and our limits would be u's. Here's the rest of this problem.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\, dt = -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}$$

$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) = \frac{1}{9}\left(33\sqrt{33}-1\right)$$

We got exactly the same answer and this time didn't have to worry about going back to t's in our answer.

So, we've seen two solution techniques for computing definite integrals that require the substitution rule. Both are valid solution methods and each have their uses. We will be using the second exclusively however since it makes the evaluation step a little easier.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw$ [Solution]

(b) $\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\, dx$ [Solution]

(c) $\int_{0}^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\, dy$ [Solution]

(d) $\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\, dz$ [Solution]

Solution

Since we've done quite a few substitution rule integrals to this time we aren't going to put a lot of effort into explaining the substitution part of things here.

(a) $\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw$

The substitution and converted limits are,

$$u = 2w + w^2 \qquad du = (2+2w)\,dw \qquad \Rightarrow \qquad (1+w)\,dw = \frac{1}{2}du$$

$$w = -1 \quad \Rightarrow \quad u = -1 \qquad\qquad w = 5 \quad \Rightarrow \quad u = 35$$

Sometimes a limit will remain the same after the substitution. Don't get excited when it happens and don't expect it to happen all the time.

Here is the integral,

$$\int_{-1}^{5}(1+w)\left(2w+w^2\right)^5 dw = \frac{1}{2}\int_{-1}^{35} u^5\,du$$
$$= \frac{1}{12}u^6\Big|_{-1}^{35} = 153188802$$

Don't get excited about large numbers for answers here. Sometime they are. That's life.

[Return to Problems]

(b) $\displaystyle\int_{-2}^{-6}\frac{4}{(1+2x)^3}-\frac{5}{1+2x}\,dx$

Here is the substitution and converted limits for this problem,

$$u = 1+2x \qquad du = 2dx \quad \Rightarrow \quad dx = \frac{1}{2}du$$
$$x=-2 \quad \Rightarrow \quad u=-3 \qquad\qquad x=-6 \quad \Rightarrow \quad u=-11$$

The integral is then,

$$\int_{-2}^{-6}\frac{4}{(1+2x)^3}-\frac{5}{1+2x}\,dx = \frac{1}{2}\int_{-3}^{-11}4u^{-3}-\frac{5}{u}\,du$$
$$= \frac{1}{2}\left(-2u^{-2}-5\ln|u|\right)\Big|_{-3}^{-11}$$
$$= \frac{1}{2}\left(-\frac{2}{121}-5\ln 11\right)-\frac{1}{2}\left(-\frac{2}{9}-5\ln 3\right)$$
$$= \frac{112}{1089}-\frac{5}{2}\ln 11+\frac{5}{2}\ln 3$$

[Return to Problems]

(c) $\displaystyle\int_0^{\frac{1}{2}}\mathbf{e}^y+2\cos(\pi y)\,dy$

This integral needs to be split into two integrals since the first term doesn't require a substitution and the second does.

$$\int_0^{\frac{1}{2}}\mathbf{e}^y+2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}}\mathbf{e}^y\,dy+\int_0^{\frac{1}{2}}2\cos(\pi y)\,dy$$

Here is the substitution and converted limits for the second term.

$$u=\pi y \qquad du=\pi dy \quad \Rightarrow \quad dy=\frac{1}{\pi}du$$
$$y=0 \quad \Rightarrow \quad u=0 \qquad\qquad y=\frac{1}{2} \quad \Rightarrow \quad u=\frac{\pi}{2}$$

Here is the integral.

$$\int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}} \mathbf{e}^y\,dy + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\,du$$

$$= \mathbf{e}^y \Big|_0^{\frac{1}{2}} + \frac{2}{\pi}\sin u \Big|_0^{\frac{\pi}{2}}$$

$$= \mathbf{e}^{\frac{1}{2}} - \mathbf{e}^0 + \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{\pi}\sin 0$$

$$= \mathbf{e}^{\frac{1}{2}} - 1 + \frac{2}{\pi}$$

[Return to Problems]

(d) $\displaystyle\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz$

This integral will require two substitutions. So first split up the integral so we can do a substitution on each term.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = \int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right)dz - \int_{\frac{\pi}{3}}^{0} 5\cos(\pi - z)\,dz$$

There are the two substitutions for these integrals.

$$u = \frac{z}{2} \qquad du = \frac{1}{2}dz \qquad \Rightarrow \qquad dz = 2du$$

$$z = \frac{\pi}{3} \quad \Rightarrow \quad u = \frac{\pi}{6} \qquad\qquad z = 0 \quad \Rightarrow \quad u = 0$$

$$v = \pi - z \qquad dv = -dz \qquad \Rightarrow \qquad dz = -dv$$

$$z = \frac{\pi}{3} \quad \Rightarrow \quad v = \frac{2\pi}{3} \qquad\qquad z = 0 \quad \Rightarrow \quad v = \pi$$

Here is the integral for this problem.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = 6\int_{\frac{\pi}{6}}^{0} \sin(u)\,du + 5\int_{\frac{2\pi}{3}}^{\pi} \cos(v)\,dv$$

$$= -6\cos(u)\Big|_{\frac{\pi}{6}}^{0} + 5\sin(v)\Big|_{\frac{2\pi}{3}}^{\pi}$$

$$= 3\sqrt{3} - 6 + \left(-\frac{5\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} - 6$$

[Return to Problems]

The next set of examples is designed to make sure that we don't forget about a very important point about definite integrals.

Example 3 Evaluate each of the following.

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2-8t^2}\,dt$ [Solution]

(b) $\displaystyle\int_{3}^{5} \frac{4t}{2-8t^2}\,dt$ [Solution]

Solution

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2-8t^2}\,dt$

Be careful with this integral. The denominator is zero at $t = \pm\frac{1}{2}$ and both of these are in the interval of integration. Therefore, this integrand is not continuous in the interval and so the integral can't be done.

