led to fractions and so, in this case, solving for *x* will probably be best.

$$x = 500 - 2y$$

Substituting this into the area function gives a function of *y*.

$$A(y) = (500 - 2y)\,y = 500y - 2y^2$$

Now we want to find the largest value this will have on the interval [0,250]. Note that the interval corresponds to taking $y = 0$ (*i.e.* no sides to the fence) and $y = 250$ (*i.e.* only two sides and no width, also if there are two sides each must be 250 ft to use the whole 500ft…).

Note that the endpoints of the interval won't make any sense from a physical standpoint if we actually want to enclose some area because they would both give zero area. They do, however, give us a set of limits on *y* and so the Extreme Value Theorem tells us that we will have a maximum value of the area somewhere between the two endpoints. Having these limits will also mean that we can use the process we discussed in the Finding Absolute Extrema section earlier in the chapter to find the maximum value of the area.

So, recall that the maximum value of a continuous function (which we've got here) on a closed interval (which we also have here) will occur at critical points and/or end points. As we've already pointed out the end points in this case will give zero area and so don't make any sense. That means our only option will be the critical points.

So let's get the derivative and find the critical points.

$$A'(y) = 500 - 4y$$

Setting this equal to zero and solving gives a lone critical point of $y = 125$. Plugging this into the area gives an area of 31250 ft^2. So according to the method from Absolute Extrema section this must be the largest possible area, since the area at either endpoint is zero.

Finally, let's not forget to get the value of x and then we'll have the dimensions since this is what the problem statement asked for. We can get the x by plugging in our y into the constraint.

$$x = 500 - 2(125) = 250$$

The dimensions of the field that will give the largest area, subject to the fact that we used exactly 500 ft of fencing material, are 250 x 125.

Don't forget to actually read the problem and give the answer that was asked for. These types of problems can take a fair amount of time/effort to solve and it's not hard to sometimes forget what the problem was actually asking for.

In the previous problem we used the method from the Finding Absolute Extrema section to find the maximum value of the function we wanted to optimize. However, as we'll see in later examples we won't always have easy to find endpoints and/or dealing with the endpoints may not be easy to deal with. Not only that, but this method requires that the function we're optimizing be continuous on the interval we're looking at, including the endpoints, and that may not always be the case.

So, before proceeding with the anymore examples let's spend a little time discussing some methods for determining if our solution is in fact the absolute minimum/maximum value that we're looking for. In some examples all of these will work while in others one or more won't be all that useful. However, we will always need to use some method for making sure that our answer is in fact that optimal value that we're after.

Method 1 : Use the method used in Finding Absolute Extrema.

This is the method used in the first example above. Recall that in order to use this method the range of possible optimal values, let's call it I, must have finite endpoints. Also, the function we're optimizing (once it's down to a single variable) must be continuous on I, including the endpoints. If these conditions are met then we know that the optimal value, either the maximum or minimum depending on the problem, will occur at either the endpoints of the range or at a critical point that is inside the range of possible solutions.

There are two main issues that will often prevent this method from being used however. First, not every problem will actually have a range of possible solutions that have finite endpoints at both ends. We'll see at least one example of this as we work through the remaining examples. Also, many of the functions we'll be optimizing will not be continuous once we reduce them down to a single variable and this will prevent us from using this method.

Method 2 : Use a variant of the First Derivative Test.

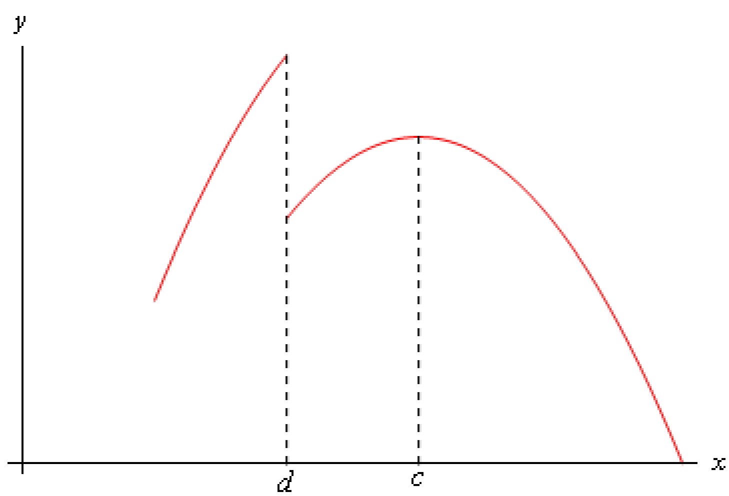
In this method we also will need a range of possible optimal values, I. However, in this case, unlike the previous method the endpoints do not need to be finite. Also, we will need to require that the function be continuous on the interior I and we will only need the function to be continuous at the end points if the endpoint is finite and the function actually exists at the endpoint. We'll see several problems where the function we're optimizing doesn't actually exist at one of the endpoints. This will not prevent this method from being used.

Let's suppose that $x = c$ is a critical point of the function we're trying to optimize, $f(x)$. We already know from the First Derivative Test that if $f'(x) > 0$ immediately to the left of $x = c$ (*i.e.* the function is increasing immediately to the left) and if $f'(x) < 0$ immediately to the right of $x = c$ (*i.e.* the function is decreasing immediately to the right) then $x = c$ will be a relative maximum for $f(x)$.

Now, this does not mean that the absolute maximum of $f(x)$ will occur at $x = c$. However, suppose that we knew a little bit more information. Suppose that in fact we knew that $f'(x) > 0$ for all x in I such that $x < c$. Likewise, suppose that we knew that $f'(x) < 0$ for all x in I such that $x > c$. In this case we know that to the left of $x = c$, provided we stay in *I* of course, the function is always increasing and to the right of $x = c$, again staying in *I*, we are always decreasing. In this case we can say that the absolute maximum of $f(x)$ in *I* will occur at $x = c$.

Similarly, if we know that to the left of $x = c$ the function is always decreasing and to the right of $x = c$ the function is always increasing then the absolute minimum of $f(x)$ in *I* will occur at $x = c$.

Before we give a summary of this method let's discuss the continuity requirement a little. Nowhere in the above discussion did the continuity requirement apparently come into play. We require that that the function we're optimizing to be continuous in *I* to prevent the following situation.



In this case, a relative maximum of the function clearly occurs at $x = c$. Also, the function is always decreasing to the right and is always increasing to the left. However, because of the discontinuity at $x = d$, we can clearly see that $f(d) > f(c)$ and so the absolute maximum of the function does not occur at $x = c$. Had the discontinuity at $x = d$ not been there this would not have happened and the absolute maximum would have occurred at $x = c$.

Here is a summary of this method.

First Derivative Test for Absolute Extrema

Let *I* be the interval of all possible optimal values of $f(x)$ and further suppose that $f(x)$ is continuous on I, except possibly at the endpoints. Finally suppose that $x = c$ is a critical point of $f(x)$ and that *c* is in the interval *I*. If we restrict *x* to values from *I* (*i.e.* we only consider possible optimal values of the function) then,

1. If $f'(x) > 0$ for all $x < c$ and if $f'(x) < 0$ for all $x > c$ then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval *I*.
2. If $f'(x) < 0$ for all $x < c$ and if $f'(x) > 0$ for all $x > c$ then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval *I*.

