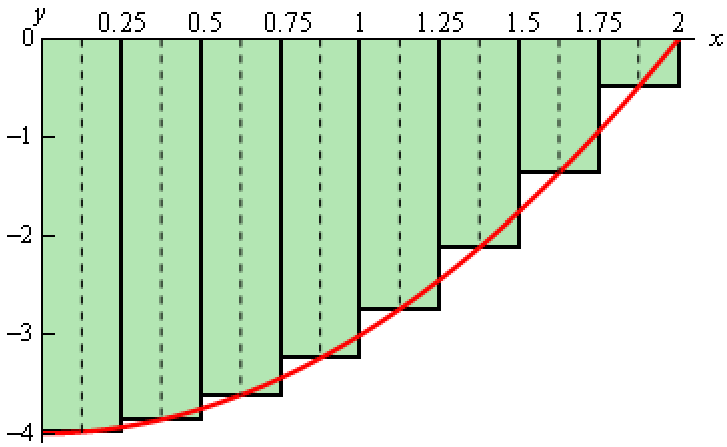
of $f(x) = x^2 - 4$ on [0,2]. If we use $n = 8$ and the midpoints for the rectangle height we get the following graph,

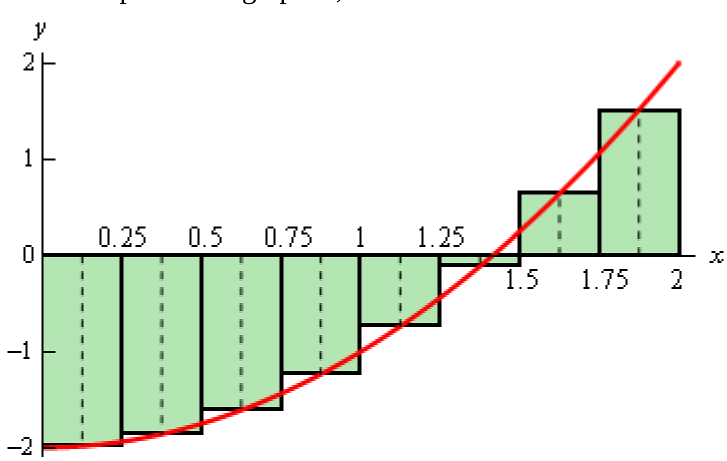


In this case let's notice that the function lies completely below the *x*-axis and hence is always negative. If we ignore the fact that the function is always negative and use the same ideas above to estimate the area between the graph and the *x*-axis we get,

$$A_m = \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) + \frac{1}{4} f\left(\frac{7}{8}\right) + \frac{1}{4} f\left(\frac{9}{8}\right) + \frac{1}{4} f\left(\frac{11}{8}\right) + \frac{1}{4} f\left(\frac{13}{8}\right) + \frac{1}{4} f\left(\frac{15}{8}\right)$$
$$= -5.34375$$

Our answer is negative as we might have expected given that all the function evaluations are negative.

So, using the technique in this section it looks like if the function is above the *x*-axis we will get a positive area and if the function is below the *x*-axis we will get a negative area. Now, what about a function that is both positive and negative in the interval? For example, $f(x) = x^2 - 2$ on [0,2]. Using $n = 8$ and midpoints the graph is,



Some of the rectangles are below the *x*-axis and so will give negative areas while some are above the *x*-axis and will give positive areas. Since more rectangles are below the *x*-axis than above it looks like we should probably get a negative area estimation for this case. In fact that is correct. Here the area estimation for this case.

$$\begin{aligned} A_m &= \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) \frac{1}{4} f\left(\frac{7}{8}\right) + \frac{1}{4} f\left(\frac{9}{8}\right) + \\ &\qquad \frac{1}{4} f\left(\frac{11}{8}\right) + \frac{1}{4} f\left(\frac{13}{8}\right) + \frac{1}{4} f\left(\frac{15}{8}\right) \\ &= -1.34375 \end{aligned}$$

In cases where the function is both above and below the *x*-axis the technique given in the section will give the *net* area between the function and the *x*-axis with areas below the *x*-axis negative and areas above the *x*-axis positive. So, if the net area is negative then there is more area under the *x*-axis than above while a positive net area will mean that more of the area is above the *x*-axis.