Be careful with definite integrals and be on the lookout for division by zero problems. In the previous section they were easy to spot since all the division by zero problems that we had there were at zero. Once we move into substitution problems however they will not always be so easy to spot so make sure that you first take a quick look at the integrand and see if there are any continuity problems with the integrand and if they occur in the interval of integration.

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(b) $\displaystyle\int_{3}^{5} \frac{4t}{2-8t^2}\,dt$

Now, in this case the integral can be done because the two points of discontinuity, $t = \pm\frac{1}{2}$, are both outside of the interval of integration. The substitution and converted limits in this case are,

$$u = 2-8t^2 \qquad du = -16t\,dt \qquad \Rightarrow \qquad dz = -\frac{1}{16}\,dt$$

$$t = 3 \quad \Rightarrow \quad u = -70 \qquad\qquad t = 5 \quad \Rightarrow \quad u = -198$$

The integral is then,

$$\begin{aligned}\int_{3}^{5} \frac{4t}{2-8t^2}\,dt &= -\frac{4}{16}\int_{-70}^{-198} \frac{1}{u}\,du \\ &= -\frac{1}{4}\ln|u|\Big|_{-70}^{-198} \\ &= -\frac{1}{4}\left(\ln(198)-\ln(70)\right)\end{aligned}$$

[Return to Problems]

Let's work another set of examples. These are a little tougher (at least in appearance) than the previous sets.

Example 4 Evaluate each of the following.

(a) $\int_0^{\ln(1-\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx$ [Solution]

(b) $\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{\left[\ln t\right]^4}{t}\, dt$ [Solution]

(c) $\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\, dP$ [Solution]

(d) $\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos\left(\sin(x)\right) dx$ [Solution]

(e) $\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\, dw$ [Solution]

Solution

(a) $\int_0^{\ln(1-\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx$

The limits are a little unusual in this case, but that will happen sometimes so don't get too excited about it. Here is the substitution.

$$u = 1-\mathbf{e}^x \qquad\qquad du = -\mathbf{e}^x dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1-\mathbf{e}^0 = 1-1 = 0$$

$$x = \ln(1-\pi) \qquad \Rightarrow \qquad u = 1-\mathbf{e}^{\ln(1-\pi)} = 1-(1-\pi) = \pi$$

The integral is then,

$$\begin{aligned}\int_0^{\ln(1-\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx &= -\int_0^{\pi} \cos u\, du \\ &= -\sin(u)\Big|_0^{\pi} \\ &= -\left(\sin\pi - \sin 0\right) = 0\end{aligned}$$

[Return to Problems]

(b) $\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{\left[\ln t\right]^4}{t}\, dt$

Here is the substitution and converted limits for this problem.

$$u = \ln t \qquad du = \frac{1}{t}\,dt$$

$$t = \mathbf{e}^2 \quad \Rightarrow \quad u = \ln \mathbf{e}^2 = 2 \qquad\qquad t = \mathbf{e}^6 \quad \Rightarrow \quad u = \ln \mathbf{e}^6 = 6$$

The integral is,

$$\begin{aligned}\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\,dt &= \int_2^6 u^4\,du \\ &= \left.\frac{1}{5}u^5\right|_2^6 \\ &= \frac{7744}{5}\end{aligned}$$

[Return to Problems]

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP$

Here is the substitution and converted limits and don't get too excited about the substitution. It's a little messy in the case, but that can happen on occasion.

$$u = 2+\sec(3P) \qquad du = 3\sec(3P)\tan(3P)\,dP \quad \Rightarrow \quad \sec(3P)\tan(3P)\,dP = \frac{1}{3}\,du$$

$$P = \frac{\pi}{12} \qquad \Rightarrow \qquad u = 2+\sec\left(\frac{\pi}{4}\right) = 2+\sqrt{2}$$

$$P = \frac{\pi}{9} \qquad \Rightarrow \qquad u = 2+\sec\left(\frac{\pi}{3}\right) = 4$$

Here is the integral,

$$\begin{aligned}\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP &= \frac{1}{3}\int_{2+\sqrt{2}}^{4} u^{-\frac{1}{3}}\,du \\ &= \left.\frac{1}{2}u^{\frac{2}{3}}\right|_{2+\sqrt{2}}^{4} \\ &= \frac{1}{2}\left(4^{\frac{3}{2}} - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right) \\ &= \frac{1}{2}\left(8 - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right)\end{aligned}$$

So, not only was the substitution messy, but we also a messy answer, but again that's life on occasion.

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(d) $\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$

This problem not as bad as it looks. Here is the substitution and converted limits.

$$u = \sin x \qquad du = \cos x\,dx$$

$$x = \frac{\pi}{2} \quad \Rightarrow \quad u = \sin\frac{\pi}{2} = 1 \qquad\qquad x = -\pi \quad \Rightarrow \quad u = \sin(-\pi) = 0$$

The cosine in the very front of the integrand will get substituted away in the differential and so this integrand actually simplifies down significantly. Here is the integral.

$$\begin{aligned}\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx &= \int_0^1 \cos u\,du \\ &= \sin(u)\Big|_0^1 \\ &= \sin(1) - \sin(0) \\ &= \sin(1)\end{aligned}$$

Don't get excited about these kinds of answers. On occasion we will end up with trig function evaluations like this.