Method 3 : Use the second derivative.

There are actually two ways to use the second derivative to help us identify the optimal value of a function and both use the Second Derivative Test to one extent or another.

The first way to use the second derivative doesn't actually help us to identify the optimal value. What it does do is allow us to potentially exclude values and knowing this can simplify our work somewhat and so is not a bad thing to do.

Suppose that we are looking for the absolute maximum of a function and after finding the critical points we find that we have multiple critical points. Let's also suppose that we run all of them through the second derivative test and determine that some of them are in fact relative minimums of the function. Since we are after the absolute maximum we know that a maximum (of any kind) can't occur at relative minimums and so we immediately know that we can exclude these points from further consideration. We could do a similar check if we were looking for the absolute minimum. Doing this may not seem like all that great of a thing to do, but it can, on occasion, lead to a nice reduction in the about of work that we need to in later steps.

The second of way using the second derivative can be used to identify the optimal value of a function and in fact is very similar to the second method above. In fact we will have the same requirements for this method as we did in that method. We need an interval of possible optimal values, *I* and the endpoint(s) may or may not be finite. We'll also need to require that the function, $f(x)$ be continuous everywhere in *I* except possibly at the endpoints as above.

Now, suppose that $x = c$ is a critical point and that $f''(c) > 0$. The second derivative test tells us that $x = c$ must be a relative minimum of the function. Suppose however that we also knew that $f''(x) > 0$ for all *x* in *I*. In this case we would know that the function was concave up in all of *I* and that would in turn mean that the absolute minimum of $f(x)$ in *I* would in fact have to be at $x = c$.

Likewise if $x = c$ is a critical point and $f''(x) < 0$ for all *x* in *I* then we would know that the function was concave down in *I* and that the absolute maximum of $f(x)$ in *I* would have to be at $x = c$.

Here is a summary of this method.

Second Derivative Test for Absolute Extrema

Let *I* be the range of all possible optimal values of $f(x)$ and further suppose that $f(x)$ is continuous on I, except possibly at the endpoints. Finally suppose that $x = c$ is a critical point of $f(x)$ and that *c* is in the interval *I*. Then,

1. If $f''(x) > 0$ for all x in I then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval *I*.
2. If $f''(x) < 0$ for all x in I then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval *I*.

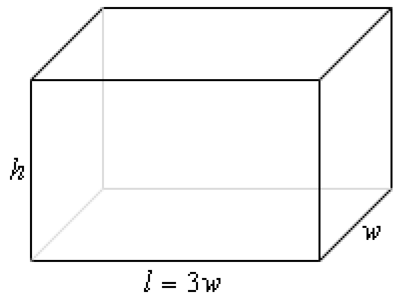
Before proceeding with some more examples we need to once again acknowledge that not every method discussed above will work for every problem and that, in some problems, more than one method will work. There are also problems were we may need to use a combination of these methods to identify the optimal value. Each problem will be different and we'll need to see what we've got once we get the critical points before we decide which method might be best to use.

Okay, let's work some more examples.

Example 2 We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50ft^3 determine the dimensions that will minimize the cost to build the box.

Solution

First, a quick figure (probably not to scale…).



We want to minimize the cost of the materials subject to the constraint that the volume must be 50ft^3. Note as well that the cost for each side is just the area of that side times the appropriate cost.

The two functions we'll be working with here this time are,

$$\begin{aligned}&\text{Minimize : } C = 10\left(2lw\right) + 6\left(2wh + 2lh\right) = 60w^2 + 48wh\\&\text{Constraint : } 50 = lwh = 3w^2h\end{aligned}$$

As with the first example, we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for *h* so let's do that.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C\left(w\right) = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w}$$

Now, let's get the first and second (we'll be needing this later…) derivatives,

$$C'\left(w\right) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \qquad\qquad C''\left(w\right) = 120 + 1600w^{-3}$$

So, it looks like we've got two critical points here. The first is obvious, $w = 0$, and it's also just as obvious that this will not be the correct value. We are building a box here and *w* is the box's width and so since it makes no sense to talk about a box with zero width we will ignore this critical point. This does not mean however that you should just get into the habit of ignoring zero when it occurs. There are other types of problems where it will be a valid point that we will need to consider.

The next critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \qquad \Rightarrow \qquad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.8821$$

So, once we throw out $w = 0$, we've got a single critical point and we now have to verify that this is in fact the value that will give the absolute minimum cost.

In this case we can't use Method 1 from above. First, the function is not continuous at one of the endpoints, $w = 0$, of our interval of possible values. Secondly, there is no theoretical upper limit to the width that will give a box with volume of 50 ft^3. If *w* is very large then we would just need to make *h* very small.

The second method listed above would work here, but that's going to involve some calculations, not difficult calculations, but more work nonetheless.

The third method however, will work quickly and simply here. First, we know that whatever the value of *w* that we get it will have to be positive and we can see second derivative above that provided $w > 0$ we will have $C''(w) > 0$ and so in the interval of possible optimal values the cost function will always be concave up and so $w = 1.8821$ must give the absolute minimum cost.

All we need to do now is to find the remaining dimensions.

$$w = 1.8821$$
$$l = 3w = 3(1.8821) = 5.6463$$
$$h = \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050$$

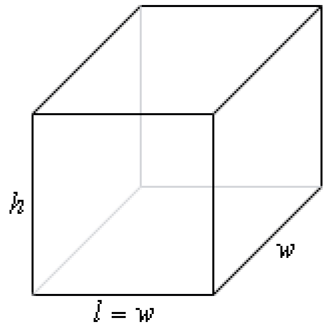
Also, even though it was not asked for, the minimum cost is : $C(1.8821) = \$637.60$.

Example 3 We want to construct a box with a square base and we only have 10 m^2 of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

Solution

This example is in many ways the exact opposite of the previous example. In this case we want to optimize the volume and the constraint this time is the amount of material used. We don't have a cost here, but if you think about it the cost is nothing more than the amount of material used times a cost and so the amount of material and cost are pretty much tied together. If you can do one you can do the other as well. Note as well that the amount of material used is really just the surface area of the box.

As always, let's start off with a quick sketch of the box.



Now, as mentioned above we want to maximize the volume and the amount of material is the constraint so here are the equations we'll need.

$$\begin{aligned}&\text{Maximize : } V = lwh = w^2 h\\ &\text{Constraint : } 10 = 2lw + 2wh + 2lh = 2w^2 + 4wh\end{aligned}$$

We'll solve the constraint for *h* and plug this into the equation for the volume.

$$h = \frac{10 - 2w^2}{4w} = \frac{5 - w^2}{2w} \qquad \Rightarrow \qquad V\left(w\right) = w^2\left(\frac{5 - w^2}{2w}\right) = \frac{1}{2}\left(5w - w^3\right)$$

Here are the first and second derivatives of the volume function.

$$V'\left(w\right) = \tfrac{1}{2}\left(5 - 3w^2\right) \qquad\qquad V''\left(w\right) = -3w$$

Note as well here that provided $w > 0$, which we know from a physical standpoint will be true, then the volume function will be concave down and so if we get a single critical point then we know that it will have to be the value that gives the absolute maximum.

