

### ***Area Problem***

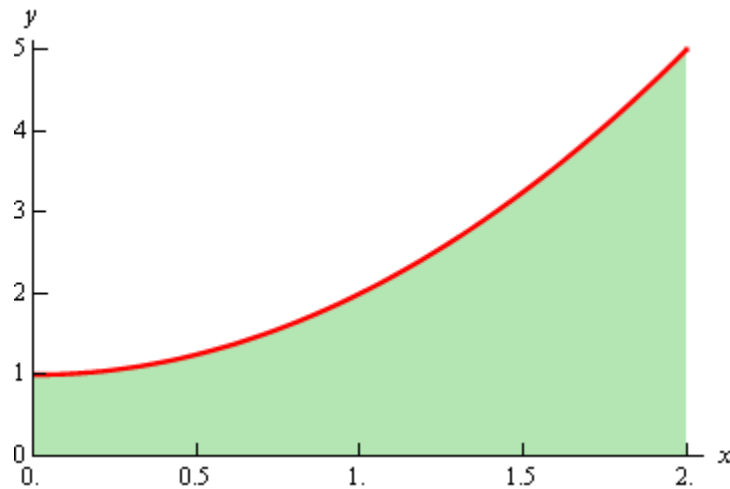
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As noted in the first section of this section there are two kinds of integrals and to this point we've looked at indefinite integrals. It is now time to start thinking about the second kind of integral : Definite Integrals. However, before we do that we're going to take a look at the Area Problem. The area problem is to definite integrals what the tangent and rate of change problems are to derivatives.

The area problem will give us one of the interpretations of a definite integral and it will lead us to the definition of the definite integral.

To start off we are going to assume that we've got a function  $f(x)$  that is positive on some interval  $[a,b]$ . What we want to do is determine the area of the region between the function and the  $x$ -axis.

It's probably easiest to see how we do this with an example. So let's determine the area between  $f(x) = x^2 + 1$  on  $[0,2]$ . In other words, we want to determine the area of the shaded region below.

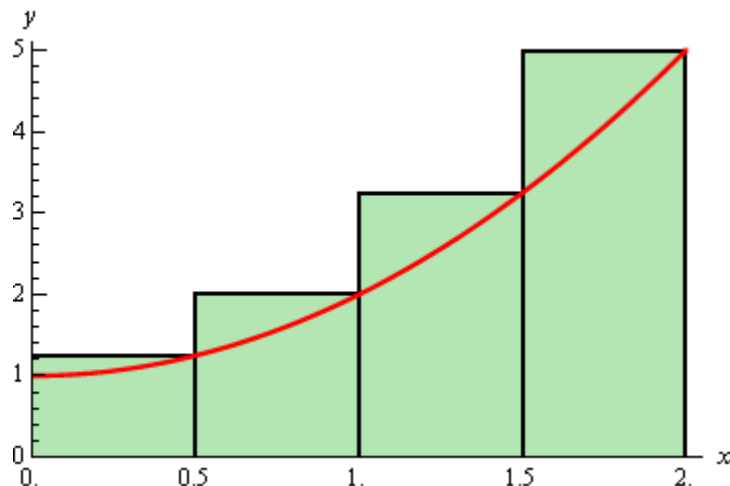


Now, at this point, we can't do this exactly. However, we can estimate the area. We will estimate the area by dividing up the interval into  $n$  subintervals each of width,

$$\Delta x = \frac{b-a}{n}$$

Then in each interval we can form a rectangle whose height is given by the function value at a specific point in the interval. We can then find the area of each of these rectangles, add them up and this will be an estimate of the area.

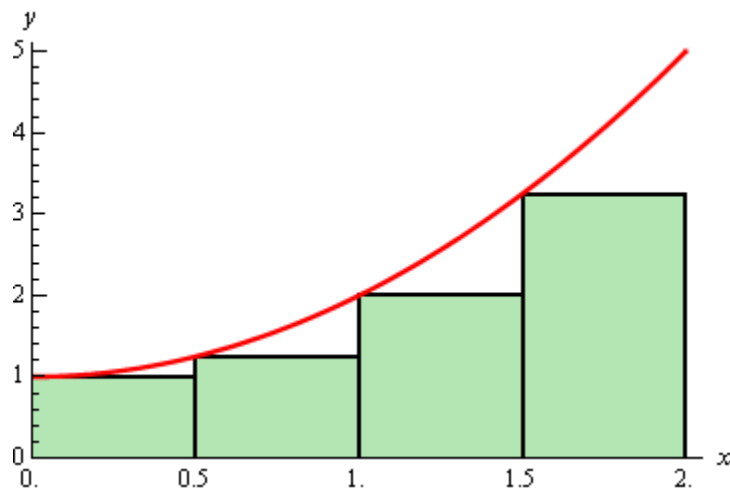
It's probably easier to see this with a sketch of the situation. So, let's divide up the interval into 4 subintervals and use the function value at the right endpoint of each interval to define the height of the rectangle. This gives,



Note that by choosing the height as we did each of the rectangles will over estimate the area since each rectangle takes in more area than the graph each time. Now let's estimate the area. First, the width of each of the rectangles is  $\frac{1}{2}$ . The height of each rectangle is determined by the function value at the right endpoint and so the height of each rectangle is nothing more than the function value at the right endpoint. Here is the estimated area.

$$\begin{aligned} A_r &= \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f(2) \\ &= \frac{1}{2} \left(\frac{5}{4}\right) + \frac{1}{2} (2) + \frac{1}{2} \left(\frac{13}{4}\right) + \frac{1}{2} (5) \\ &= 5.75 \end{aligned}$$

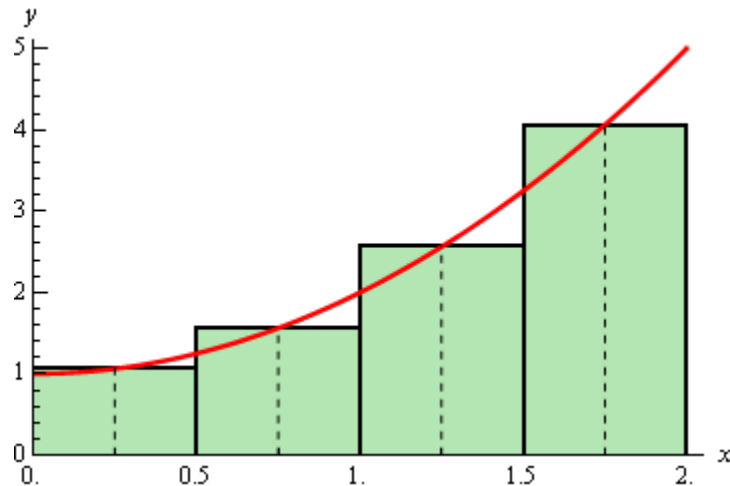
Of course taking the rectangle heights to be the function value at the right endpoint is not our only option. We could have taken the rectangle heights to be the function value at the left endpoint. Using the left endpoints as the heights of the rectangles will give the following graph and estimated area.



$$\begin{aligned}
 A_l &= \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) \\
 &= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) \\
 &= 3.75
 \end{aligned}$$

In this case we can see that the estimation will be an underestimation since each rectangle misses some of the area each time.

There is one more common point for getting the heights of the rectangles that is often more accurate. Instead of using the right or left endpoints of each sub interval we could take the midpoint of each subinterval as the height of each rectangle. Here is the graph for this case.



So, it looks like each rectangle will over and under estimate the area. This means that the approximation this time should be much better than the previous two choices of points. Here is the estimation for this case.

$$\begin{aligned}
 A_m &= \frac{1}{2}f\left(\frac{1}{4}\right) + \frac{1}{2}f\left(\frac{3}{4}\right) + \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f\left(\frac{7}{4}\right) \\
 &= \frac{1}{2}\left(\frac{17}{16}\right) + \frac{1}{2}\left(\frac{25}{16}\right) + \frac{1}{2}\left(\frac{41}{16}\right) + \frac{1}{2}\left(\frac{65}{16}\right) \\
 &= 4.625
 \end{aligned}$$

We've now got three estimates. For comparison's sake the exact area is

$$A = \frac{14}{3} = 4.\overline{666}$$

So, both the right and left endpoint estimation did not do all that great of a job at the estimation. The midpoint estimation however did quite well.