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(e) $\displaystyle\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw$

This is also a tricky substitution (at least until you see it). Here it is,

$$u = \frac{2}{w} \qquad du = -\frac{2}{w^2}\,dw \qquad \Rightarrow \qquad \frac{1}{w^2}\,dw = -\frac{1}{2}\,du$$

$$w = 2 \quad \Rightarrow \quad u = 1 \qquad\qquad w = \frac{1}{50} \quad \Rightarrow \quad u = 100$$

Here is the integral.

$$\begin{aligned}\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw &= -\frac{1}{2}\int_{100}^{1} \mathbf{e}^u\,du \\ &= -\frac{1}{2}\mathbf{e}^u\Big|_{100}^{1} \\ &= -\frac{1}{2}\left(\mathbf{e}^1 - \mathbf{e}^{100}\right)\end{aligned}$$

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In this last set of examples we saw some tricky substitutions and messy limits, but these are a fact of life with some substitution problems and so we need to be prepared for dealing with them when the happen.

Even and Odd Functions

This is the last topic that we need to discuss in this chapter. It is probably better suited in the previous section, but that section has already gotten fairly large so I decided to put it here.

First, recall that an even function is any function which satisfies,

$$f\left(-x\right)=f\left(x\right)$$

Typical examples of even functions are,

$$f\left(x\right)=x^2 \qquad\qquad f\left(x\right)=\cos\left(x\right)$$

An odd function is any function which satisfies,

$$f\left(-x\right)=-f\left(x\right)$$

The typical examples of odd functions are,

$$f\left(x\right)=x^3 \qquad\qquad f\left(x\right)=\sin\left(x\right)$$

There are a couple of nice facts about integrating even and odd functions over the interval [-*a*,*a*]. If *f(x)* is an even function then,

$$\int_{-a}^{a} f\left(x\right)dx=2\int_{0}^{a} f\left(x\right)dx$$

Likewise, if *f(x)* is an odd function then,

$$\int_{-a}^{a} f\left(x\right)dx=0$$

Note that in order to use these facts the limit of integration must be the same number, but opposite signs!

Example 5 Integrate each of the following.

(a) $\int_{-2}^{2} 4x^4-x^2+1\,dx$ [Solution]

(b) $\int_{-10}^{10} x^5+\sin\left(x\right)dx$ [Solution]

Solution

Neither of these are terribly difficult integrals, but we can use the facts on them anyway.

(a) $\int_{-2}^{2} 4x^4-x^2+1\,dx$

In this case the integrand is even and the interval is correct so,

$$\int_{-2}^{2} 4x^4 - x^2 + 1\,dx = 2\int_{0}^{2} 4x^4 - x^2 + 1\,dx$$

$$= 2\left(\frac{4}{5}x^5 - \frac{1}{3}x^3 + x\right)\Big|_0^2$$

$$= \frac{748}{15}$$

So, using the fact cut the evaluation in half (in essence since one of the new limits was zero).

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(b) $\int_{-10}^{10} x^5 + \sin(x)\,dx$

The integrand in this case is odd and the interval is in the correct form and so we don't even need to integrate. Just use the fact.

$$\int_{-10}^{10} x^5 + \sin(x)\,dx = 0$$

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Note that the limits of integration are important here. Take the last integral as an example. A small change to the limits will not give us zero.

$$\int_{-10}^{9} x^5 + \sin(x)\,dx = \cos(10) - \cos(9) - \frac{468559}{6} = -78093.09461$$

The moral here is to be careful and not misuse these facts.

Applications of Integrals

Introduction

In this last chapter of this course we will be taking a look at a couple of applications of integrals. There are many other applications, however many of them require integration techniques that are typically taught in Calculus II. We will therefore be focusing on applications that can be done only with knowledge taught in this course.

Because this chapter is focused on the applications of integrals it is assumed in all the examples that you are capable of doing the integrals. There will not be as much detail in the integration process in the examples in this chapter as there was in the examples in the previous chapter.

Here is a listing of applications covered in this chapter.

Average Function Value – We can use integrals to determine the average value of a function.

Area Between Two Curves – In this section we'll take a look at determining the area between two curves.

Volumes of Solids of Revolution / Method of Rings – This is the first of two sections devoted to find the volume of a solid of revolution. In this section we look that the method of rings/disks.

Volumes of Solids of Revolution / Method of Cylinders – This is the second section devoted to finding the volume of a solid of revolution. Here we will look at the method of cylinders.

Work – The final application we will look at is determining the amount of work required to move an object.

Average Function Value

The first application of integrals that we'll take a look at is the average value of a function. The following fact tells us how to compute this.

Average Function Value

The average value of a function $f(x)$ over the interval [*a*,*b*] is given by,

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

To see a justification of this formula see the Proof of Various Integral Properties section of the Extras chapter.

Let's work a couple of quick examples.

Example 1 Determine the average value of each of the following functions on the given interval.

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$ [Solution]

(b) $R(z) = \sin(2z)\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$ [Solution]

Solution

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$

There's really not a whole lot to do in this problem other than just use the formula.

$$\begin{aligned} f_{avg} &= \frac{1}{\frac{5}{2}-(-1)}\int_{-1}^{\frac{5}{2}} t^2 - 5t + 6\cos(\pi t)\,dt \\ &= \frac{2}{7}\left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + \frac{6}{\pi}\sin(\pi t)\right)\Bigg|_{-1}^{\frac{5}{2}} \\ &= \frac{12}{7\pi} - \frac{13}{6} \\ &= -1.620993 \end{aligned}$$

You caught the substitution needed for the third term right?

So, the average value of this function of the given interval is -1.620993.