Setting the first derivative equal to zero and solving gives us the two critical points,

$$w = \pm\sqrt{\frac{5}{3}} = \pm 1.2910$$

In this case we can exclude the negative critical point since we are dealing with a length of a box and we know that these must be positive. Do not however get into the habit of just excluding any negative critical point. There are problems where negative critical points are perfectly valid possible solutions.

Now, as noted above we got a single critical point, 1.2910, and so this must be the value that gives the maximum volume and since the maximum volume is all that was asked for in the problem statement the answer is then : $V\left(1.2910\right) = 2.1517\,\mathrm{m}$

Note that we could also have noted here that if $w < 1.2910$ then $V'\left(w\right) > 0$ and likewise if

$w > 1.2910$ then $V'(w) < 0$ and so if we are to the left of the critical point the volume is always increasing and if we are to the right of the critical point the volume is always decreasing and so by the Method 2 above we can also see that the single critical point must give the absolute maximum of the volume.

Finally, even though these weren't asked for here are the dimension of the box that gives the maximum volume.

$$l = w = 1.2910 \qquad\qquad h = \frac{5 - 1.2910^2}{2(1.2910)} = 1.2910$$

So, it looks like in this case we actually have a perfect cube.

In the last two examples we've seen that many of these optimization problems can in both directions so to speak. In both examples we have essentially the same two equations: volume and surface area. However, in Example 2 the volume was the constraint and the cost (which is directly related to the surface area) was the function we were trying to optimize. In Example 3, on the other hand, we were trying to optimize the volume and the surface area was the constraint.

It is important to not get so locked into one way of doing these problems that we can't do it in the opposite direction as needed as well. This is one of the more common mistakes that students make with these kinds of problems. They see one problem and then try to make every other problem that seems to be the same conform to that one solution even if the problem needs to be worked differently. Keep an open mind with these problems and make sure that you understand what is being optimized and what the constraint is before you jump into the solution.

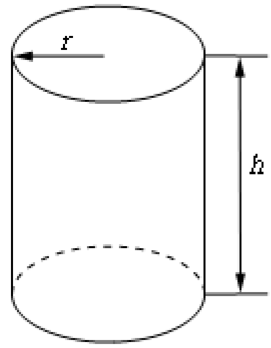
Also, as seen in the last example we used two different methods of verifying that we did get the optimal value. Do not get too locked into one method of doing this verification that you forget about the other methods.

Let's work some another example that this time doesn't involve a rectangle or box.

Example 4 A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

Solution

In this problem the constraint is the volume and we want to minimize the amount of material used. This means that what we want to minimize is the surface area of the can and we'll need to include both the walls of the can as well as the top and bottom "caps". Here is a quick sketch to get us started off.



We'll need the surface area of this can and that will be the surface area of the walls of the can (which is really just a cylinder) and the area of the top and bottom caps (which are just disks, and don't forget that there are two of them).

Note that if you think of a cylinder of height *h* and radius *r* as just a bunch of disks/circles of radius *r* stacked on top of each other the equations for the surface area and volume are pretty simple to remember. The volume is just the area of each of the disks times the height. Similarly, the surface area is just the circumference of the each circle times the. The equations for the volume and surface area of a cylinder are then,

$$V = \left(\pi r^2\right)\left(h\right) = \pi r^2 h \qquad\qquad A = \left(2\pi r\right)\left(h\right) = 2\pi r h$$

Next, we're also going to need the required volume in a better set of units than liters. Since we want length measurements for the radius and height we'll need to use the fact that 1 Liter = 1000 cm^3 to convert the 1.5 liters into 1500 cm^3. This will in turn give a radius and height in terms of centimeters.

Here are the equations that we'll need for this problem and don't forget that there two caps and so we'll need the area from each.

$$\text{Minimize} : A = 2\pi r h + 2\pi r^2$$
$$\text{Constraint} : 1500 = \pi r^2 h$$

In this case it looks like our best option is to solve the constraint for *h* and plug this into the area function.

$$h = \frac{1500}{\pi r^2} \qquad \Rightarrow \qquad A\left(r\right) = 2\pi r\left(\frac{1500}{\pi r^2}\right) + 2\pi r^2 = 2\pi r^2 + \frac{3000}{r}$$

Notice that this formula will only make sense from a physical standpoint if $r > 0$ which is a good thing as it is not defined at $r = 0$.

Next, let's get the first derivative.

$$A'\left(r\right) = 4\pi r - \frac{3000}{r^2} = \frac{4\pi r^3 - 3000}{r^2}$$

From this we can see that we have two critical points : $r = 0$ and $r = \sqrt[3]{\frac{750}{\pi}} = 6.2035$. The first critical point doesn't make sense from a physical standpoint and so we can ignore that one.

So we only have a single critical point to deal with here and notice that 6.2035 is the only value for which the derivative will be zero and hence the only place (with $r > 0$ of course) that the derivative may change sign. It's not difficult to check that if $r < 6.2035$ then $A'(r) < 0$ and likewise if $r > 6.2035$ then $A'(r) > 0$. The variant of the First Derivative Test above then tells us that the absolute minimum value of the area (for $r > 0$) must occur at $r = 6.2035$.

All we need to do this is determine height of the can and we'll be done.

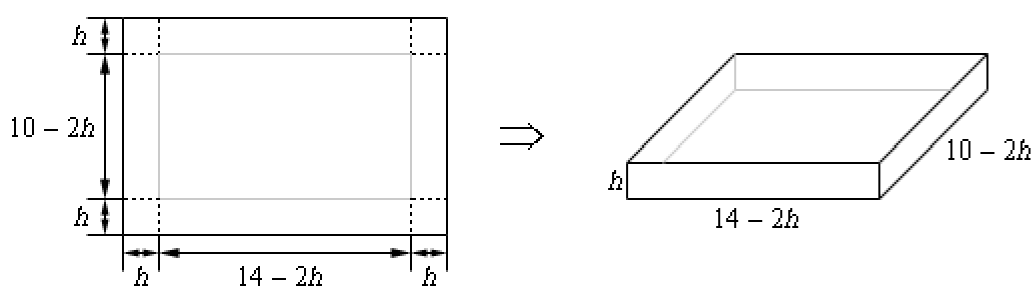
$$h = \frac{1500}{\pi(6.2035)^2} = 12.4070$$

Therefore if the manufacturer makes the can with a radius of 6.2035 cm and a height of 12.4070 cm the least amount of material will be used to make the can.

As an interesting side problem and extension to the above example you might want to show that for a given volume, *L*, the minimum material will be used if $h = 2r$ regardless of the volume of the can.

In the examples to this point we've put in quite a bit of discussion in the solution. In the remaining problems we won't be putting in quite as much discussion and leave it to you to fill in any missing details.