The Definition of the Definite Integral

In this section we will formally define the definite integral and give many of the properties of definite integrals. Let's start off with the definition of a definite integral.

Definite Integral

Given a function $f\left(x\right)$ that is continuous on the interval [*a*,*b*] we divide the interval into *n* subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of *f(x)* from *a* to *b*** is

$$\int_a^b f\left(x\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

The definite integral is defined to be exactly the limit and summation that we looked at in the last section to find the net area between a function and the *x*-axis. Also note that the notation for the definite integral is very similar to the notation for an indefinite integral. The reason for this will be apparent eventually.

There is also a little bit of terminology that we should get out of the way here. The number "*a*" that is at the bottom of the integral sign is called the **lower limit** of the integral and the number "*b*" at the top of the integral sign is called the **upper limit** of the integral. Also, despite the fact that *a* and *b* were given as an interval the lower limit does not necessarily need to be smaller than the upper limit. Collectively we'll often call *a* and *b* the **interval of integration**.

Let's work a quick example. This example will use many of the properties and facts from the brief review of summation notation in the Extras chapter.

Example 1 Using the definition of the definite integral compute the following.

$$\int_0^2 x^2 + 1\,dx$$

Solution

First, we can't actually use the definition unless we determine which points in each interval that well use for x_i^*. In order to make our life easier we'll use the right endpoints of each interval.

From the previous section we know that for a general *n* the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$\left[0,\frac{2}{n}\right],\ \left[\frac{2}{n},\frac{4}{n}\right],\ \left[\frac{4}{n},\frac{6}{n}\right],\ \ldots,\ \left[\frac{2\left(i-1\right)}{n},\frac{2i}{n}\right],\ \ldots,\ \left[\frac{2\left(n-1\right)}{n},2\right]$$

As we can see the right endpoint of the i^{th} subinterval is

$$x_i^* = \frac{2i}{n}$$

The summation in the definition of the definite integral is then,

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2 + 1\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{2}{n}\right)\end{aligned}$$

Now, we are going to have to take a limit of this. That means that we are going to need to "evaluate" this summation. In other words, we are going to have to use the formulas given in the summation notation review to eliminate the actual summation and get a formula for this for a general *n*.

To do this we will need to recognize that *n* is a constant as far as the summation notation is concerned. As we cycle through the integers from 1 to *n* in the summation only *i* changes and so anything that isn't an *i* will be a constant and can be factored out of the summation. In particular any *n* that is in the summation can be factored out if we need to.

Here is the summation "evaluation".

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n}\frac{8i^2}{n^3} + \sum_{i=1}^{n}\frac{2}{n} \\ &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 2 \\ &= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(2n) \\ &= \frac{4(n+1)(2n+1)}{3n^2} + 2 \\ &= \frac{14n^2 + 12n + 4}{3n^2}\end{aligned}$$

We can now compute the definite integral.

$$\begin{aligned}\int_0^2 x^2+1\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x\\ &= \lim_{n\to\infty}\frac{14n^2+12n+4}{3n^2}\\ &= \frac{14}{3}\end{aligned}$$

We've seen several methods for dealing with the limit in this problem so I'll leave it to you to verify the results.

Wow, that was a lot of work for a fairly simple function. There is a much simpler way of evaluating these and we will get to it eventually. The main purpose to this section is to get the main properties and facts about the definite integral out of the way. We'll discuss how we compute these in practice starting with the next section.

So, let's start taking a look at some of the properties of the definite integral.

Properties

1. $\int_a^b f\left(x\right)dx = -\int_b^a f\left(x\right)dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f\left(x\right)dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b cf\left(x\right)dx = c\int_a^b f\left(x\right)dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b f\left(x\right)\pm g\left(x\right)dx = \int_a^b f\left(x\right)dx \pm \int_a^b g\left(x\right)dx$. We can break up definite integrals across a sum or difference.