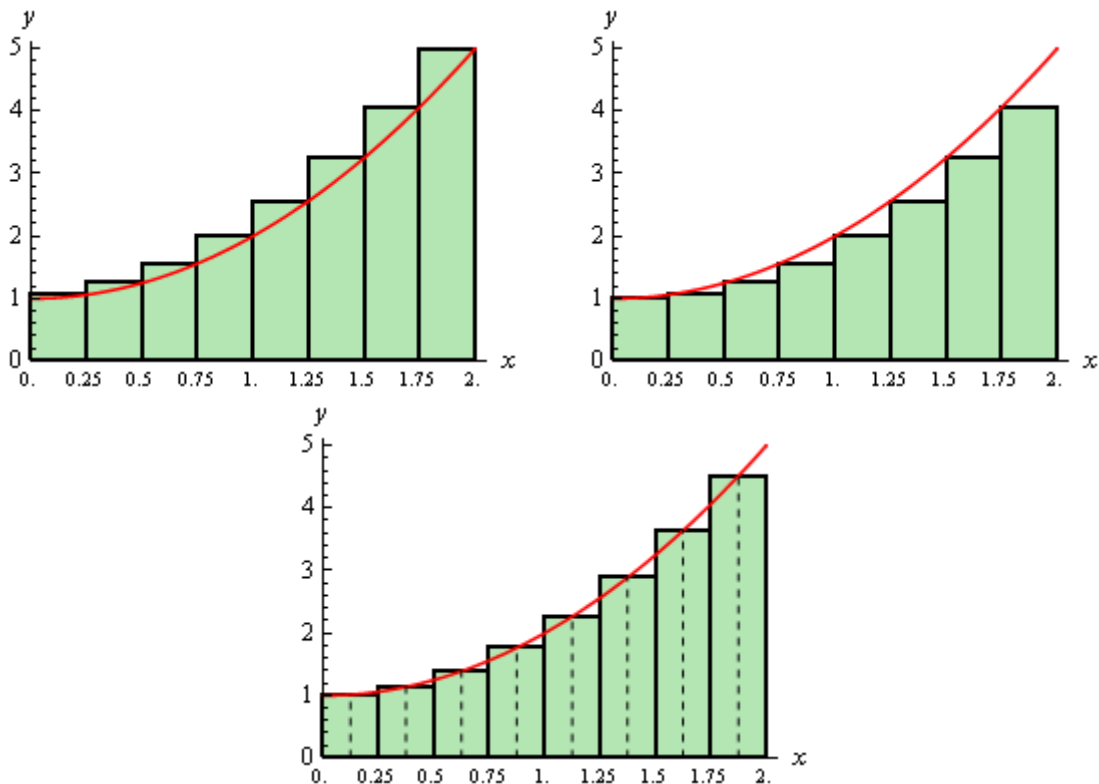
Be careful to not draw any conclusion about how choosing each of the points will affect our estimation. In this case, because we are working with an increasing function choosing the right endpoints will overestimate and choosing left endpoint will underestimate.

If we were to work with a decreasing function we would get the opposite results. For decreasing functions the right endpoints will underestimate and the left endpoints will overestimate.

Also, if we had a function that both increased and decreased in the interval we would, in all likelihood, not even be able to determine if we would get an overestimation or underestimation.

Now, let's suppose that we want a better estimation, because none of the estimations above really did all that great of a job at estimating the area. We could try to find a different point to use for the height of each rectangle but that would be cumbersome and there wouldn't be any guarantee that the estimation would in fact be better. Also, we would like a method for getting better approximations that would work for any function we would chose to work with and if we just pick new points that may not work for other functions.

The easiest way to get a better approximation is to take more rectangles (*i.e.* increase  $n$ ). Let's double the number of rectangles that we used and see what happens. Here are the graphs showing the eight rectangles and the estimations for each of the three choices for rectangle heights that we used above.



Here are the area estimations for each of these cases.

$$A_r = 5.1875$$

$$A_l = 4.1875$$

$$A_m = 4.65625$$

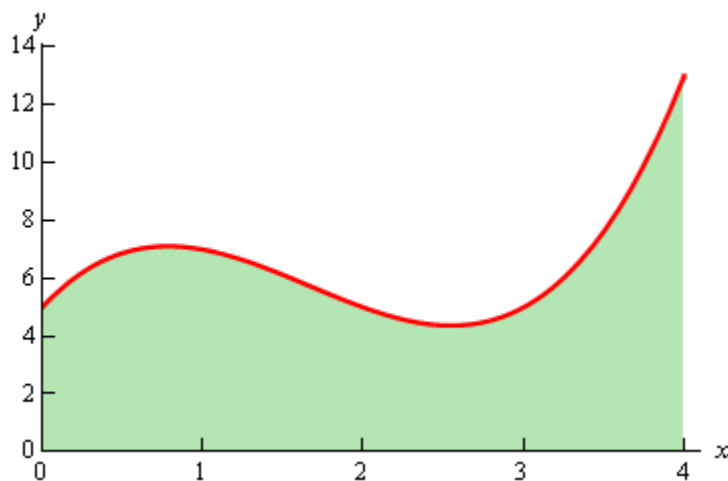
So, increasing the number of rectangles did improve the accuracy of the estimation as we'd guessed that it would.

Let's work a slightly more complicated example.

**Example 1** Estimate the area between  $f(x) = x^3 - 5x^2 + 6x + 5$  and the  $x$ -axis using  $n = 5$  subintervals and all three cases above for the heights of each rectangle.

**Solution**

First, let's get the graph to make sure that the function is positive.



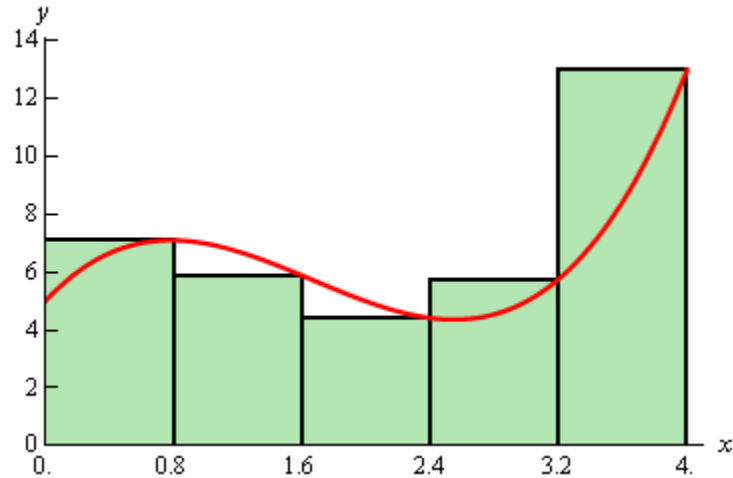
So, the graph is positive and the width of each subinterval will be,

$$\Delta x = \frac{4}{5} = 0.8$$

This means that the endpoints of the subintervals are,

$$0, 0.8, 1.6, 2.4, 3.2, 4$$

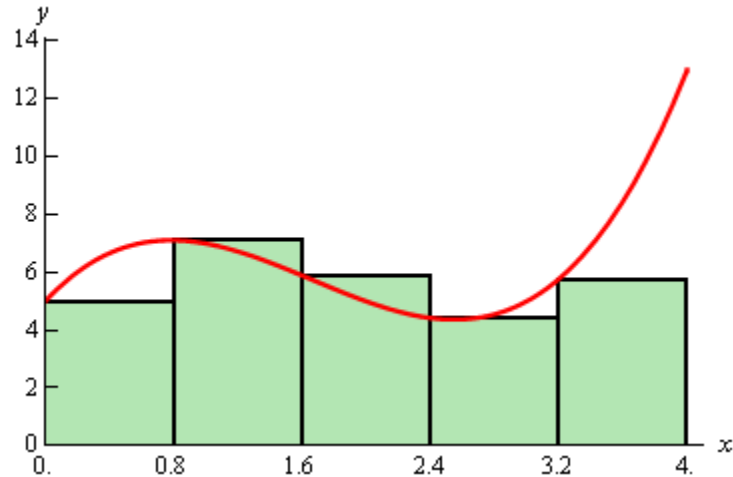
Let's first look at using the right endpoints for the function height. Here is the graph for this case.