Example 5 We have a piece of cardboard that is 14 inches by 10 inches and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



Solution

In this example, for the first time, we've run into a problem where the constraint doesn't really have an equation. The constraint is simply the size of the piece of cardboard and has already been factored into the figure above. This will happen on occasion and so don't get excited about it when it does. This just means that we have one less equation to worry about. In this case we want to maximize the volume. Here is the volume, in terms of *h* and its first derivative.

$$V(h) = h(14-2h)(10-2h) = 140h - 48h^2 + 4h^3 \qquad\qquad V'(h) = 140 - 96h + 12h^2$$

Setting the first derivative equal to zero and solving gives the following two critical points,

$$h = \frac{12 \pm \sqrt{39}}{3} = 1.9183, \quad 6.0817$$

We now have an apparent problem. We have two critical points and we'll need to determine which one is the value we need. In this case, this is easier than it looks. Go back to the figure in the problem statement and notice that we can quite easily find limits on *h*. The smallest *h* can be is $h = 0$ even though this doesn't make much sense as we won't get a box in this case. Also from the 10 inch side we can see that the largest *h* can be is $h = 5$ although again, this doesn't make much sense physically.

So, knowing that whatever *h* is it must be in the range $0 \le h \le 5$ we can see that the second critical point is outside this range and so the only critical point that we need to worry about is 1.9183.

Finally, since the volume is defined and continuous on $0 \le h \le 5$ all we need to do is plug in the critical points and endpoints into the volume to determine which gives the largest volume. Here are those function evaluations.

$$V(0) = 0 \qquad\qquad V(1.9183) = 120.1644 \qquad\qquad V(5) = 0$$

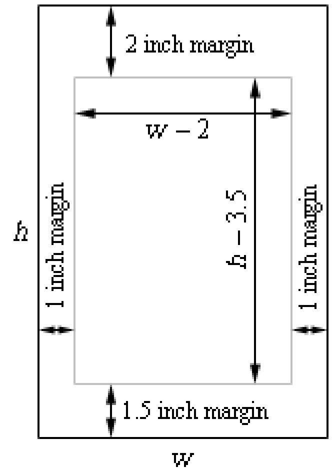
So, if we take $h = 1.9183$ we get a maximum volume.

Example 6 A printer need to make a poster that will have a total area of 200 in^2 and will have 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. What dimensions will give the largest printed area?

Solution

This problem is a little different from the previous problems. Both the constraint and the function we are going to optimize are areas. The constraint is that the overall area of the poster must be 200 in^2 while we want to optimize the printed area (*i.e.* the area of the poster with the margins taken out).

Here is a sketch of the poster and we can see that once we've taken the margins into account the width of the printed area is $w-2$ and the height of the printer area is $h-3.5$.



Here are the equations that we'll be working with.

$$\text{Maximize : } A = (w-2)(h-3.5)$$
$$\text{Constraint : } 200 = wh$$

Solving the constraint for h and plugging into the equation for the printed area gives,

$$A(w) = (w-2)\left(\frac{200}{w} - 3.5\right) = 207 - 3.5w - \frac{400}{w}$$

The first and second derivatives are,

$$A'(w) = -3.5 + \frac{400}{w^2} = \frac{400 - 3.5w^2}{w^2} \qquad\qquad A''(w) = -\frac{800}{w^3}$$

From the first derivative we have the following three critical points.

$$w = 0 \qquad\qquad w = \pm\sqrt{\tfrac{400}{3.5}} = \pm 10.6904$$

However, since we're dealing with the dimensions of a piece of paper we know that we must have $w > 0$ and so only 10.6904 will make sense.

Also notice that provided $w > 0$ the second derivative will always be negative and so in the range of possible optimal values of the width the area function is always concave down and so we know that the maximum printed area will be at $w = 10.6904 \text{ inches}$.

The height of the paper that gives the maximum printed area is then,

$$h = \frac{200}{10.6904} = 18.7084 \text{ inches}$$

We've worked quite a few examples to this point and we have quite a few more to work. However this section has gotten quite lengthy so let's continue our examples in the next section. This is being done mostly because these notes are also being presented on the web and this will help to keep the load times on the pages down somewhat.

More Optimization Problems

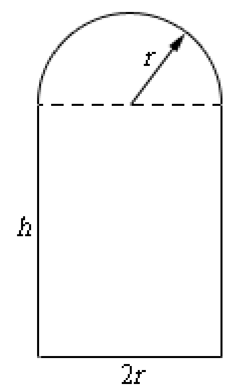
Because these notes are also being presented on the web we've broken the optimization examples up into several sections to keep the load times to a minimum. Do not forget the various methods for verifying that we have the optimal value that we looked at in the previous section. In this section we'll just use them without acknowledging so make sure you understand them and can use them. So let's get going on some more examples.

Example 1 A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing materials what must the dimensions of the window be to let in the most light?

Solution

Okay, let's ask this question again is slightly easier to understand terms. We want a window in the shape described above to have a maximum area (and hence let in the most light) and have a perimeter of 12 m (because we have 12 m of framing material). Little bit easier to understand in those terms.

Here's a sketch of the window. The height of the rectangular portion is h and because the semicircle is on top we can think of the width of the rectangular portion at $2r$.



The perimeter (our constraint) is the lengths of the three sides on the rectangular portion plus half the circumference of a circle of radius r. The area (what we want to maximize) is the area of the rectangle plus half the area of a circle of radius r. Here are the equations we'll be working with in this example.

$$\text{Maximize} : A = 2hr + \tfrac{1}{2}\pi r^2$$
$$\text{Constraint} : 12 = 2h + 2r + \pi r$$

In this case we'll solve the constraint for h and plug that into the area equation.

$$h = 6 - r - \tfrac{1}{2}\pi r \qquad \Rightarrow \qquad A(r) = 2r\left(6 - r - \tfrac{1}{2}\pi r\right) + \tfrac{1}{2}\pi r^2 = 12r - 2r^2 - \tfrac{1}{2}\pi r^2$$

The first and second derivatives are,

$$A'(r) = 12 - r(4+\pi) \qquad\qquad A''(r) = -4 - \pi$$

We can see that the only critical point is,

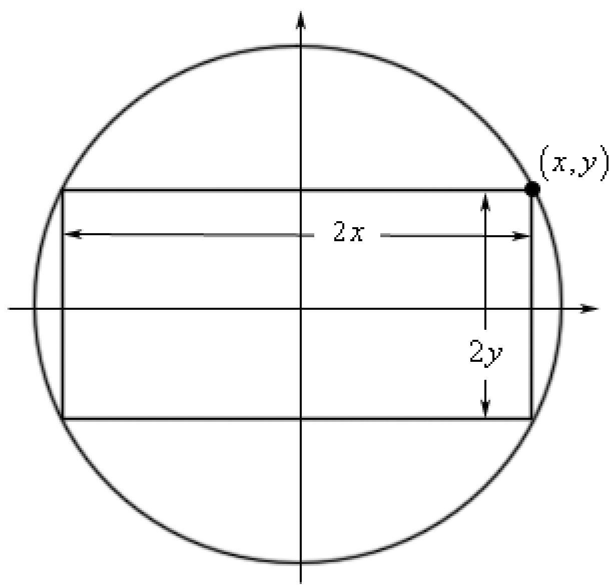
$$r = \frac{12}{4+\pi} = 1.6803$$

We can also see that the second derivative is always negative (in fact it's a constant) and so we can see that the maximum area must occur at this point. So, for the maximum area the semicircle on top must have a radius of 1.6803 and the rectangle must have the dimensions 3.3606 x 1.6803 (h x $2r$).