5. $\int_a^b f\left(x\right)dx = \int_a^c f\left(x\right)dx + \int_c^b f\left(x\right)dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, [a,c] and [c,b]. Note however that c doesn't need to be between a and b.

6. $\int_a^b f\left(x\right)dx = \int_a^b f\left(t\right)dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

See the Proof of Various Integral Properties section of the Extras chapter for the proof of properties 1 – 4. Property 5 is not easy to prove and so is not shown there. Property is not really a property in the full sense of the word. It is only here to acknowledge that as long as the function and limits are the same it doesn't matter what letter we use for the variable. The answer will be the same.

Let's do a couple of examples dealing with these properties.

Example 2 Use the results from the first example to evaluate each of the following.

(a) $\int_{2}^{0} x^2 + 1\,dx$ [Solution]

(b) $\int_{0}^{2} 10x^2 + 10\,dx$ [Solution]

(c) $\int_{0}^{2} t^2 + 1\,dt$ [Solution]

Solution

All of the solutions to these problems will rely on the fact we proved in the first example. Namely that,

$$\int_{0}^{2} x^2 + 1\,dx = \frac{14}{3}$$

(a) $\int_{2}^{0} x^2 + 1\,dx$

In this case the only difference between the two is that the limits have interchanged. So, using the first property gives,

$$\begin{aligned}\int_{2}^{0} x^2 + 1\,dx &= -\int_{0}^{2} x^2 + 1\,dx \\ &= -\frac{14}{3}\end{aligned}$$

[Return to Problems]

(b) $\int_{0}^{2} 10x^2 + 10\,dx$

For this part notice that we can factor a 10 out of both terms and then out of the integral using the third property.

$$\begin{aligned}\int_{0}^{2} 10x^2 + 10\,dx &= \int_{0}^{2} 10\left(x^2 + 1\right)dx \\ &= 10\int_{0}^{2} x^2 + 1\,dx \\ &= 10\left(\frac{14}{3}\right) \\ &= \frac{140}{3}\end{aligned}$$

[Return to Problems]

(c) $\int_0^2 t^2 + 1\,dt$

In this case the only difference is the letter used and so this is just going to use property 6.

$$\int_0^2 t^2 + 1\,dt = \int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

[Return to Problems]

Here are a couple of examples using the other properties.

Example 3 Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2 + 1}\,dx$$

Solution

There really isn't anything to do with this integral once we notice that the limits are the same. Using the second property this is,

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2 + 1}\,dx = 0$$

Example 4 Given that $\int_6^{-10} f(x)\,dx = 23$ and $\int_{-10}^{6} g(x)\,dx = -9$ determine the value of

$$\int_{-10}^{6} 2f(x) - 10g(x)\,dx$$

Solution

We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\,dx &= \int_{-10}^{6} 2f(x)\,dx - \int_{-10}^{6} 10g(x)\,dx \\ &= 2\int_{-10}^{6} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx\end{aligned}$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above (and adding a minus sign of course). Once this is done we can plug in the known values of the integrals.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\,dx &= -2\int_{6}^{-10} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx \\ &= -2(23) - 10(-9) \\ &= 44\end{aligned}$$

Example 5 Given that $\int_{12}^{-10} f(x)\,dx = 6$, $\int_{100}^{-10} f(x)\,dx = -2$, and $\int_{100}^{-5} f(x)\,dx = 4$ determine the value of $\int_{-5}^{12} f(x)\,dx$.

Solution
This example is mostly an example of property 5 although there are a couple of uses of property 1 in the solution as well.