Notice, that unlike the first area we looked at, the choosing the right endpoints here will both over and underestimate the area depending on where we are on the curve. This will often be the case with a more general curve that the one we initially looked at. The area estimation using the right endpoints of each interval for the rectangle height is,

$$\begin{aligned} A_r &= 0.8f(0.8) + 0.8f(1.6) + 0.8f(2.4) + 0.8f(3.2) + 0.8f(4) \\ &= 28.96 \end{aligned}$$

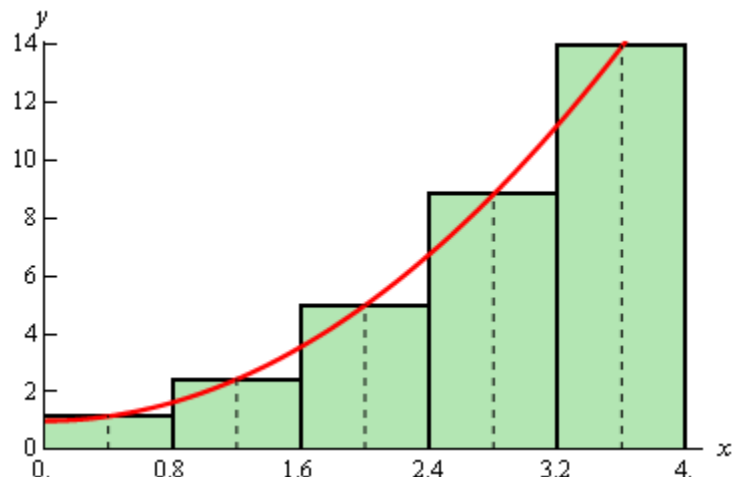
Now let's take a look at left endpoints for the function height. Here is the graph.



The area estimation using the left endpoints of each interval for the rectangle height is,

$$\begin{aligned} A_r &= 0.8f(0) + 0.8f(0.8) + 0.8f(1.6) + 0.8f(2.4) + 0.8f(3.2) \\ &= 22.56 \end{aligned}$$

Finally, let's take a look at the midpoints for the heights of each rectangle. Here is the graph,



The area estimation using the midpoint is then,

$$\begin{aligned} A_r &= 0.8f(0.4) + 0.8f(1.2) + 0.8f(2.0) + 0.8f(2.8) + 0.8f(3.6) \\ &= 25.12 \end{aligned}$$

For comparison purposes the exact area is,

$$A = \frac{76}{3} = 25.33\bar{3}$$

So, again the midpoint did a better job than the other two. While this will be the case more often than not, it won't always be the case and so don't expect this to always happen.

Now, let's move on to the general case. Let's start out with  $f(x) \geq 0$  on  $[a, b]$  and we'll divide the interval into  $n$  subintervals each of length,

$$\Delta x = \frac{b-a}{n}$$

Note that the subintervals don't have to be equal length, but it will make our work significantly easier. The endpoints of each subinterval are,

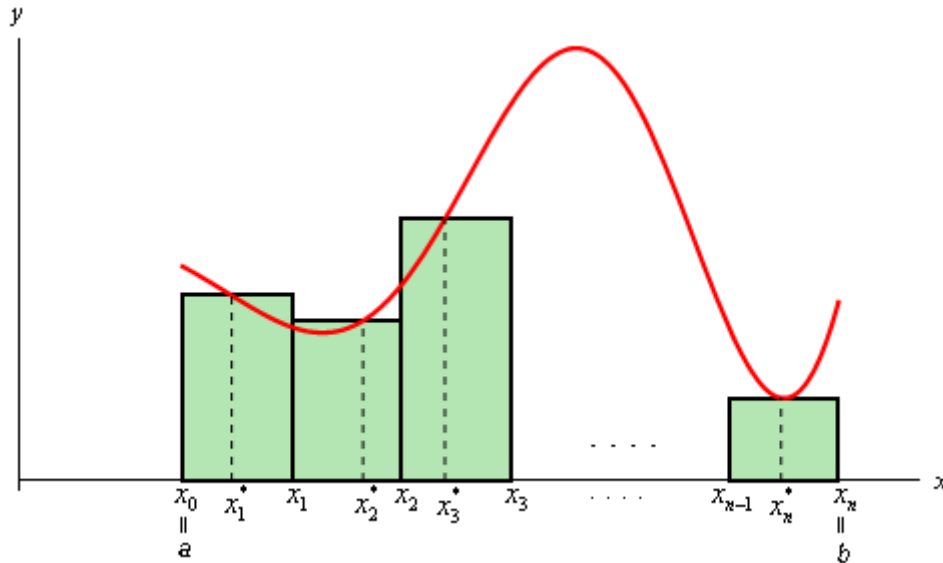
$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_i &= a + i\Delta x \\ &\vdots \\ x_{n-1} &= a + (n-1)\Delta x \\ x_n &= a + n\Delta x = b \end{aligned}$$

Next in each interval,

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

we choose a point  $x_1^*, x_2^*, \dots, x_i^*, \dots, x_n^*$ . These points will define the height of the rectangle in each subinterval. Note as well that these points do not have to occur at the same point in each subinterval.

Here is a sketch of this situation.



The area under the curve on the given interval is then approximately,

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_i^*)\Delta x + \dots + f(x_n^*)\Delta x$$

We will use **summation notation** or **sigma notation** at this point to simplify up our notation a little. If you need a refresher on summation notation check out the [section](#) devoted to this in the Extras chapter.

Using summation notation the area estimation is,

$$A \approx \sum_{i=1}^n f(x_i^*)\Delta x$$

The summation in the above equation is called a **Riemann Sum**.

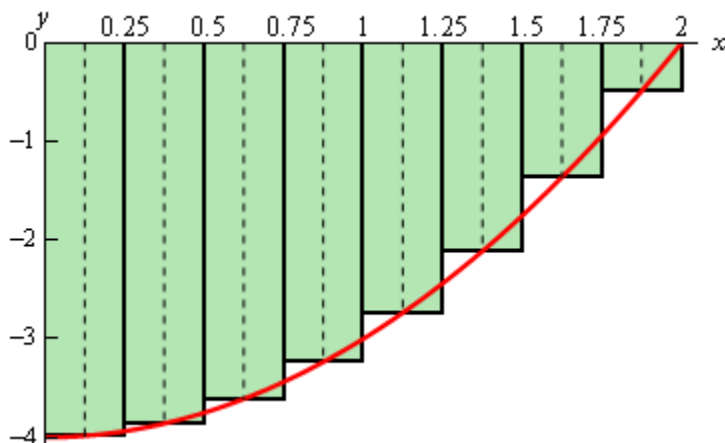
To get a better estimation we will take  $n$  larger and larger. In fact, if we let  $n$  go out to infinity we will get the exact area. In other words,

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

Before leaving this section let's address one more issue. To this point we've required the function to be positive in our work. Many functions are not positive however. Consider the case



of  $f(x) = x^2 - 4$  on  $[0,2]$ . If we use  $n = 8$  and the midpoints for the rectangle height we get the following graph,

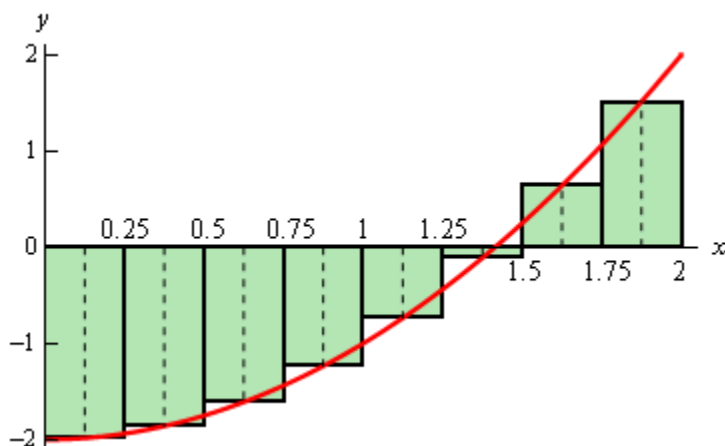


In this case let's notice that the function lies completely below the  $x$ -axis and hence is always negative. If we ignore the fact that the function is always negative and use the same ideas above to estimate the area between the graph and the  $x$ -axis we get,

$$\begin{aligned}
 A_m &= \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) + \frac{1}{4} f\left(\frac{7}{8}\right) + \frac{1}{4} f\left(\frac{9}{8}\right) + \\
 &\quad \frac{1}{4} f\left(\frac{11}{8}\right) + \frac{1}{4} f\left(\frac{13}{8}\right) + \frac{1}{4} f\left(\frac{15}{8}\right) \\
 &= -5.34375
 \end{aligned}$$

Our answer is negative as we might have expected given that all the function evaluations are negative.