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(b) $R(z) = \sin(2z)\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$

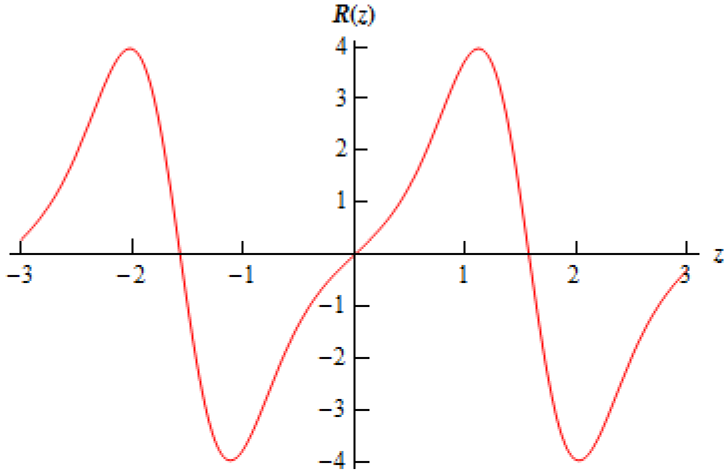
Again, not much to do here other than use the formula. Note that the integral will need the following substitution.

$$u = 1 - \cos(2z)$$

Here is the average value of this function,

$$\begin{aligned} R_{avg} &= \frac{1}{\pi - (-\pi)} \int_{\pi}^{-\pi} \sin(2z) \mathbf{e}^{1-\cos(2z)}\, dz \\ &= \left.\frac{1}{2} \mathbf{e}^{1-\cos(2z)}\right|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

So, in this case the average function value is zero. Do not get excited about getting zero here. It will happen on occasion. In fact, if you look at the graph of the function on this interval it's not too hard to see that this is the correct answer.



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There is also a theorem that is related to the average function value.

The Mean Value Theorem for Integrals

If $f(x)$ is a continuous function on $[a,b]$ then there is a number c in $[a,b]$ such that,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Note that this is very similar to the Mean Value Theorem that we saw in the Derivatives Applications chapter. See the Proof of Various Integral Properties section of the Extras chapter for the proof.

Note that one way to think of this theorem is the following. First rewrite the result as,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

and from this we can see that this theorem is telling us that there is a number $a < c < b$ such that $f_{avg} = f\left(c\right)$. Or, in other words, if $f\left(x\right)$ is a continuous function then somewhere in [a,b] the function will take on its average value.

Let's take a quick look at an example using this theorem.

Example 2 Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f\left(x\right) = x^2 + 3x + 2$ on the interval [1,4]

Solution

First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. So, let's do that.

$$\int_1^4 x^2 + 3x + 2\,dx = \left(c^2 + 3c + 2\right)\left(4 - 1\right)$$

$$\left.\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\right|_1^4 = 3\left(c^3 + 3c + 2\right)$$

$$\frac{99}{2} = 3c^3 + 9c + 6$$

$$0 = 3c^3 + 9c - \frac{87}{2}$$

This is a quadratic equation that we can solve. Using the quadratic formula we get the following two solutions,

$$c = \frac{-3 + \sqrt{67}}{2} = 2.593$$

$$c = \frac{-3 - \sqrt{67}}{2} = -5.593$$

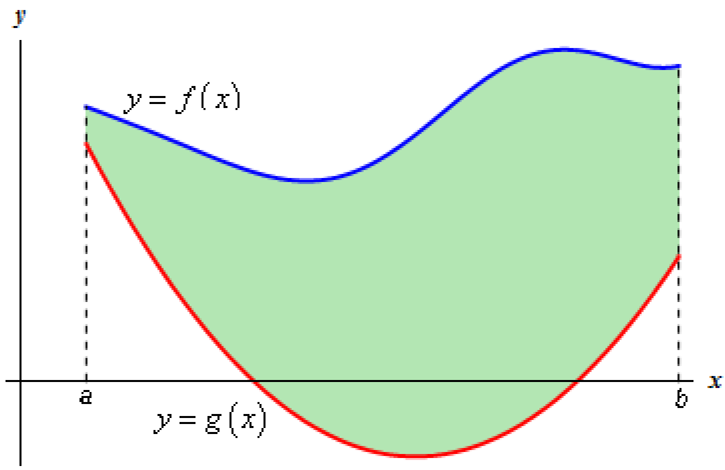
Clearly the second number is not in the interval and so that isn't the one that we're after. The first however is in the interval and so that's the number we want.

Note that it is possible for both numbers to be in the interval so don't expect only one to be in the interval.

Area Between Curves

In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

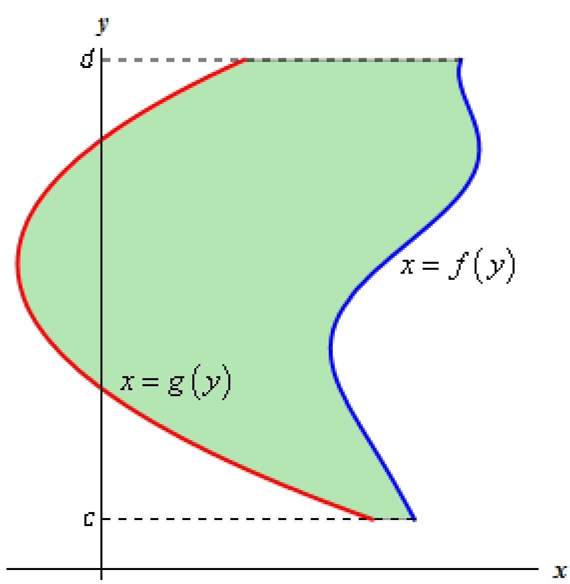
In the first case we are want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval [*a*,*b*]. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



In the [Area and Volume Formulas](http://tutorial.math.lamar.edu) section of the Extras chapter we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x)\,dx \qquad (1)$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval [*c*,*d*] with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\,dy \qquad (2)$$

Now (1) and (2) are perfectly serviceable formulas, however, it is sometimes easy to forget that these always require the first function to be the larger of the two functions. So, instead of these formulas we will instead use the following "word" formulas to make sure that we remember that the formulas area always the "larger" function minus the "smaller" function.