We need to figure out how to correctly break up the integral using property 5 to allow us to use the given pieces of information. First we'll note that there is an integral that has a "-5" in one of the limits. It's not the lower limit, but we can use property 1 to correct that eventually. The other limit is 100 so this is the number *c* that we'll use in property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{12} f(x)\,dx$$

We'll be able to get the value of the first integral, but the second still isn't in the list of know integrals. However, we do have second limit that has a limit of 100 in it. The other limit for this second integral is -10 and this will be *c* in this application of property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{-10} f(x)\,dx + \int_{-10}^{12} f(x)\,dx$$

At this point all that we need to do is use the property 1 on the first and third integral to get the limits to match up with the known integrals. After that we can plug in for the known integrals.

$$\begin{aligned}\int_{-5}^{12} f(x)\,dx &= -\int_{100}^{-5} f(x)\,dx + \int_{100}^{-10} f(x)\,dx - \int_{12}^{-10} f(x)\,dx \\ &= -4 - 2 - 6 \\ &= -12\end{aligned}$$

There are also some nice properties that we can use in comparing the general size of definite integrals. Here they are.

More Properties

7. $\int_a^b c\,dx = c(b-a)$, c is any number.
8. If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$.
9. If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.
10. If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.
11. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$

See the Proof of Various Integral Properties section of the Extras chapter for the proof of these properties.

Interpretations of Definite Integral

There are a couple of quick interpretations of the definite integral that we can give here.

First, as we alluded to in the previous section one possible interpretation of the definite integral is to give the net area between the graph of $f(x)$ and the *x*-axis on the interval [*a*,*b*]. So, the net area between the graph of $f(x) = x^2 + 1$ and the *x*-axis on [0,2] is,

$$\int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

If you look back in the last section this was the exact area that was given for the initial set of problems that we looked at in this area.

Another interpretation is sometimes called the Net Change Theorem. This interpretation says that if $f(x)$ is some quantity (so $f'(x)$ is the rate of change of $f(x)$, then,

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

is the net change in $f(x)$ on the interval [*a*,*b*]. In other words, compute the definite integral of a rate of change and you'll get the net change in the quantity. We can see that the value of the definite integral, $f(b) - f(a)$, does in fact give use the net change in $f(x)$ and so there really isn't anything to prove with this statement. This is really just an acknowledgment of what the definite integral of a rate of change tells us.

So as a quick example, if $V(t)$ is the volume of water in a tank then,

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the net change in the volume as we go from time t_1 to time t_2.

Likewise, if $s(t)$ is the function giving the position of some object at time *t* we know that the velocity of the object at any time *t* is : $v(t) = s'(t)$. Therefore the displacement of the object time t_1 to time t_2 is,

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

Note that in this case if $v(t)$ is both positive and negative (*i.e.* the object moves to both the right and left) in the time frame this will NOT give the total distance traveled. It will only give the displacement, *i.e.* the difference between where the object started and where it ended up. To get the total distance traveled by an object we'd have to compute,

$$\int_{t_1}^{t_2} \left| v(t) \right| dt$$

It is important to note here that the Net Change Theorem only really makes sense if we're integrating a derivative of a function.

Fundamental Theorem of Calculus, Part I

As noted by the title above this is only the first part to the Fundamental Theorem of Calculus. We will give the second part in the next section as it is the key to easily computing definite integrals and that is the subject of the next section.

The first part of the Fundamental Theorem of Calculus tells us how to differentiate certain types of definite integrals and it also tells us about the very close relationship between integrals and derivatives.

Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a,b]$ then,

$$g(x) = \int_a^x f(t)\, dt$$

is continuous on $[a,b]$ and it is differentiable on (a,b) and that,

$$g'(x) = f(x)$$

An alternate notation for the derivative portion of this is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

To see the proof of this see the Proof of Various Integral Properties section of the Extras chapter.

Let's check out a couple of quick examples using this.

Example 6 Differentiate each of the following.

(a) $g(x) = \int_{-4}^{x} \mathbf{e}^{2t}\cos^2(1-5t)\,dt$ [Solution]

(b) $\displaystyle\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt$ [Solution]

Solution

(a) $g(x) = \int_{-4}^{x} \mathbf{e}^{2t}\cos^2(1-5t)\,dt$

This one is nothing more than a quick application of the Fundamental Theorem of Calculus.

$$g'(x) = \mathbf{e}^{2x}\cos^2(1-5x)$$

[Return to Problems]