So, using the technique in this section it looks like if the function is above the  $x$ -axis we will get a positive area and if the function is below the  $x$ -axis we will get a negative area. Now, what about a function that is both positive and negative in the interval? For example,  $f(x) = x^2 - 2$  on  $[0,2]$ . Using  $n = 8$  and midpoints the graph is,



Some of the rectangles are below the  $x$ -axis and so will give negative areas while some are above the  $x$ -axis and will give positive areas. Since more rectangles are below the  $x$ -axis than above it looks like we should probably get a negative area estimation for this case. In fact that is correct. Here the area estimation for this case.

$$\begin{aligned}
 A_m &= \frac{1}{4}f\left(\frac{1}{8}\right) + \frac{1}{4}f\left(\frac{3}{8}\right) + \frac{1}{4}f\left(\frac{5}{8}\right) + \frac{1}{4}f\left(\frac{7}{8}\right) + \frac{1}{4}f\left(\frac{9}{8}\right) + \\
 &\quad \frac{1}{4}f\left(\frac{11}{8}\right) + \frac{1}{4}f\left(\frac{13}{8}\right) + \frac{1}{4}f\left(\frac{15}{8}\right) \\
 &= -1.34375
 \end{aligned}$$

In cases where the function is both above and below the  $x$ -axis the technique given in the section will give the *net* area between the function and the  $x$ -axis with areas below the  $x$ -axis negative and areas above the  $x$ -axis positive. So, if the net area is negative then there is more area under the  $x$ -axis than above while a positive net area will mean that more of the area is above the  $x$ -axis.

## ***The Definition of the Definite Integral***

In this section we will formally define the definite integral and give many of the properties of definite integrals. Let's start off with the definition of a definite integral.

### **Definite Integral**

Given a function  $f(x)$  that is continuous on the interval  $[a, b]$  we divide the interval into  $n$  subintervals of equal width,  $\Delta x$ , and from each interval choose a point,  $x_i^*$ . Then the **definite integral of  $f(x)$  from  $a$  to  $b$**  is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

The definite integral is defined to be exactly the limit and summation that we looked at in the last section to find the net area between a function and the  $x$ -axis. Also note that the notation for the definite integral is very similar to the notation for an indefinite integral. The reason for this will be apparent eventually.

There is also a little bit of terminology that we should get out of the way here. The number " $a$ " that is at the bottom of the integral sign is called the **lower limit** of the integral and the number " $b$ " at the top of the integral sign is called the **upper limit** of the integral. Also, despite the fact that  $a$  and  $b$  were given as an interval the lower limit does not necessarily need to be smaller than the upper limit. Collectively we'll often call  $a$  and  $b$  the **interval of integration**.

Let's work a quick example. This example will use many of the properties and facts from the brief review of [summation notation](#) in the Extras chapter.

**Example 1** Using the definition of the definite integral compute the following.

$$\int_0^2 x^2 + 1 dx$$

#### **Solution**

First, we can't actually use the definition unless we determine which points in each interval that we'll use for  $x_i^*$ . In order to make our life easier we'll use the right endpoints of each interval.

From the previous section we know that for a general  $n$  the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$\left[0, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{4}{n}\right], \left[\frac{4}{n}, \frac{6}{n}\right], \dots, \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right], \dots, \left[\frac{2(n-1)}{n}, 2\right]$$

As we can see the right endpoint of the  $i^{\text{th}}$  subinterval is

$$x_i^* = \frac{2i}{n}$$

The summation in the definition of the definite integral is then,

$$\begin{aligned} \sum_{i=1}^n f(x_i^*) \Delta x &= \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left( \left(\frac{2i}{n}\right)^2 + 1 \right) \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left( \frac{8i^2}{n^3} + \frac{2}{n} \right) \end{aligned}$$

Now, we are going to have to take a limit of this. That means that we are going to need to “evaluate” this summation. In other words, we are going to have to use the formulas given in the [summation notation](#) review to eliminate the actual summation and get a formula for this for a general  $n$ .

To do this we will need to recognize that  $n$  is a constant as far as the summation notation is concerned. As we cycle through the integers from 1 to  $n$  in the summation only  $i$  changes and so anything that isn't an  $i$  will be a constant and can be factored out of the summation. In particular any  $n$  that is in the summation can be factored out if we need to.

Here is the summation “evaluation”.

$$\begin{aligned} \sum_{i=1}^n f(x_i^*) \Delta x &= \sum_{i=1}^n \frac{8i^2}{n^3} + \sum_{i=1}^n \frac{2}{n} \\ &= \frac{8}{n^3} \sum_{i=1}^n i^2 + \frac{1}{n} \sum_{i=1}^n 2 \\ &= \frac{8}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right) + \frac{1}{n} (2n) \\ &= \frac{4(n+1)(2n+1)}{3n^2} + 2 \\ &= \frac{14n^2 + 12n + 4}{3n^2} \end{aligned}$$

We can now compute the definite integral.

$$\begin{aligned}\int_0^2 x^2 + 1 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \lim_{n \rightarrow \infty} \frac{14n^2 + 12n + 4}{3n^2} \\ &= \frac{14}{3}\end{aligned}$$

We've seen several methods for dealing with the limit in this problem so I'll leave it to you to verify the results.

Wow, that was a lot of work for a fairly simple function. There is a much simpler way of evaluating these and we will get to it eventually. The main purpose to this section is to get the main properties and facts about the definite integral out of the way. We'll discuss how we compute these in practice starting with the next section.

So, let's start taking a look at some of the properties of the definite integral.

### Properties

1.  $\int_a^b f(x) dx = -\int_b^a f(x) dx$ . We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.
2.  $\int_a^a f(x) dx = 0$ . If the upper and lower limits are the same then there is no work to do, the integral is zero.
3.  $\int_a^b cf(x) dx = c \int_a^b f(x) dx$ , where  $c$  is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.
4.  $\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$ . We can break up definite integrals across a sum or difference.
5.  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$  where  $c$  is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals,  $[a, c]$  and  $[c, b]$ . Note however that  $c$  doesn't need to be between  $a$  and  $b$ .
6.  $\int_a^b f(x) dx = \int_a^b f(t) dt$ . The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

See the [Proof of Various Integral Properties](#) section of the Extras chapter for the proof of properties 1 – 4. Property 5 is not easy to prove and so is not shown there. Property is not really a property in the full sense of the word. It is only here to acknowledge that as long as the function and limits are the same it doesn't matter what letter we use for the variable. The answer will be the same.

Let's do a couple of examples dealing with these properties.

**Example 2** Use the results from the first example to evaluate each of the following.

(a)  $\int_2^0 x^2 + 1 dx$  [\[Solution\]](#)

(b)  $\int_0^2 10x^2 + 10 dx$  [\[Solution\]](#)

(c)  $\int_0^2 t^2 + 1 dt$  [\[Solution\]](#)

**Solution**

All of the solutions to these problems will rely on the fact we proved in the first example. Namely that,

$$\int_0^2 x^2 + 1 dx = \frac{14}{3}$$

(a)  $\int_2^0 x^2 + 1 dx$

In this case the only difference between the two is that the limits have interchanged. So, using the first property gives,

$$\begin{aligned} \int_2^0 x^2 + 1 dx &= -\int_0^2 x^2 + 1 dx \\ &= -\frac{14}{3} \end{aligned}$$

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(b)  $\int_0^2 10x^2 + 10 dx$

For this part notice that we can factor a 10 out of both terms and then out of the integral using the third property.

$$\begin{aligned} \int_0^2 10x^2 + 10 dx &= \int_0^2 10(x^2 + 1) dx \\ &= 10 \int_0^2 x^2 + 1 dx \\ &= 10 \left( \frac{14}{3} \right) \\ &= \frac{140}{3} \end{aligned}$$

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$$(c) \int_0^2 t^2 + 1 dt$$

In this case the only difference is the letter used and so this is just going to use property 6.

$$\int_0^2 t^2 + 1 dt = \int_0^2 x^2 + 1 dx = \frac{14}{3}$$

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Here are a couple of examples using the other properties.

**Example 3** Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x \sin(x) + \cos(x)}{x^2 + 1} dx$$

**Solution**

There really isn't anything to do with this integral once we notice that the limits are the same. Using the second property this is,

$$\int_{130}^{130} \frac{x^3 - x \sin(x) + \cos(x)}{x^2 + 1} dx = 0$$

**Example 4** Given that  $\int_6^{-10} f(x) dx = 23$  and  $\int_{-10}^6 g(x) dx = -9$  determine the value of

$$\int_{-10}^6 2f(x) - 10g(x) dx$$

**Solution**

We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\begin{aligned} \int_{-10}^6 2f(x) - 10g(x) dx &= \int_{-10}^6 2f(x) dx - \int_{-10}^6 10g(x) dx \\ &= 2 \int_{-10}^6 f(x) dx - 10 \int_{-10}^6 g(x) dx \end{aligned}$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above (and adding a minus sign of course). Once this is done we can plug in the known values of the integrals.

$$\begin{aligned} \int_{-10}^6 2f(x) - 10g(x) dx &= -2 \int_6^{-10} f(x) dx - 10 \int_{-10}^6 g(x) dx \\ &= -2(23) - 10(-9) \\ &= 44 \end{aligned}$$

**Example 5** Given that  $\int_{12}^{-10} f(x) dx = 6$ ,  $\int_{100}^{-10} f(x) dx = -2$ , and  $\int_{100}^{-5} f(x) dx = 4$  determine the value of  $\int_{-5}^{12} f(x) dx$ .