Example 2 Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

Solution

Huh? This problem is best described with a sketch. Here is what we're looking for.



We want the area of the largest rectangle that we can fit inside a circle and have all of its corners touching the circle.

To do this problem it's easiest to assume that the circle (and hence the rectangle) is centered at the origin. Doing this we know that the equation of the circle will be

$$x^2 + y^2 = 16$$

and that the right upper corner of the rectangle will have the coordinates (x, y). This means that the width of the rectangle will be $2x$ and the height of the rectangle will be $2y$. The area of the rectangle will then be,

$$A = (2x)(2y) = 4xy$$

So, we've got the function we want to maximize (the area), but what is the constraint? Well since the coordinates of the upper right corner must be on the circle we know that *x* and *y* must satisfy the equation of the circle. In other words, the equation of the circle is the constraint.

The first thing to do then is to solve the constraint for one of the variables.

$$y = \pm\sqrt{16 - x^2}$$

Since the point that we're looking at is in the first quadrant we know that *y* must be positive and so we can take the "+" part of this. Plugging this into the area and computing the first derivative gives,

$$A(x) = 4x\sqrt{16 - x^2}$$
$$A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = \frac{64 - 8x^2}{\sqrt{16 - x^2}}$$

Before getting the critical points let's notice that we can limit *x* to the range $0 \le x \le 4$ since we are assuming that *x* is in the first quadrant and must stay inside the circle. Now the four critical points we get (two from the numerator and two from the denominator) are,

$$16 - x^2 = 0 \qquad \Rightarrow \qquad x = \pm 4$$
$$64 - 8x^2 = 0 \qquad \Rightarrow \qquad x = \pm 2\sqrt{2}$$

We only want critical points that are in the range of possible optimal values so that means that we have two critical points to deal with : $x = 2\sqrt{2}$ and $x = 4$. Notice however that the second critical point is also one of the endpoints of our interval.

Now, area function is continuous and we have an interval of possible solution with finite endpoints so,

$$A(0) = 0 \qquad A\left(2\sqrt{2}\right) = 32 \qquad A(4) = 0$$

So, we can see that we'll get the maximum area if $x = 2\sqrt{2}$ and the corresponding value of *y* is,

$$y = \sqrt{16 - \left(2\sqrt{2}\right)^2} = \sqrt{8} = 2\sqrt{2}$$

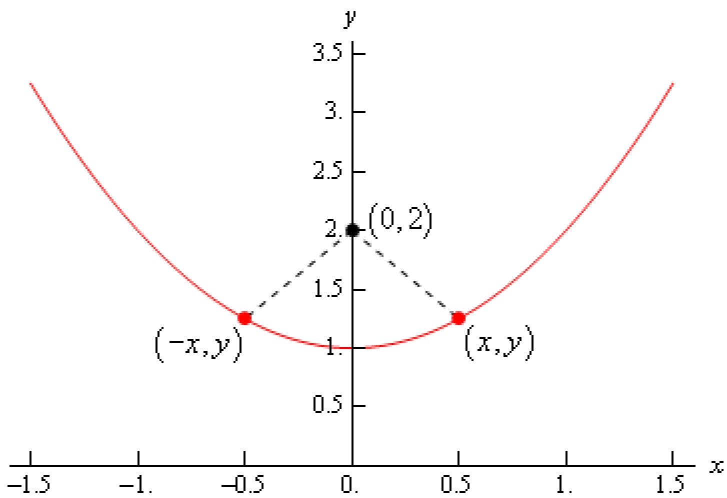
It looks like the maximum area will be found if the inscribed rectangle is in fact a square.

We need to again make a point that was made several times in the previous section. We excluded several critical points in the work above. Do not always expect to do that. There will often be physical reasons to exclude zero and/or negative critical points, however, there will be problems where these are perfectly acceptable values. You should always write down every possible critical point and then exclude any that can't be possible solutions. This keeps you in the habit of finding all the critical points and then deciding which ones you actually need and that in turn will make it less likely that you'll miss one when it is actually needed.

Example 3 Determine the point(s) on $y = x^2 + 1$ that are closest to (0,2).

Solution

Here's a quick sketch of the situation.



So, we're looking for the shortest length of the dashed line. Notice as well that if the shortest distance isn't at $x = 0$ there will be two points on the graph, as we've shown above, that will give the shortest distance. This is because the parabola is symmetric to the *y*-axis and the point in question is on the *y*-axis. This won't always be the case of course so don't always expect two points in these kinds of problems.

In this case we need to minimize the distance between the point (0,2) and any point that is one the graph (*x*,*y*). Or,

$$d = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$$

If you think about the situation here it makes sense that the point that minimizes the distance will also minimize the square of the distance and so since it will be easier to work with we will use the square of the distance and minimize that. So, the function that we're going to minimize is,

$$D = d^2 = x^2 + (y-2)^2$$

The constraint in this case is the function itself since the point must lie on the graph of the function.

At this point there are two methods for proceeding. One of which will require significantly more work than the other. Let's take a look at both of them.

Solution 1
In this case we will use the constraint in probably the most obvious way. We already have the constraint solved for *y* so let's plug that into the square of the distance and get the derivatives.

$$D(x) = x^2 + \left(x^2 + 1 - 2\right)^2 = x^4 - x^2 + 1$$
$$D'(x) = 4x^3 - 2x = 2x\left(2x^2 - 1\right)$$
$$D''(x) = 12x^2 - 2$$

So, it looks like there are three critical points for the square of the distance and notice that this time, unlike pretty much every previous example we've worked, we can't exclude zero or negative numbers. They are perfectly valid possible optimal values this time.

$$x = 0, \quad x = \pm\frac{1}{\sqrt{2}}$$

Before going any farther, let's check these in the second derivative to see if they are all relative minimums.

$$D''(0) = -2 < 0 \qquad D''\left(\frac{1}{\sqrt{2}}\right) = 4 \qquad D''\left(-\frac{1}{\sqrt{2}}\right) = 4$$

So, $x = 0$ is a relative maximum and so can't possibly be the minimum distance. That means that we've got two critical points. The question is how do we verify that these give the minimum distance and yes we did mean to say that both will give the minimum distance. Recall from our sketch above that if *x* gives the minimum distance then so will –*x* and so if gives the minimum distance then the other should as well.

None of the methods we discussed in the previous section will really work here. We don't have an interval of possible solutions with finite endpoints and both the first and second derivative change sign. In this case however, we can still verify that they are the points that give the minimum distance.

First, notice that if we are working on the interval $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ then the endpoints of this interval (which are also the critical points) are in fact where the absolute minimum of the function occurs in this interval.

Next we can see that if $x < -\frac{1}{\sqrt{2}}$ then $D'(x) < 0$. Or in other words, if $x < -\frac{1}{\sqrt{2}}$ the function is decreasing until it hits $x = -\frac{1}{\sqrt{2}}$ and so must always be larger than the function at $x = -\frac{1}{\sqrt{2}}$.