In the first case we will use,

$$A = \int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx, \qquad a \le x \le b \qquad (3)$$

In the second case we will use,

$$A = \int_c^d \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right) dy, \qquad c \le y \le d \qquad (4)$$

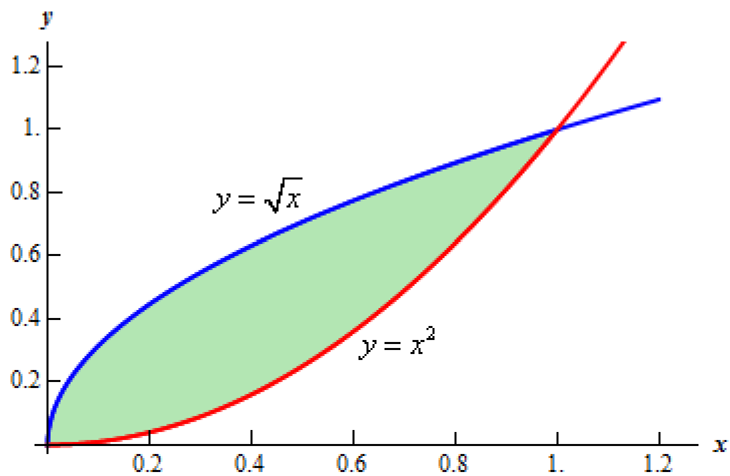
Using these formulas will always force us to think about what is going on with each problem and to make sure that we've got the correct order of functions when we go to use the formula.

Let's work an example.

Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution

First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Note that we don't take any part of the region to the right of the intersection point of these two graphs. In this region there is no boundary on the right side and so is not part of the enclosed area. Remember that one of the given functions must be on the each boundary of the enclosed region.

Also from this graph it's clear that the upper function will be dependent on the range of *x*'s that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the *x*'s. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x = 0$ and $x = 1$ so these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$\begin{aligned} A &= \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx \\ &= \int_0^1 \sqrt{x} - x^2 \, dx \\ &= \left. \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right) \right|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Before moving on to the next example, there are a couple of important things to note.

First, in almost all of these problems a graph is pretty much required. Often the bounding region, which will give the limits of integration, is difficult to determine without a graph.

Also, it can often be difficult to determine which of the functions is the upper function and with is the lower function without a graph. This is especially true in cases like the last example where the answer to that question actually depended upon the range of *x*'s that we were using.

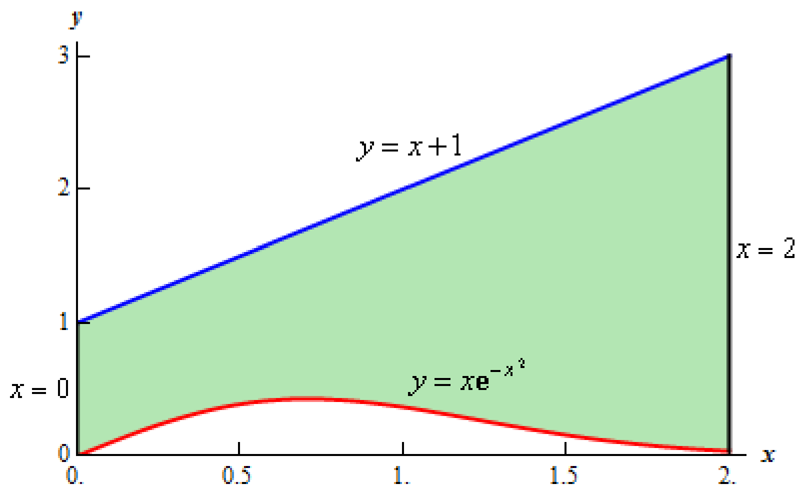
Finally, unlike the area under a curve that we looked at in the previous chapter the area between two curves will always be positive. If we get a negative number or zero we can be sure that we've made a mistake somewhere and will need to go back and find it.

Note as well that sometimes instead of saying region enclosed by we will say region bounded by. They mean the same thing.

Let's work some more examples.

Example 2 Determine the area of the region bounded by $y = x\mathbf{e}^{-x^2}$, $y = x+1$, $x = 2$, and the *y*-axis.

Solution
In this case the last two pieces of information, $x = 2$ and the *y*-axis, tell us the right and left boundaries of the region. Also, recall that the *y*-axis is given by the line $x = 0$. Here is the graph with the enclosed region shaded in.



Here, unlike the first example, the two curves don't meet. Instead we rely on two vertical lines to bound the left and right sides of the region as we noted above

Here is the integral that will give the area.

$$A = \int_a^b \begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix} dx$$

$$= \int_0^2 x+1-x\mathbf{e}^{-x^2}\,dx$$

$$= \left(\frac{1}{21}x^2 + x + \frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_0^2$$