**Solution**

This example is mostly an example of property 5 although there are a couple of uses of property 1 in the solution as well.

We need to figure out how to correctly break up the integral using property 5 to allow us to use the given pieces of information. First we'll note that there is an integral that has a "-5" in one of the limits. It's not the lower limit, but we can use property 1 to correct that eventually. The other limit is 100 so this is the number  $c$  that we'll use in property 5.

$$\int_{-5}^{12} f(x) dx = \int_{-5}^{100} f(x) dx + \int_{100}^{12} f(x) dx$$

We'll be able to get the value of the first integral, but the second still isn't in the list of known integrals. However, we do have second limit that has a limit of 100 in it. The other limit for this second integral is -10 and this will be  $c$  in this application of property 5.

$$\int_{-5}^{12} f(x) dx = \int_{-5}^{100} f(x) dx + \int_{100}^{-10} f(x) dx + \int_{-10}^{12} f(x) dx$$

At this point all that we need to do is use the property 1 on the first and third integral to get the limits to match up with the known integrals. After that we can plug in for the known integrals.

$$\begin{aligned} \int_{-5}^{12} f(x) dx &= -\int_{100}^{-5} f(x) dx + \int_{100}^{-10} f(x) dx - \int_{12}^{-10} f(x) dx \\ &= -4 - 2 - 6 \\ &= -12 \end{aligned}$$

There are also some nice properties that we can use in comparing the general size of definite integrals. Here they are.

**More Properties**

7.  $\int_a^b c dx = c(b-a)$ ,  $c$  is any number.

8. If  $f(x) \geq 0$  for  $a \leq x \leq b$  then  $\int_a^b f(x) dx \geq 0$ .

9. If  $f(x) \geq g(x)$  for  $a \leq x \leq b$  then  $\int_a^b f(x) dx \geq \int_a^b g(x) dx$ .

10. If  $m \leq f(x) \leq M$  for  $a \leq x \leq b$  then  $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$ .

11.  $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$



See the [Proof of Various Integral Properties](#) section of the Extras chapter for the proof of these properties.

### Interpretations of Definite Integral

There are a couple of quick interpretations of the definite integral that we can give here.

First, as we alluded to in the previous section one possible interpretation of the definite integral is to give the net area between the graph of  $f(x)$  and the  $x$ -axis on the interval  $[a,b]$ . So, the net area between the graph of  $f(x) = x^2 + 1$  and the  $x$ -axis on  $[0,2]$  is,

$$\int_0^2 x^2 + 1 dx = \frac{14}{3}$$

If you look back in the last section this was the exact area that was given for the initial set of problems that we looked at in this area.

Another interpretation is sometimes called the Net Change Theorem. This interpretation says that if  $f(x)$  is some quantity (so  $f'(x)$  is the rate of change of  $f(x)$ ), then,

$$\int_a^b f'(x) dx = f(b) - f(a)$$

is the net change in  $f(x)$  on the interval  $[a,b]$ . In other words, compute the definite integral of a rate of change and you'll get the net change in the quantity. We can see that the value of the definite integral,  $f(b) - f(a)$ , does in fact give us the net change in  $f(x)$  and so there really isn't anything to prove with this statement. This is really just an acknowledgment of what the definite integral of a rate of change tells us.

So as a quick example, if  $V(t)$  is the volume of water in a tank then,

$$\int_{t_1}^{t_2} V'(t) dt = V(t_2) - V(t_1)$$

is the net change in the volume as we go from time  $t_1$  to time  $t_2$ .

Likewise, if  $s(t)$  is the function giving the position of some object at time  $t$  we know that the velocity of the object at any time  $t$  is:  $v(t) = s'(t)$ . Therefore the displacement of the object time  $t_1$  to time  $t_2$  is,

$$\int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1)$$

Note that in this case if  $v(t)$  is both positive and negative (*i.e.* the object moves to both the right and left) in the time frame this will NOT give the total distance traveled. It will only give the displacement, *i.e.* the difference between where the object started and where it ended up. To get the total distance traveled by an object we'd have to compute,

$$\int_{t_1}^{t_2} |v(t)| dt$$

It is important to note here that the Net Change Theorem only really makes sense if we're integrating a derivative of a function.

### Fundamental Theorem of Calculus, Part I

As noted by the title above this is only the first part to the Fundamental Theorem of Calculus. We will give the second part in the next section as it is the key to easily computing definite integrals and that is the subject of the next section.

The first part of the Fundamental Theorem of Calculus tells us how to differentiate certain types of definite integrals and it also tells us about the very close relationship between integrals and derivatives.

### Fundamental Theorem of Calculus, Part I

If  $f(x)$  is continuous on  $[a,b]$  then,

$$g(x) = \int_a^x f(t) dt$$

is continuous on  $[a,b]$  and it is differentiable on  $(a,b)$  and that,

$$g'(x) = f(x)$$

An alternate notation for the derivative portion of this is,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

To see the proof of this see the [Proof of Various Integral Properties](#) section of the Extras chapter.

Let's check out a couple of quick examples using this.

**Example 6** Differentiate each of the following.

(a)  $g(x) = \int_{-4}^x e^{2t} \cos^2(1-5t) dt$  [\[Solution\]](#)

(b)  $\int_{x^2}^1 \frac{t^4 + 1}{t^2 + 1} dt$  [\[Solution\]](#)

**Solution**

(a)  $g(x) = \int_{-4}^x e^{2t} \cos^2(1-5t) dt$

This one is nothing more than a quick application of the Fundamental Theorem of Calculus.

$$g'(x) = e^{2x} \cos^2(1-5x)$$

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$$(b) \int_{x^2}^1 \frac{t^4 + 1}{t^2 + 1} dt$$

This one needs a little work before we can use the Fundamental Theorem of Calculus. The first thing to notice is that the FToC requires the lower limit to be a constant and the upper limit to be the variable. So, using a property of definite integrals we can interchange the limits of the integral we just need to remember to add in a minus sign after we do that. Doing this gives,

$$\frac{d}{dx} \int_{x^2}^1 \frac{t^4 + 1}{t^2 + 1} dt = \frac{d}{dx} \left( - \int_1^{x^2} \frac{t^4 + 1}{t^2 + 1} dt \right) = - \frac{d}{dx} \int_1^{x^2} \frac{t^4 + 1}{t^2 + 1} dt$$

The next thing to notice is that the FToC also requires an  $x$  in the upper limit of integration and we've got  $x^2$ . To do this derivative we're going to need the following version of the [chain rule](#).

$$\frac{d}{dx}(g(u)) = \frac{d}{du}(g(u)) \frac{du}{dx} \quad \text{where } u = f(x)$$

So, if we let  $u = x^2$  we use the chain rule to get,

$$\begin{aligned} \frac{d}{dx} \int_{x^2}^1 \frac{t^4 + 1}{t^2 + 1} dt &= - \frac{d}{dx} \int_1^{x^2} \frac{t^4 + 1}{t^2 + 1} dt \\ &= - \frac{d}{du} \int_1^u \frac{t^4 + 1}{t^2 + 1} dt \frac{du}{dx} \quad \text{where } u = x^2 \\ &= - \frac{u^4 + 1}{u^2 + 1} (2x) \\ &= -2x \frac{u^4 + 1}{u^2 + 1} \end{aligned}$$

The final step is to get everything back in terms of  $x$ .

$$\begin{aligned} \frac{d}{dx} \int_{x^2}^1 \frac{t^4 + 1}{t^2 + 1} dt &= -2x \frac{(x^2)^4 + 1}{(x^2)^2 + 1} \\ &= -2x \frac{x^8 + 1}{x^4 + 1} \end{aligned}$$

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Using the chain rule as we did in the last part of this example we can derive some general formulas for some more complicated problems.

First,

$$\frac{d}{dx} \int_a^{u(x)} f(t) dt = u'(x) f(u(x))$$

This is simply the chain rule for these kinds of problems.