(b) $\displaystyle\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt$

This one needs a little work before we can use the Fundamental Theorem of Calculus. The first thing to notice is that the FToC requires the lower limit to be a constant and the upper limit to be the variable. So, using a property of definite integrals we can interchange the limits of the integral we just need to remember to add in a minus sign after we do that. Doing this gives,

$$\frac{d}{dx}\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt = \frac{d}{dx}\left(-\int_{1}^{x^2} \frac{t^4+1}{t^2+1}\,dt\right) = -\frac{d}{dx}\int_{1}^{x^2} \frac{t^4+1}{t^2+1}\,dt$$

The next thing to notice is that the FToC also requires an x in the upper limit of integration and we've got x^2. To do this derivative we're going to need the following version of the chain rule.

$$\frac{d}{dx}\big(g(u)\big) = \frac{d}{du}\big(g(u)\big)\,\frac{du}{dx} \qquad \text{where } u = f(x)$$

So, if we let $u = x^2$ we use the chain rule to get,

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt &= -\frac{d}{dx}\int_{1}^{x^2} \frac{t^4+1}{t^2+1}\,dt \\ &= -\frac{d}{du}\int_{1}^{u} \frac{t^4+1}{t^2+1}\,dt\;\frac{du}{dx} \qquad \text{where } u = x^2 \\ &= -\frac{u^4+1}{u^2+1}(2x) \\ &= -2x\frac{u^4+1}{u^2+1}\end{aligned}$$

The final step is to get everything back in terms of x.

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt &= -2x\frac{\left(x^2\right)^4+1}{\left(x^2\right)^2+1} \\ &= -2x\frac{x^8+1}{x^4+1}\end{aligned}$$

[Return to Problems]

Using the chain rule as we did in the last part of this example we can derive some general formulas for some more complicated problems.

First,

$$\frac{d}{dx}\int_{a}^{u(x)} f(t)\,dt = u'(x)\,f\big(u(x)\big)$$

This is simply the chain rule for these kinds of problems.

Next, we can get a formula for integrals in which the upper limit is a constant and the lower limit is a function of *x*. All we need to do here is interchange the limits on the integral (adding in a minus sign of course) and then using the formula above to get,

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -\frac{d}{dx}\int_{b}^{v(x)} f(t)\,dt = -v'(x)\, f\left(v(x)\right)$$

Finally, we can also get a version for both limits being functions of *x*. In this case we'll need to use Property 5 above to break up the integral as follows,

$$\int_{v(x)}^{u(x)} f(t)\,dt = \int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt$$

We can use pretty much any value of *a* when we break up the integral. The only thing that we need to avoid is to make sure that $f(a)$ exists. So, assuming that $f(a)$ exists after we break up the integral we can then differentiate and use the two formulas above to get,

$$\begin{aligned}\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt &= \frac{d}{dx}\left(\int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt\right)\\ &= -v'(x)\, f\left(v(x)\right) + u'(x)\, f\left(u(x)\right)\end{aligned}$$

Let's work a quick example.

Example 7 Differentiate the following integral.

$$\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right) dt$$

Solution

This will use the final formula that we derived above.

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right) dt &= -\frac{1}{2}x^{-\frac{1}{2}}\left(\sqrt{x}\right)^2 \sin\left(1+\left(\sqrt{x}\right)^2\right) + (3)(3x)^2 \sin\left(1+(3x)^2\right)\\ &= -\frac{1}{2}\sqrt{x}\sin(1+x) + 27x^2\sin\left(1+9x^2\right)\end{aligned}$$

Computing Definite Integrals

In this section we are going to concentrate on how we actually evaluate definite integrals in practice. To do this we will need the Fundamental Theorem of Calculus, Part II.

Fundamental Theorem of Calculus, Part II

Suppose $f(x)$ is a continuous function on [a,b] and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

To see the proof of this see the Proof of Various Integral Properties section of the Extras chapter.

Recall that when we talk about an anti-derivative for a function we are really talking about the indefinite integral for the function. So, to evaluate a definite integral the first thing that we're going to do is evaluate the indefinite integral for the function. This should explain the similarity in the notations for the indefinite and definite integrals.