Similarly, $x > \frac{1}{\sqrt{2}}$ then $D'(x) > 0$ and so the function is always increasing to the right of $x = -\frac{1}{\sqrt{2}}$ and so must be larger than the function at $x = -\frac{1}{\sqrt{2}}$.

So, putting all of this together tells us that we do in fact have an absolute minimum at $x = \pm\frac{1}{\sqrt{2}}$.

All that we need to do is to find the value of *y* for these points.

$$x = \frac{1}{\sqrt{2}} \quad : \qquad y = \frac{3}{2}$$
$$x = -\frac{1}{\sqrt{2}} : \qquad y = \frac{3}{2}$$

So, the points on the graph that are closest to (0,2) are,

$$\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right) \qquad \left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$$

Solution 2

The first solution that we worked was actually the long solution. There is a much shorter solution to this problem. Instead of plugging *y* into the square of the distance let's plug in *x*. From the constraint we get,

$$x^2 = y - 1$$

and notice that the only place *x* show up in the square of the distance it shows up as x^2 and let's just plug this into the square of the distance. Doing this gives,

$$D(y) = y - 1 + (y-2)^2 = y^2 - 3y + 3$$
$$D'(y) = 2y - 3$$
$$D''(y) = 2$$

There is now a single critical point, $y = \frac{3}{2}$, and since the second derivative is always positive we know that this point must give the absolute minimum. So all that we need to do at this point is find the value(s) of *x* that go with this value of *y*.

$$x^2 = \frac{3}{2} - 1 = \frac{1}{2} \qquad \Rightarrow \qquad x = \pm\frac{1}{\sqrt{2}}$$

The points are then,

$$\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right) \qquad \left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$$

So, for significantly less work we got exactly the same answer.

This previous example had a couple of nice points. First, as pointed out in the problem, we couldn't exclude zero or negative critical points this time as we've done in all the previous examples. Again, be careful to not get into the habit of always excluding them as we do many of the examples we'll work.

Next, some of these problems will have multiple solution methods and sometimes one will be significantly easier than the other. The method you use is up to you and often the difficulty of any particular method is dependent upon the person doing the problem. One person may find one way easier and other person may find a different method easier.

Finally, as we saw in the first solution method sometimes we'll need to use a combination of the optimal value verification methods we discussed in the previous section.

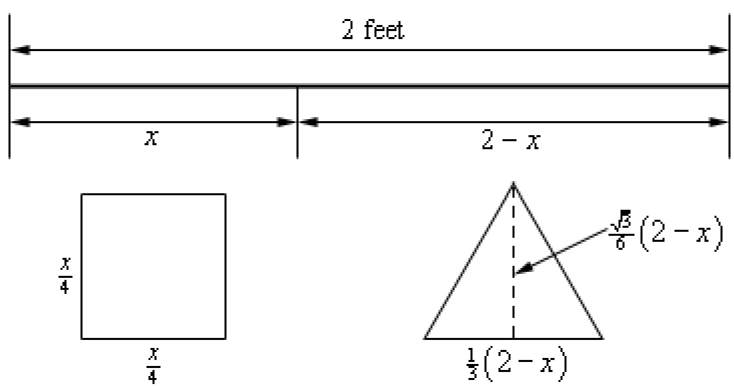
Let's work some more examples.

Example 4 A 2 feet piece of wire is cut into two pieces and once piece is bent into a square and the other is bent into an equilateral triangle. Where should the wire cut so that the total area enclosed by both is minimum and maximum?

Solution
Before starting the solution recall that an equilateral triangle is a triangle with three equal sides and each of the interior angles are $\frac{\pi}{3}$ (or $60°$).

Now, this is another problem where the constraint isn't really going to be given by an equation, it is simply that there is 2 ft of wire to work with and this will be taken into account in our work.

So, let's cut the wire into two pieces. The first piece will have length x which we'll bend into a square and each side will have length $\frac{x}{4}$. The second piece will then have length $2-x$ (we just used the constraint here…) and we'll bend this into an equilateral triangle and each side will have length $\frac{1}{3}(2-x)$. Here is a sketch of all this.



As noted in the sketch above we also will need the height of the triangle. This is easy to get if you realize that the dashed line divides the equilateral triangle into two other triangles. Let's look at the right one. The hypotenuse is $\frac{1}{3}(2-x)$ while the lower right angle is $\frac{\pi}{3}$. Finally the height is then the opposite side to the lower right angle so using basic right triangle trig we arrive at the height of the triangle as follows.

$$\sin\left(\frac{\pi}{3}\right) = \frac{opp}{hyp} \qquad \Rightarrow \qquad opp = \frac{1}{3}\left(2-x\right)\sin\left(\frac{\pi}{3}\right) = \frac{1}{3}\left(2-x\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{6}\left(2-x\right)$$

So, the total area of both objects is then,

$$A\left(x\right) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\left(2-x\right)\right)\left(\frac{\sqrt{3}}{6}\left(2-x\right)\right) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}\left(2-x\right)^2$$

Here's the first derivative of the area.

$$A'\left(x\right) = \frac{x}{8} + \frac{\sqrt{3}}{36}\left(2\right)\left(2-x\right)\left(-1\right) = \frac{x}{8} - \frac{\sqrt{3}}{9} + \frac{\sqrt{3}}{18}x$$

Setting this equal to zero and solving gives the single critical point of,

$$x = \frac{8\sqrt{3}}{9+4\sqrt{3}} = 0.8699$$

Now, let's notice that the problem statement asked for both the minimum and maximum enclosed area and we got a single critical point. This clearly can't be the answer to both, but this is not the problem that it might seem to be.

Let's notice that x must be in the range $0 \le x \le 2$ and since the area function is continuous we use the basic process for finding absolute extrema of a function.

$$A\left(0\right) = 0.1925 \qquad A\left(0.8699\right) = 0.1087 \qquad A\left(2\right) = 0.25$$

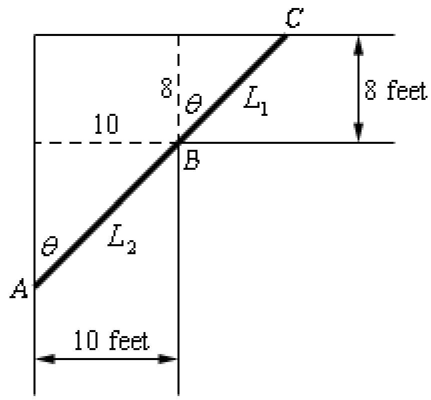
So, it looks like the minimum area will arise if we take $x = 0.8699$ while the maximum area will arise if we take the whole piece of wire and bend it into a square.

As the previous problem illustrated we can't get too locked into the answers always occurring at the critical points as they have to this point. That will often happen, but one of the extrema in the previous problem was at an endpoint and that will happen on occasion.

Example 5 A piece of pipe is being carried down a hallway that is 10 feet wide. At the end of the hallway the there is a right-angled turn and the hallway narrows down to 8 feet wide. What is the longest pipe that can be carried (always keeping it horizontal) around the turn in the hallway?

Solution

Let's start off with a sketch of the situation so we can get a grip on what's going on and how we're going to have to go about solving this.