Next, we can get a formula for integrals in which the upper limit is a constant and the lower limit is a function of  $x$ . All we need to do here is interchange the limits on the integral (adding in a minus sign of course) and then using the formula above to get,

$$\frac{d}{dx} \int_{v(x)}^b f(t) dt = -\frac{d}{dx} \int_b^{v(x)} f(t) dt = -v'(x) f(v(x))$$

Finally, we can also get a version for both limits being functions of  $x$ . In this case we'll need to use Property 5 above to break up the integral as follows,

$$\int_{v(x)}^{u(x)} f(t) dt = \int_{v(x)}^a f(t) dt + \int_a^{u(x)} f(t) dt$$

We can use pretty much any value of  $a$  when we break up the integral. The only thing that we need to avoid is to make sure that  $f(a)$  exists. So, assuming that  $f(a)$  exists after we break up the integral we can then differentiate and use the two formulas above to get,

$$\begin{aligned} \frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt &= \frac{d}{dx} \left( \int_{v(x)}^a f(t) dt + \int_a^{u(x)} f(t) dt \right) \\ &= -v'(x) f(v(x)) + u'(x) f(u(x)) \end{aligned}$$

Let's work a quick example.

**Example 7** Differentiate the following integral.

$$\int_{\sqrt{x}}^{3x} t^2 \sin(1+t^2) dt$$

**Solution**

This will use the final formula that we derived above.

$$\begin{aligned} \frac{d}{dx} \int_{\sqrt{x}}^{3x} t^2 \sin(1+t^2) dt &= -\frac{1}{2} x^{-\frac{1}{2}} (\sqrt{x})^2 \sin(1+(\sqrt{x})^2) + (3)(3x)^2 \sin(1+(3x)^2) \\ &= -\frac{1}{2} \sqrt{x} \sin(1+x) + 27x^2 \sin(1+9x^2) \end{aligned}$$

## ***Computing Definite Integrals***

In this section we are going to concentrate on how we actually evaluate definite integrals in practice. To do this we will need the Fundamental Theorem of Calculus, Part II.

### **Fundamental Theorem of Calculus, Part II**

Suppose  $f(x)$  is a continuous function on  $[a,b]$  and also suppose that  $F(x)$  is any anti-derivative for  $f(x)$ . Then,

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

To see the proof of this see the [Proof of Various Integral Properties](#) section of the Extras chapter.

Recall that when we talk about an anti-derivative for a function we are really talking about the indefinite integral for the function. So, to evaluate a definite integral the first thing that we're going to do is evaluate the indefinite integral for the function. This should explain the similarity in the notations for the indefinite and definite integrals.

Also notice that we require the function to be continuous in the interval of integration. This was also a requirement in the definition of the definite integral. We didn't make a big deal about this in the last section. In this section however, we will need to keep this condition in mind as we do our evaluations.

Next let's address the fact that we can use any anti-derivative of  $f(x)$  in the evaluation. Let's take a final look at the following integral.

$$\int_0^2 x^2 + 1 dx$$

Both of the following are anti-derivatives of the integrand.

$$F(x) = \frac{1}{3}x^3 + x \quad \text{and} \quad F(x) = \frac{1}{3}x^3 + x - \frac{18}{31}$$

Using the FToC to evaluate this integral with the first anti-derivatives gives,

$$\begin{aligned} \int_0^2 x^2 + 1 dx &= \left( \frac{1}{3}x^3 + x \right) \Big|_0^2 \\ &= \frac{1}{3}(2)^3 + 2 - \left( \frac{1}{3}(0)^3 + 0 \right) \\ &= \frac{14}{3} \end{aligned}$$

Much easier than [using the definition](#) wasn't it? Let's now use the second anti-derivative to evaluate this definite integral.

$$\begin{aligned}\int_0^2 x^2 + 1 dx &= \left( \frac{1}{3}x^3 + x - \frac{18}{31} \right) \Big|_0^2 \\ &= \frac{1}{3}(2)^3 + 2 - \frac{18}{31} - \left( \frac{1}{3}(0)^3 + 0 - \frac{18}{31} \right) \\ &= \frac{14}{3} - \frac{18}{31} + \frac{18}{31} \\ &= \frac{14}{3}\end{aligned}$$

The constant that we tacked onto the second anti-derivative canceled in the evaluation step. So, when choosing the anti-derivative to use in the evaluation process make your life easier and don't bother with the constant as it will only end up canceling in the long run.

Also, note that we're going to have to be very careful with minus signs and parenthesis with these problems. It's very easy to get in a hurry and mess them up.

Let's start our examples with the following set designed to make a couple of quick points that are very important.

**Example 1** Evaluate each of the following.

(a)  $\int y^2 + y^{-2} dy$  [\[Solution\]](#)

(b)  $\int_1^2 y^2 + y^{-2} dy$  [\[Solution\]](#)

(c)  $\int_{-1}^2 y^2 + y^{-2} dy$  [\[Solution\]](#)

**Solution**

(a)  $\int y^2 + y^{-2} dy$

This is the only indefinite integral in this section and by now we should be getting pretty good with these so we won't spend a lot of time on this part. This is here only to make sure that we understand the difference between an indefinite and a definite integral. The integral is,

$$\int y^2 + y^{-2} dy = \frac{1}{3}y^3 - y^{-1} + c$$

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(b)  $\int_1^2 y^2 + y^{-2} dy$

Recall from our first example above that all we really need here is any anti-derivative of the integrand. We just computed the most general anti-derivative in the first part so we can use that if we want to. However, recall that as we noted above any constants we tack on will just cancel

in the long run and so we'll use the answer from (a) without the "+c".

Here's the integral,

$$\begin{aligned}\int_1^2 y^2 + y^{-2} dy &= \left( \frac{1}{3}y^3 - \frac{1}{y} \right) \Big|_1^2 \\ &= \frac{1}{3}(2)^3 - \frac{1}{2} - \left( \frac{1}{3}(1)^3 - \frac{1}{1} \right) \\ &= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \\ &= \frac{17}{6}\end{aligned}$$

Remember that the evaluation is always done in the order of evaluation at the upper limit minus evaluation at the lower limit. Also be very careful with minus signs and parenthesis. It's very easy to forget them or mishandle them and get the wrong answer.

Notice as well that, in order to help with the evaluation, we rewrote the indefinite integral a little. In particular we got rid of the negative exponent on the second term. It's generally easier to evaluate the term with positive exponents.

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(c)  $\int_{-1}^2 y^2 + y^{-2} dy$

This integral is here to make a point. Recall that in order for us to do an integral the integrand must be continuous in the range of the limits. In this case the second term will have division by zero at  $y = 0$  and since  $y = 0$  is in the interval of integration, *i.e.* it is between the lower and upper limit, this integrand is not continuous in the interval of integration and so we can't do this integral.

Note that this problem will not prevent us from doing the integral in (b) since  $y = 0$  is not in the interval of integration.

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So what have we learned from this example?

First, in order to do a definite integral the first thing that we need to do is the indefinite integral. So we aren't going to get out of doing indefinite integrals, they will be in every integral that we'll be doing in the rest of this course so make sure that you're getting good at computing them.

Second, we need to be on the lookout for functions that aren't continuous at any point between the limits of integration. Also, it's important to note that this will only be a problem if the

point(s) of discontinuity occur between the limits of integration or at the limits themselves. If the point of discontinuity occurs outside of the limits of integration the integral can still be evaluated.

In the following sets of examples we won't make too much of an issue with continuity problems, or lack of continuity problems, unless it affects the evaluation of the integral. Do not let this convince you that you don't need to worry about this idea. It arises often enough that it can cause real problems if you aren't on the lookout for it.

Finally, note the difference between indefinite and definite integrals. Indefinite integrals are functions while definite integrals are numbers.

Let's work some more examples.