$$= \frac{7}{2} + \frac{\mathbf{e}^{-4}}{2} = 3.5092$$

Example 3 Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

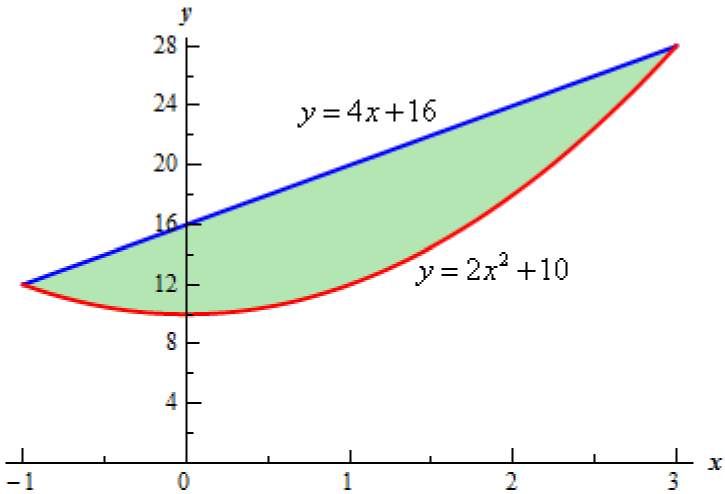
Solution
In this case the intersection points (which we'll need eventually) are not going to be easily identified from the graph so let's go ahead and get them now. Note that for most of these problems you'll not be able to accurately identify the intersection points from the graph and so you'll need to be able to determine them by hand. In this case we can get the intersection points

by setting the two equations equal.

$$\begin{aligned} 2x^2 + 10 &= 4x + 16 \\ 2x^2 - 4x - 6 &= 0 \\ 2(x+1)(x-3) &= 0 \end{aligned}$$

So it looks like the two curves will intersect at $x = -1$ and $x = 3$. If we need them we can get the *y* values corresponding to each of these by plugging the values back into either of the equations. We'll leave it to you to verify that the coordinates of the two intersection points on the graph are (-1,12) and (3,28).

Note as well that if you aren't good at graphing knowing the intersection points can help in at least getting the graph started. Here is a graph of the region.



With the graph we can now identify the upper and lower function and so we can now find the enclosed area.

$$\begin{aligned} A &= \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx \\ &= \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) dx \\ &= \int_{-1}^{3} -2x^2 + 4x + 6 \, dx \\ &= \left(-\frac{2}{3}x^3 + 2x^2 + 6x \right) \Big|_{-1}^{3} \\ &= \frac{64}{3} \end{aligned}$$

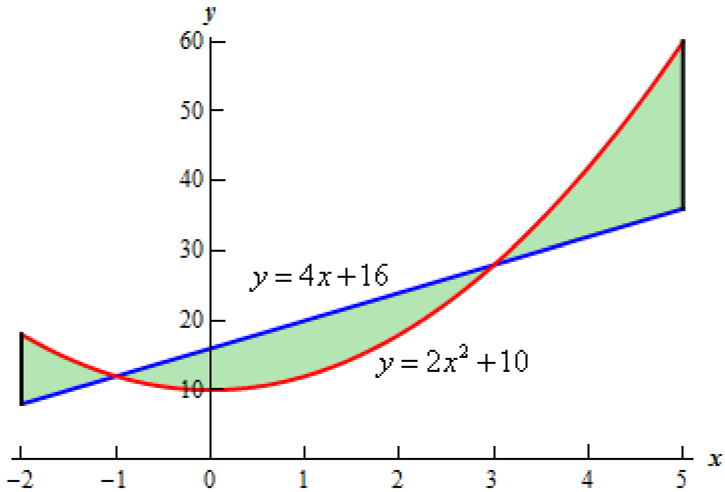
Be careful with parenthesis in these problems. One of the more common mistakes students make with these problems is to neglect parenthesis on the second term.

Example 4 Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$

Solution

So, the functions used in this problem are identical to the functions from the first problem. The difference is that we've extended the bounded region out from the intersection points. Since these are the same functions we used in the previous example we won't bother finding the intersection points again.

Here is a graph of this region.



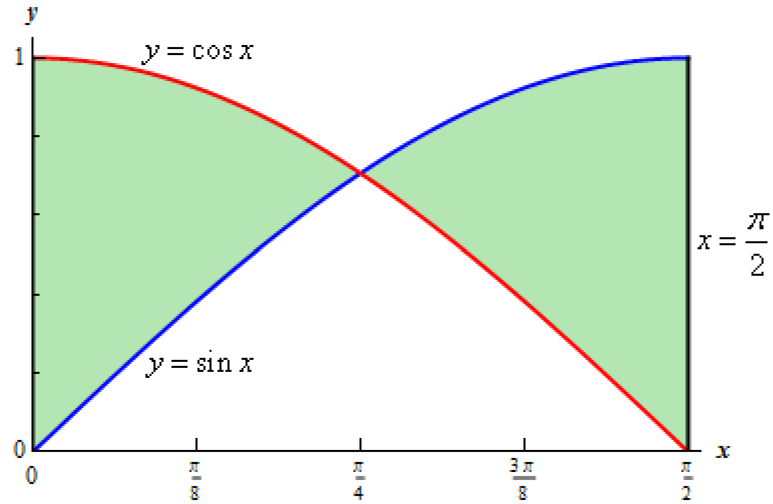
Okay, we have a small problem here. Our formula requires that one function always be the upper function and the other function always be the lower function and we clearly do not have that here. However, this actually isn't the problem that it might at first appear to be. There are three regions in which one function is always the upper function and the other is always the lower function. So, all that we need to do is find the area of each of the three regions, which we can do, and then add them all up.

Here is the area.

$$\begin{aligned}
A &= \int_{-2}^{-1} 2x^2 + 10 - (4x+16)\,dx + \int_{-1}^{3} 4x + 16 - (2x^2 + 10)\,dx + \int_{3}^{5} 2x^2 + 10 - (4x+16)\,dx \\
&= \int_{-2}^{-1} 2x^2 - 4x - 6\,dx + \int_{-1}^{3} -2x^2 + 4x + 6\,dx + \int_{3}^{5} 2x^2 - 4x - 6\,dx \\
&= \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{3}^{5} \\
&= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} \\
&= \frac{142}{3}
\end{aligned}$$

Example 5 Determine the area of the region enclosed by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$, and the y-axis.