The largest pipe that can go around the turn will do so in the position shown above. One end will be touching the outer wall of the hall way at *A* and *C* and the pipe will touch the inner corner at *B*. Let's assume that the length of the pipe in the small hallway is L_1 while L_2 is the length of the pipe in the large hallway. The pipe then has a length of $L = L_1 + L_2$.

Now, if $\theta = 0$ then the pipe is completely in the wider hallway and we can see that as $\theta \to 0$ then $L \to \infty$. Likewise, if $\theta = \frac{\pi}{2}$ the pipe is completely in the narrow hallway and as $\theta \to \frac{\pi}{2}$ we also have $L \to \infty$. So, somewhere in the interval $0 < \theta < \frac{\pi}{2}$ is an angle that will minimize *L* and oddly enough that is the length that we're after. The largest pipe that will fit around the turn will in fact be the minimum value of *L*.

The constraint for this problem is not so obvious and there are actually two of them. The constraints for this problem are the widths of the hallways. We'll use these to get an equation for *L* in terms of θ and then we'll minimize this new equation.

So, using basic right triangle trig we can see that,

$$L_1 = 8\sec\theta \qquad L_2 = 10\csc\theta \qquad \Rightarrow \qquad L = 8\sec\theta + 10\csc\theta$$

So, differentiating *L* gives,

$$L' = 8\sec\theta\tan\theta - 10\csc\theta\cot\theta$$

Setting this equal to zero and solving gives,

$$\begin{aligned} 8\sec\theta\tan\theta &= 10\csc\theta\cot\theta \\ \frac{\sec\theta\tan\theta}{\csc\theta\cot\theta} &= \frac{10}{8} \\ \frac{\sin\theta\tan^2\theta}{\cos\theta} &= \frac{5}{4} \qquad \Rightarrow \qquad \tan^3\theta = 1.25 \end{aligned}$$

Solving for θ gives,

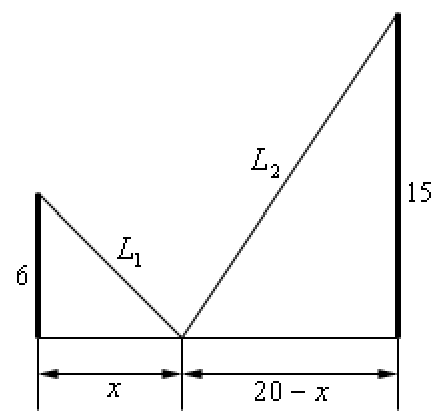
$$\tan\theta = \sqrt[3]{1.25} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\sqrt[3]{1.25}\right) = 0.8226$$

So, if $\theta = 0.8226$ radians then the pipe will have a minimum length and will just fit around the turn. Anything larger will not fit around the turn and so the largest pipe that can be carried around the turn is,

$$L = 8\sec\left(0.8226\right) + 10\csc\left(0.8226\right) = 25.4033 \text{ feet}$$

Example 6 Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?

Solution
As always let's start off with a sketch of this situation.



The total length of the wire is $L = L_1 + L_2$ and we need to determine the value of *x* that will minimize this. The constraint in this problem is that the poles must be 20 meters apart and that *x* must be in the range $0 \le x \le 20$. The first thing that we'll need to do here is to get the length of wire in terms of *x*, which is fairly simple to do using the Pythagorean Theorem.

$$L_1 = \sqrt{36 + x^2} \qquad L_2 = \sqrt{225 + \left(20 - x\right)^2} \qquad L = \sqrt{36 + x^2} + \sqrt{625 - 40x + x^2}$$

Not the nicest function we've had to work with but there it is. Note however, that it is a continuous function and we've got an interval with finite endpoints and so finding the absolute minimum won't require much more work than just getting the critical points of this function. So, let's do that. Here's the derivative.

$$L' = \frac{x}{\sqrt{36 + x^2}} + \frac{x - 20}{\sqrt{625 - 40x + x^2}}$$

Setting this equal to zero gives,

$$\frac{x}{\sqrt{36+x^2}}+\frac{x-20}{\sqrt{625-40x+x^2}}=0$$

$$x\sqrt{625-40x+x^2}=-(x-20)\sqrt{36+x^2}$$

It's probably been quite a while since you've been asked to solve something like this. To solve this we'll need to square both sides to get rid of the roots, but this will cause problems as well soon see. Let's first just square both sides and solve that equation.

$$x^2\left(625-40x+x^2\right)=\left(x-20\right)^2\left(36+x^2\right)$$

$$625x^2-40x^3+x^4=14400-1440x+436x^2-40x^3+x^4$$

$$189x^2+1440x-14400=0$$

$$9\left(3x+40\right)\left(7x-40\right)=0 \qquad \Rightarrow \qquad x=-\tfrac{40}{3},\quad x=\tfrac{40}{7}$$

Note that if you can't do that factoring done worry, you can always just use the quadratic formula and you'll get the same answers.

Okay two issues that we need to discuss briefly here. The first solution above (note that I didn't call it a critical point…) doesn't make any sense because it is negative and outside of the range of possible solutions and so we can ignore it.

Secondly, and maybe more importantly, if you were to plug $x=-\frac{40}{3}$ into the derivative you would not get zero and so is not even a critical point. How is this possible? It is a solution after all. We'll recall that we squared both sides of the equation above and it was mentioned at the time that this would cause problems. We'll we've hit those problems. In squaring both sides we've inadvertently introduced a new solution to the equation. When you do something like this you should ALWAYS go back and verify that the solutions that you are in fact solutions to the original equation. In this case we were lucky and the "bad" solution also happened to be outside the interval of solutions we were interested in but that won't always be the case.

So, if we go back and do a quick verification we can in fact see that the only critical point is $x=\frac{40}{7}=5.7143$ and this is nicely in our range of acceptable solutions.

Now all that we need to do is plug this critical point and the endpoints of the wire into the length formula and identify the one that gives the minimum value.

$$L(0)=31 \qquad L\left(\tfrac{40}{7}\right)=29 \qquad L(20)=35.8806$$

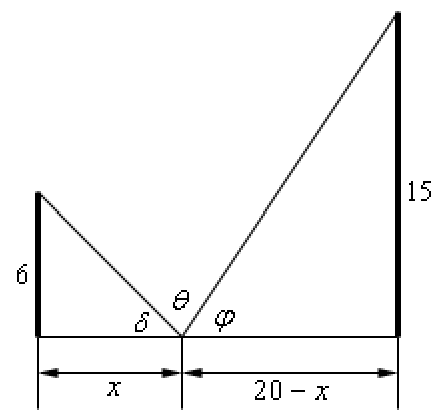
So, we will get the minimum length of wire if we stake it to the ground $\frac{40}{7}$ feet from the smaller pole.

Let's do a modification of the above problem that asks a completely different question.

Example 7 Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the angle formed by the two pieces of wire at the stake is a maximum?

Solution

Here's a sketch for this example.