**Example 2** Evaluate each of the following.

(a)  $\int_{-3}^1 6x^2 - 5x + 2 \, dx$  [\[Solution\]](#)

(b)  $\int_4^0 \sqrt{t}(t-2) \, dt$  [\[Solution\]](#)

(c)  $\int_1^2 \frac{2w^5 - w + 3}{w^2} \, dw$  [\[Solution\]](#)

(d)  $\int_{25}^{-10} dR$  [\[Solution\]](#)

**Solution**

(a)  $\int_{-3}^1 6x^2 - 5x + 2 \, dx$

There isn't a lot to this one other than simply doing the work.

$$\begin{aligned} \int_{-3}^1 6x^2 - 5x + 2 \, dx &= \left( 2x^3 - \frac{5}{2}x^2 + 2x \right) \Big|_{-3}^1 \\ &= \left( 2 - \frac{5}{2} + 2 \right) - \left( -54 - \frac{45}{2} - 6 \right) \\ &= 84 \end{aligned}$$

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(b)  $\int_4^0 \sqrt{t}(t-2) \, dt$

Recall that we can't integrate products as a product of integrals and so we first need to multiply the integrand out before integrating, just as we did in the indefinite integral case.



$$\begin{aligned}
 \int_4^0 \sqrt{t}(t-2) dt &= \int_4^0 t^{\frac{3}{2}} - 2t^{\frac{1}{2}} dt \\
 &= \left( \frac{2}{5} t^{\frac{5}{2}} - \frac{4}{3} t^{\frac{3}{2}} \right) \Big|_4^0 \\
 &= 0 - \left( \frac{2}{5} (4)^{\frac{5}{2}} - \frac{4}{3} (2)^{\frac{3}{2}} \right) \\
 &= -\frac{32}{15}
 \end{aligned}$$

In the evaluation process recall that,

$$\begin{aligned}
 (4)^{\frac{5}{2}} &= \left( (4)^{\frac{1}{2}} \right)^5 = (2)^5 = 32 \\
 (4)^{\frac{3}{2}} &= \left( (4)^{\frac{1}{2}} \right)^3 = (2)^3 = 8
 \end{aligned}$$

Also, don't get excited about the fact that the lower limit of integration is larger than the upper limit of integration. That will happen on occasion and there is absolutely nothing wrong with this.

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$$(c) \int_1^2 \frac{2w^5 - w + 3}{w^2} dw$$

First, notice that we will have a division by zero issue at  $w = 0$ , but since this isn't in the interval of integration we won't have to worry about it.

Next again recall that we can't integrate quotients as a quotient of integrals and so the first step that we'll need to do is break up the quotient so we can integrate the function.

$$\begin{aligned}
 \int_1^2 \frac{2w^5 - w + 3}{w^2} dw &= \int_1^2 2w^3 - \frac{1}{w} + 3w^{-2} dw \\
 &= \left( \frac{1}{2} w^4 - \ln|w| - \frac{3}{w} \right) \Big|_1^2 \\
 &= \left( 8 - \ln 2 - \frac{3}{2} \right) - \left( \frac{1}{2} - \ln 1 - 3 \right) \\
 &= 9 - \ln 2
 \end{aligned}$$

Don't get excited about answers that don't come down to a simple integer or fraction. Often times they won't. Also don't forget that  $\ln(1) = 0$ .

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$$(d) \int_{25}^{-10} dR$$

This one is actually pretty easy. Recall that we're just integrating 1!

$$\begin{aligned} \int_{25}^{-10} dR &= R \Big|_{25}^{-10} \\ &= -10 - 25 \\ &= -35 \end{aligned}$$

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The last set of examples dealt exclusively with integrating powers of  $x$ . Let's work a couple of examples that involve other functions.

**Example 3** Evaluate each of the following.

$$(a) \int_0^1 4x - 6\sqrt[3]{x^2} dx \quad \text{[Solution]}$$

$$(b) \int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta d\theta \quad \text{[Solution]}$$

$$(c) \int_{\pi/6}^{\pi/4} 5 - 2\sec z \tan z dz \quad \text{[Solution]}$$

$$(d) \int_{-20}^{-1} \frac{3}{e^{-z}} - \frac{1}{3z} dz \quad \text{[Solution]}$$

$$(e) \int_{-2}^3 5t^6 - 10t + \frac{1}{t} dt \quad \text{[Solution]}$$

**Solution**

$$(a) \int_0^1 4x - 6\sqrt[3]{x^2} dx.$$

This one is here mostly here to contrast with the next example.

$$\begin{aligned} \int_0^1 4x - 6\sqrt[3]{x^2} dx &= \int_0^1 4x - 6x^{\frac{2}{3}} dx \\ &= \left( 2x^2 - \frac{18}{5}x^{\frac{5}{3}} \right) \Big|_0^1 \\ &= 2 - \frac{18}{5} - (0) \\ &= -\frac{8}{5} \end{aligned}$$

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$$(b) \int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta d\theta$$

Be careful with signs with this one. Recall from the indefinite integral sections that it's easy to mess up the signs when integrating sine and cosine.

$$\begin{aligned}
 \int_0^{\pi/3} 2 \sin \theta - 5 \cos \theta \, d\theta &= (-2 \cos \theta - 5 \sin \theta) \Big|_0^{\pi/3} \\
 &= -2 \cos\left(\frac{\pi}{3}\right) - 5 \sin\left(\frac{\pi}{3}\right) - (-2 \cos 0 - 5 \sin 0) \\
 &= -1 - \frac{5\sqrt{3}}{2} + 2 \\
 &= 1 - \frac{5\sqrt{3}}{2}
 \end{aligned}$$

Compare this answer to the previous answer, especially the evaluation at zero. It's very easy to get into the habit of just writing down zero when evaluating a function at zero. This is especially a problem when many of the functions that we integrate involve only  $x$ 's raised to positive integers and in these cases evaluate is zero of course. After evaluating many of these kinds of definite integrals it's easy to get into the habit of just writing down zero when you evaluate at zero. However, there are many functions out there that aren't zero when evaluated at zero so be careful.

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(c)  $\int_{\pi/6}^{\pi/4} 5 - 2 \sec z \tan z \, dz$

Not much to do other than do the integral.

$$\begin{aligned}
 \int_{\pi/6}^{\pi/4} 5 - 2 \sec z \tan z \, dz &= (5z - 2 \sec z) \Big|_{\pi/6}^{\pi/4} \\
 &= 5\left(\frac{\pi}{4}\right) - 2 \sec\left(\frac{\pi}{4}\right) - \left(5\left(\frac{\pi}{6}\right) - 2 \sec\left(\frac{\pi}{6}\right)\right) \\
 &= \frac{5\pi}{12} - 2\sqrt{2} + \frac{4}{\sqrt{3}}
 \end{aligned}$$

For the evaluation, recall that

$$\sec z = \frac{1}{\cos z}$$

and so if we can evaluate cosine at these angles we can evaluate secant at these angles.

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(d)  $\int_{-20}^{-1} \frac{3}{e^{-z}} - \frac{1}{3z} \, dz$

In order to do this one will need to rewrite both of the terms in the integral a little as follows,

$$\int_{-20}^{-1} \frac{3}{e^{-z}} - \frac{1}{3z} \, dz = \int_{-20}^{-1} 3e^z - \frac{1}{3z} \, dz$$

For the first term recall we used the following fact about exponents.

$$x^{-a} = \frac{1}{x^a} \qquad \frac{1}{x^{-a}} = x^a$$

In the second term, taking the 3 out of the denominator will just make integrating that term easier.

Now the integral.

$$\begin{aligned} \int_{-20}^{-1} \frac{3}{e^{-z}} - \frac{1}{3z} dz &= \left( 3e^z - \frac{1}{3} \ln|z| \right) \Big|_{-20}^{-1} \\ &= 3e^{-1} - \frac{1}{3} \ln|-1| - \left( 3e^{-20} - \frac{1}{3} \ln|-20| \right) \\ &= 3e^{-1} - 3e^{-20} + \frac{1}{3} \ln|20| \end{aligned}$$

Just leave the answer like this. It's messy, but it's also exact.

Note that the absolute value bars on the logarithm are required here. Without them we couldn't have done the evaluation.

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$$(e) \int_{-2}^3 5t^6 - 10t + \frac{1}{t} dt$$

This integral can't be done. There is division by zero in the third term at  $t = 0$  and  $t = 0$  lies in the interval of integration. The fact that the first two terms can be integrated doesn't matter. If even one term in the integral can't be integrated then the whole integral can't be done.

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So, we've computed a fair number of definite integrals at this point. Remember that the vast majority of the work in computing them is first finding the indefinite integral. Once we've found that the rest is just some number crunching.

There are a couple of particularly tricky definite integrals that we need to take a look at next. Actually they are only tricky until you see how to do them, so don't get too excited about them. The first one involves integrating a piecewise function.