Solution
First let's get a graph of the region.



So, we have another situation where we will need to do two integrals to get the area. The intersection point will be where

$$\sin x = \cos x$$

in the interval. We'll leave it to you to verify that this will be $x = \frac{\pi}{4}$. The area is then,

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx \\ &= \left(\sin x + \cos x\right)\Big|_0^{\frac{\pi}{4}} + \left(-\cos x - \sin x\right)\Big|_{\pi/4}^{\pi/2} \\ &= \sqrt{2} - 1 + \left(\sqrt{2} - 1\right) \\ &= 2\sqrt{2} - 2 = 0.828427 \end{aligned}$$

We will need to be careful with this next example.

Example 6 Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

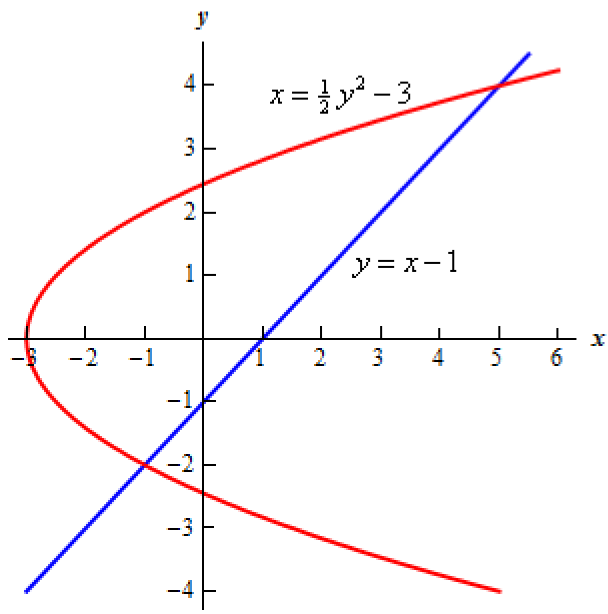
Solution
Don't let the first equation get you upset. We will have to deal with these kinds of equations occasionally so we'll need to get used to dealing with them.

As always, it will help if we have the intersection points for the two curves. In this case we'll get the intersection points by solving the second equation for *x* and the setting them equal. Here is that work,

$$\begin{aligned} y+1 &= \frac{1}{2}y^2 - 3 \\ 2y+2 &= y^2 - 6 \\ 0 &= y^2 - 2y - 8 \\ 0 &= (y-4)(y+2) \end{aligned}$$

So, it looks like the two curves will intersect at $y=-2$ and $y=4$ or if we need the full coordinates they will be : (-1,-2) and (5,4).

Here is a sketch of the two curves.



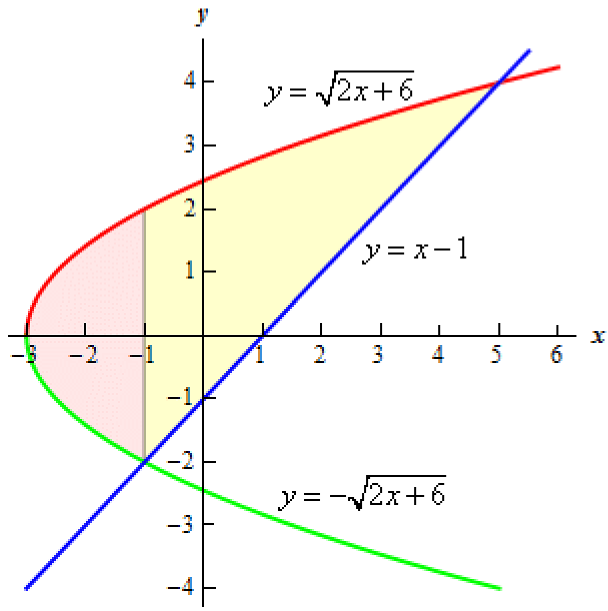
Now, we will have a serious problem at this point if we aren't careful. To this point we've been using an upper function and a lower function. To do that here notice that there are actually two portions of the region that will have different lower functions. In the range [-2,-1] the parabola is actually both the upper and the lower function.

To use the formula that we've been using to this point we need to solve the parabola for *y*. This gives,

$$y = \pm\sqrt{2x+6}$$

where the "+" gives the upper portion of the parabola and the "-" gives the lower portion.

Here is a sketch of the complete area with each region shaded that we'd need if we were going to use the first formula.



The integrals for the area would then be,

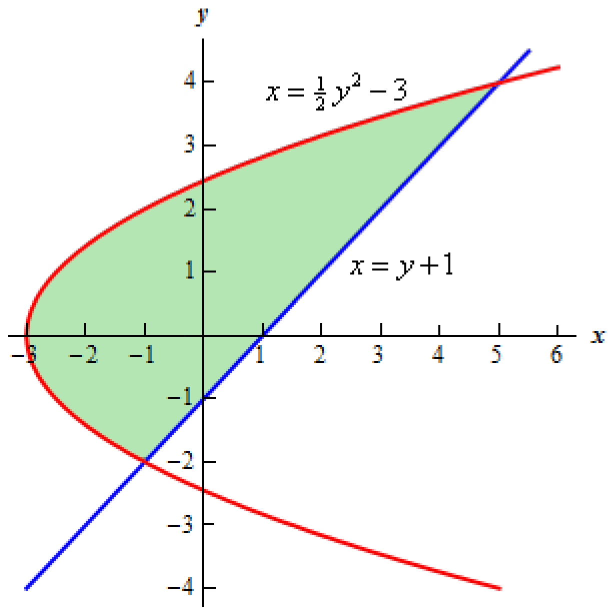
$$\begin{aligned}
A &= \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx + \int_{-1}^{5} \sqrt{2x+6} - \left(x-1\right) dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6} - x + 1 dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6}\, dx + \int_{-1}^{5} -x + 1 dx \\
&= \frac{2}{3} u^{\frac{3}{2}} \Big|_{0}^{4} + \frac{1}{3} u^{\frac{3}{2}} \Big|_{4}^{16} + \left(-\frac{1}{2}x^2 + x\right)\Big|_{-1}^{5} \\
&= 18
\end{aligned}$$

While these integrals aren't terribly difficult they are more difficult than they need to be.