The equation that we're going to need to work with here is not obvious. Let's start with the following fact.

$$\delta + \theta + \varphi = 180 = \pi$$

Note that we need to make sure that the equation is equal to π because of how we're going to work this problem. Now, basic right triangle trig tells us the following,

$$\tan\delta = \tfrac{6}{x} \qquad \Rightarrow \qquad \delta = \tan^{-1}\left(\tfrac{6}{x}\right)$$

$$\tan\varphi = \tfrac{15}{20-x} \qquad \Rightarrow \qquad \varphi = \tan^{-1}\left(\tfrac{15}{20-x}\right)$$

Plugging these into the equation above and solving for θ gives,

$$\theta = \pi - \tan^{-1}\left(\tfrac{6}{x}\right) - \tan^{-1}\left(\tfrac{15}{20-x}\right)$$

Note that this is the reason for the π in our equation. The inverse tangents give angles that are in radians and so can't use the 180 that we're used to in this kind of equation.

Next we'll need the derivative so hopefully you'll recall how to differentiate inverse tangents.

$$\theta' = -\frac{1}{1+\left(\frac{6}{x}\right)^2}\left(-\frac{6}{x^2}\right) - \frac{1}{1+\left(\frac{15}{20-x}\right)^2}\left(\frac{15}{\left(20-x\right)^2}\right)$$
$$= \frac{6}{x^2+36} - \frac{15}{\left(20-x\right)^2+225}$$
$$= \frac{6}{x^2+36} - \frac{15}{x^2-40x+625} = \frac{-3\left(3x^2+8x-1070\right)}{\left(x^2+36\right)\left(x^2-40x+625\right)}$$

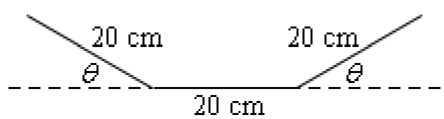
Setting this equal to zero and solving give the following two critical points.

$$x = \frac{-4\pm\sqrt{3226}}{3} = -20.2660, \quad 17.5993$$

The first critical point is not in the interval of possible solutions and so we can exclude it.

It's not difficult to show that if $0 \le x \le 17.5993$ that $\theta' > 0$ and if $17.5993 \le x \le 20$ that $\theta' < 0$ and so when $x = 17.5993$ we will get the maximum value of θ.

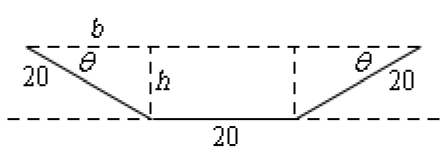
Example 8 A trough for holding water is be formed by taking a piece of sheet metal 60 cm wide and folding the 20 cm on either end up as shown below. Determine the angle θ that will maximize the amount of water that the trough can hold.



Solution

Now, in this case we are being asked to maximize the volume that a trough can hold, but if you think about it the volume of a trough in this shape is nothing more than the cross-sectional area times the length of the trough. So for a given length in order to maximize the volume all you really need to do is maximize the cross-sectional area.

To get a formula for the cross-sectional area let's redo the sketch above a little.



We can think of the cross-sectional area as a rectangle in the middle with width 20 and height *h* and two identical triangles on either end with height *h*, base *b* and hypotenuse 20. Also note that basic geometry tells us that the angle between the hypotenuse and the base must also be the same angle θ that we had in our original sketch.

Also, basic right triangle trig tells us that the base and height can be written as,

$$b = 20\cos\theta \qquad\qquad h = 20\sin\theta$$

The cross-sectional area for the whole trough, in terms of θ, is then,

$$A = 20h + 2\left(\tfrac{1}{2}bh\right) = 400\sin\theta + \left(20\cos\theta\right)\left(20\sin\theta\right) = 400\left(\sin\theta + \sin\theta\cos\theta\right)$$

The derivative of the area is,

$$\begin{aligned} A'\left(\theta\right) &= 400\left(\cos\theta + \cos^2\theta - \sin^2\theta\right) \\ &= 400\left(\cos\theta + \cos^2\theta - \left(1 - \cos^2\theta\right)\right) \\ &= 400\left(2\cos^2\theta + \cos\theta - 1\right) \\ &= 400\left(2\cos\theta - 1\right)\left(\cos\theta + 1\right) \end{aligned}$$

So, we have either,

$$\begin{array}{ccccc} 2\cos\theta - 1 = 0 & \Rightarrow & \cos\theta = \tfrac{1}{2} & \Rightarrow & \theta = \tfrac{\pi}{3} \\ \cos\theta + 1 = 0 & \Rightarrow & \cos\theta = -1 & \Rightarrow & \theta = \pi \end{array}$$

However, we can see that θ must be in the interval $0 \le \theta \le \frac{\pi}{2}$ or we won't get a trough in the proper shape. Therefore, the second critical point makes no sense and also note that we don't need to add on the standard "$+2\pi n$" for the same reason.

Finally, since the equation for the area is continuous all we need to do is plug in the critical point and the end points to find the one that gives the maximum area.

$$A\left(0\right) = 0 \qquad A\left(\tfrac{\pi}{3}\right) = 519.6152 \qquad A\left(\tfrac{\pi}{2}\right) = 400$$

So, we will get a maximum cross-sectional area, and hence a maximum volume, when $\theta = \frac{\pi}{3}$.

Indeterminate Forms and L'Hospital's Rule

Back in the chapter on Limits we saw methods for dealing with the following limits.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} \qquad\qquad \lim_{x \to \infty} \frac{4x^2 - 5x}{1 - 3x^2}$$

In the first limit if we plugged in $x = 4$ we would get 0/0 and in the second limit if we "plugged" in infinity we would get $\infty / -\infty$ (recall that as x goes to infinity a polynomial will behave in the same fashion that it's largest power behaves). Both of these are called **indeterminate forms**. In both of these cases there are competing interests or rules and it's not clear which will win out.

In the case of 0/0 we typically think of a fraction that has a numerator of zero as being zero. However, we also tend to think of fractions in which the denominator is going to zero as infinity or might not exist at all. Likewise, we tend to think of a fraction in which the numerator and denominator are the same as one. So, which will win out? Or will neither win out and they all "cancel out" and the limit will reach some other value?

In the case of $\infty / -\infty$ we have a similar set of problems. If the numerator of a fraction is going to infinity we tend to think of the whole fraction going to infinity. Also if the denominator is going to infinity we tend to think of the fraction as going to zero. We also have the case of a fraction in which the number and denominator are the same (ignoring the minus sign) and so we might get -1. Again, it's not clear which of these will win out, if any of them will win out.

With the second limit there is the further problem that infinity isn't really a number and so we really shouldn't even treat it like a number. Much of the time it simply won't behave as we would expect it to if it was a number. To look a little more into this check out the Types of Infinity section in the Extras chapter at the end of this document.

This is the problem with indeterminate forms. It's just not clear what is happening in the limit. There are other types of indeterminate forms as well. Some other types are,

$$(0)(\pm\infty) \qquad 1^{\infty} \qquad 0^0 \qquad \infty^0 \qquad \infty - \infty$$

These all have competing interests or rules that tell us what should happen and it's just not clear which, if any, of the interests or rules will win out. The topic of this section is how to deal with these kinds of limits.

As already pointed out we do know how to deal with some kinds of indeterminate forms already. For the two limits above we work them as follows.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4) = 8$$