**Example 4** Given,

$$f(x) = \begin{cases} 6 & \text{if } x > 1 \\ 3x^2 & \text{if } x \leq 1 \end{cases}$$

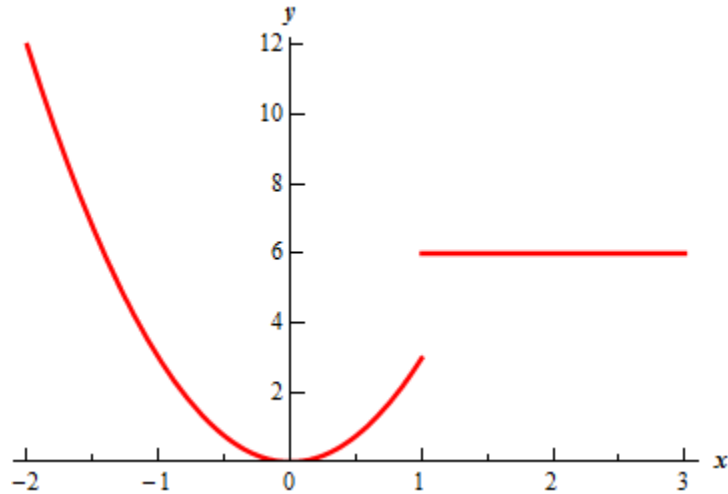
Evaluate each of the following integrals.

$$(a) \int_{10}^{22} f(x) dx \quad \text{[Solution]}$$

$$(b) \int_{-2}^3 f(x) dx \quad \text{[Solution]}$$

**Solution**

Let's first start with a graph of this function.



The graph reveals a problem. This function is not continuous at  $x = 1$  and we're going to have to watch out for that.

(a)  $\int_{10}^{22} f(x) dx$

For this integral notice that  $x = 1$  is not in the interval of integration and so that is something that we'll not need to worry about in this part.

Also note the limits for the integral lie entirely in the range for the first function. What this means for us is that when we do the integral all we need to do is plug in the first function into the integral.

Here is the integral.

$$\begin{aligned} \int_{10}^{22} f(x) dx &= \int_{10}^{22} 6 dx \\ &= 6x \Big|_{10}^{22} \\ &= 132 - 60 \\ &= 72 \end{aligned}$$

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(b)  $\int_{-2}^3 f(x) dx$

In this part  $x = 1$  is between the limits of integration. This means that the integrand is no longer continuous in the interval of integration and that is a show stopper as far we're concerned. As noted above we simply can't integrate functions that aren't continuous in the interval of integration.

Also, even if the function was continuous at  $x = 1$  we would still have the problem that the function is actually two different equations depending where we are in the interval of integration.

Let's first address the problem of the function not beginning continuous at  $x = 1$ . As we'll see, in this case, if we can find a way around this problem the second problem will also get taken care of at the same time.

In the previous examples where we had functions that weren't continuous we had division by zero and no matter how hard we try we can't get rid of that problem. Division by zero is a real problem and we can't really avoid it. In this case the discontinuity does not stem from problems with the function not existing at  $x = 1$ . Instead the function is not continuous because it takes on different values on either sides of  $x = 1$ . We can "remove" this problem by recalling [Property 5](#) from the previous section. This property tells us that we can write the integral as follows,

$$\int_{-2}^3 f(x) dx = \int_{-2}^1 f(x) dx + \int_1^3 f(x) dx$$

On each of these intervals the function is continuous. In fact we can say more. In the first integral we will have  $x$  between -2 and 1 and this means that we can use the second equation for  $f(x)$  and likewise for the second integral  $x$  will be between 1 and 3 and so we can use the first function for  $f(x)$ . The integral in this case is then,

$$\begin{aligned} \int_{-2}^3 f(x) dx &= \int_{-2}^1 f(x) dx + \int_1^3 f(x) dx \\ &= \int_{-2}^1 3x^2 dx + \int_1^3 6 dx \\ &= x^3 \Big|_{-2}^1 + 6x \Big|_1^3 \\ &= 1 - (-8) + (18 - 6) \\ &= 21 \end{aligned}$$

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So, to integrate a piecewise function, all we need to do is break up the integral at the break point(s) that happen to occur in the interval of integration and then integrate each piece.

Next we need to look at is how to integrate an absolute value function.

**Example 5** Evaluate the following integral.

$$\int_0^3 |3t - 5| dt$$

**Solution**

Recall that the point behind indefinite integration (which we'll need to do in this problem) is to determine what function we differentiated to get the integrand. To this point we've not seen any functions that will differentiate to get an absolute value nor will we ever see a function that will differentiate to get an absolute value.

The only way that we can do this problem is to get rid of the absolute value. To do this we need to recall the definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Once we remember that we can define absolute value as a piecewise function we can use the work from Example 4 as a guide for doing this integral.

What we need to do is determine where the quantity on the inside of the absolute value bars is negative and where it is positive. It looks like if  $t > \frac{5}{3}$  the quantity inside the absolute value is positive and if  $t < \frac{5}{3}$  the quantity inside the absolute value is negative.

Next, note that  $t = \frac{5}{3}$  is in the interval of integration and so, if we break up the integral at this point we get,

$$\int_0^3 |3t - 5| dt = \int_0^{\frac{5}{3}} |3t - 5| dt + \int_{\frac{5}{3}}^3 |3t - 5| dt$$

Now, in the first integrals we have  $t < \frac{5}{3}$  and so  $3t - 5 < 0$  in this interval of integration. That means we can drop the absolute value bars if we put in a minus sign. Likewise in the second integral we have  $t > \frac{5}{3}$  which means that in this interval of integration we have  $3t - 5 > 0$  and so we can just drop the absolute value bars in this integral.

After getting rid of the absolute value bars in each integral we can do each integral. So, doing the integration gives,

$$\begin{aligned} \int_0^3 |3t - 5| dt &= \int_0^{\frac{5}{3}} -(3t - 5) dt + \int_{\frac{5}{3}}^3 3t - 5 dt \\ &= \int_0^{\frac{5}{3}} -3t + 5 dt + \int_{\frac{5}{3}}^3 3t - 5 dt \\ &= \left( -\frac{3}{2}t^2 + 5t \right) \Big|_0^{\frac{5}{3}} + \left( \frac{3}{2}t^2 - 5t \right) \Big|_{\frac{5}{3}}^3 \\ &= -\frac{3}{2} \left( \frac{5}{3} \right)^2 + 5 \left( \frac{5}{3} \right) - (0) + \left( \frac{3}{2}(3)^2 - 5(3) - \left( \frac{3}{2} \left( \frac{5}{3} \right)^2 - 5 \left( \frac{5}{3} \right) \right) \right) \\ &= \frac{25}{6} + \frac{8}{3} \\ &= \frac{41}{6} \end{aligned}$$

Integrating absolute value functions isn't too bad. It's a little more work than the "standard" definite integral, but it's not really all that much more work. First, determine where the quantity inside the absolute value bars is negative and where it is positive. When we've determined that point all we need to do is break up the integral so that in each range of limits the quantity inside the absolute value bars is always positive or always negative. Once this is done we can drop the absolute value bars (adding negative signs when the quantity is negative) and then we can do the integral as we've always done.



### ***Substitution Rule for Definite Integrals***

We now need to go back and revisit the substitution rule as it applies to definite integrals. At some level there really isn't a lot to do in this section. Recall that the first step in doing a definite integral is to compute the indefinite integral and that hasn't changed. We will still compute the indefinite integral first. This means that we already know how to do these. We use the substitution rule to find the indefinite integral and then do the evaluation.

There are however, two ways to deal with the evaluation step. One of the ways of doing the evaluation is the probably the most obvious at this point, but also has a point in the process where we can get in trouble if we aren't paying attention.

Let's work an example illustrating both ways of doing the evaluation step.

**Example 1** Evaluate the following definite integral.

$$\int_{-2}^0 2t^2 \sqrt{1-4t^3} dt$$

#### ***Solution***

Let's start off looking at the first way of dealing with the evaluation step. We'll need to be careful with this method as there is a point in the process where if we aren't paying attention we'll get the wrong answer.

#### ***Solution 1 :***

We'll first need to compute the indefinite integral using the substitution rule. Note however, that we will constantly remind ourselves that this is a definite integral by putting the limits on the integral at each step. Without the limits it's easy to forget that we had a definite integral when we've gotten the indefinite integral computed.

In this case the substitution is,

$$u = 1 - 4t^3 \quad du = -12t^2 dt \quad \Rightarrow \quad t^2 dt = -\frac{1}{12} du$$

Plugging this into the integral gives,

$$\begin{aligned} \int_{-2}^0 2t^2 \sqrt{1-4t^3} dt &= -\frac{1}{6} \int_{-2}^0 u^{\frac{1}{2}} du \\ &= -\frac{1}{9} u^{\frac{3}{2}} \Big|_{-2}^0 \end{aligned}$$

Notice that we didn't do the evaluation yet. This is where the potential problem arises with this solution method. The limits given here are from the original integral and hence are values of  $t$ . We have  $u$ 's in our solution. We can't plug values of  $t$  in for  $u$ .

Therefore, we will have to go back to  $t$ 's before we do the substitution. This is the standard step