Recall that there is another formula for determining the area. It is,

$$A = \int_{c}^{d} \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad c \le y \le d$$

and in our case we do have one function that is always on the left and the other is always on the right. So, in this case this is definitely the way to go. Note that we will need to rewrite the equation of the line since it will need to be in the form $x = f\left(y\right)$ but that is easy enough to do. Here is the graph for using this formula.



The area is,

$$\begin{aligned} A &= \int_c^d \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right) dy \\ &= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy \\ &= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy \\ &= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Bigg|_{-2}^{4} \\ &= 18 \end{aligned}$$

This is the same that we got using the first formula and this was definitely easier than the first method.

So, in this last example we've seen a case where we could use either formula to find the area. However, the second was definitely easier.

Students often come into a calculus class with the idea that the only easy way to work with functions is to use them in the form $y = f(x)$. However, as we've seen in this previous example there are definitely times when it will be easier to work with functions in the form $x = f(y)$. In fact, there are going to be occasions when this will be the only way in which a problem can be worked so make sure that you can deal with functions in this form.

Let's take a look at one more example to make sure we can deal with functions in this form.

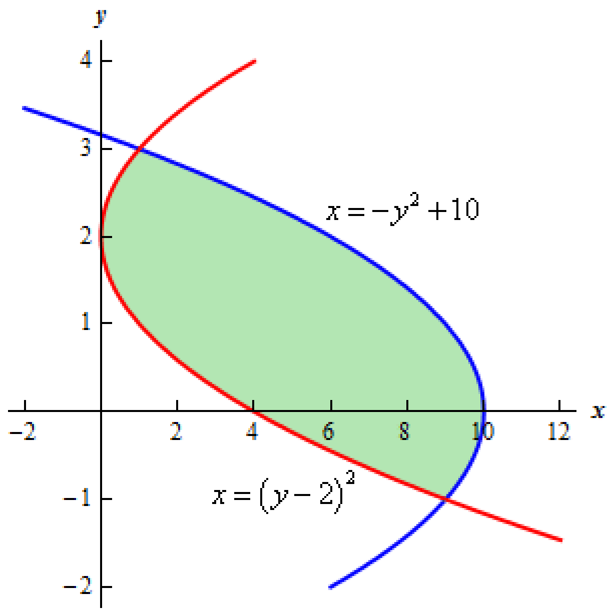
Example 7 Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.

Solution

First, we will need intersection points.

$$\begin{aligned} -y^2 + 10 &= (y-2)^2 \\ -y^2 + 10 &= y^2 - 4y + 4 \\ 0 &= 2y^2 - 4y - 6 \\ 0 &= 2(y+1)(y-3) \end{aligned}$$

The intersection points are $y = -1$ and $y = 3$. Here is a sketch of the region.



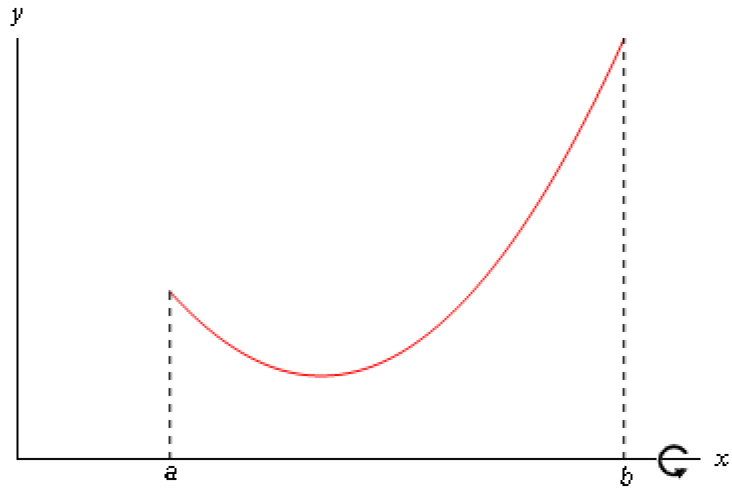
This is definitely a region where the second area formula will be easier. If we used the first formula there would be three different regions that we'd have to look at.

The area in this case is,

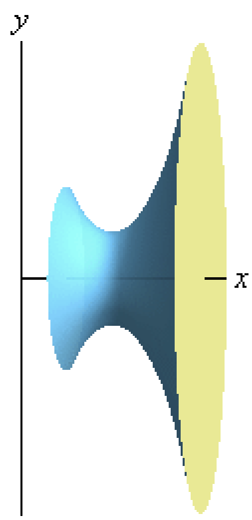
$$\begin{aligned} A &= \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy \\ &= \int_{-1}^{3} -y^2 + 10 - (y-2)^2 \, dy \\ &= \int_{-1}^{3} -2y^2 + 4y + 6 \, dy \\ &= \left(-\frac{2}{3}y^3 + 2y^2 + 6y \right) \Bigg|_{-1}^{3} = \frac{64}{3} \end{aligned}$$

Volumes of Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval [a,b].



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the x-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.



What we want to do over the course of the next two sections is to determine the volume of this object.

In the final the Area and Volume Formulas section of the Extras chapter we derived the following formulas for the volume of this solid.