Also notice that we require the function to be continuous in the interval of integration. This was also a requirement in the definition of the definite integral. We didn't make a big deal about this in the last section. In this section however, we will need to keep this condition in mind as we do our evaluations.

Next let's address the fact that we can use any anti-derivative of $f(x)$ in the evaluation. Let's take a final look at the following integral.

$$\int_0^2 x^2 + 1\,dx$$

Both of the following are anti-derivatives of the integrand.

$$F(x) = \frac{1}{3}x^3 + x \qquad \text{and} \qquad F(x) = \frac{1}{3}x^3 + x - \frac{18}{31}$$

Using the FToC to evaluate this integral with the first anti-derivatives gives,

$$\begin{aligned}\int_0^2 x^2 + 1\,dx &= \left(\frac{1}{3}x^3 + x\right)\Bigg|_0^2 \\ &= \frac{1}{3}(2)^3 + 2 - \left(\frac{1}{3}(0)^3 + 0\right) \\ &= \frac{14}{3}\end{aligned}$$

Much easier than using the definition wasn't it? Let's now use the second anti-derivative to evaluate this definite integral.

$$\begin{aligned}\int_0^2 x^2+1\,dx &= \left(\frac{1}{3}x^3+x-\frac{18}{31}\right)\Bigg|_0^2 \\ &= \frac{1}{3}(2)^3+2-\frac{18}{31}-\left(\frac{1}{3}(0)^3+0-\frac{18}{31}\right) \\ &= \frac{14}{3}-\frac{18}{31}+\frac{18}{31} \\ &= \frac{14}{3}\end{aligned}$$

The constant that we tacked onto the second anti-derivative canceled in the evaluation step. So, when choosing the anti-derivative to use in the evaluation process make your life easier and don't bother with the constant as it will only end up canceling in the long run.

Also, note that we're going to have to be very careful with minus signs and parenthesis with these problems. It's very easy to get in a hurry and mess them up.

Let's start our examples with the following set designed to make a couple of quick points that are very important.

Example 1 Evaluate each of the following.

(a) $\int y^2+y^{-2}\,dy$ [Solution]

(b) $\int_1^2 y^2+y^{-2}\,dy$ [Solution]

(c) $\int_{-1}^2 y^2+y^{-2}\,dy$ [Solution]

Solution

(a) $\int y^2+y^{-2}\,dy$

This is the only indefinite integral in this section and by now we should be getting pretty good with these so we won't spend a lot of time on this part. This is here only to make sure that we understand the difference between an indefinite and a definite integral. The integral is,

$$\int y^2+y^{-2}\,dy=\frac{1}{3}y^3-y^{-1}+c$$

[Return to Problems]

(b) $\int_1^2 y^2+y^{-2}\,dy$

Recall from our first example above that all we really need here is any anti-derivative of the integrand. We just computed the most general anti-derivative in the first part so we can use that if we want to. However, recall that as we noted above any constants we tack on will just cancel in the long run and so we'll use the answer from (a) without the "+*c*".

Here's the integral,

$$\begin{aligned}\int_1^2 y^2 + y^{-2}\,dy &= \left(\frac{1}{3}y^3 - \frac{1}{y}\right)\Bigg|_1^2 \\ &= \frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1}\right) \\ &= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \\ &= \frac{17}{6}\end{aligned}$$

Remember that the evaluation is always done in the order of evaluation at the upper limit minus evaluation at the lower limit. Also be very careful with minus signs and parenthesis. It's very easy to forget them or mishandle them and get the wrong answer.

Notice as well that, in order to help with the evaluation, we rewrote the indefinite integral a little. In particular we got rid of the negative exponent on the second term. It's generally easier to evaluate the term with positive exponents.

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(c) $\int_{-1}^{2} y^2 + y^{-2}\,dy$

This integral is here to make a point. Recall that in order for us to do an integral the integrand must be continuous in the range of the limits. In this case the second term will have division by zero at $y = 0$ and since $y = 0$ is in the interval of integration, *i.e.* it is between the lower and upper limit, this integrand is not continuous in the interval of integration and so we can't do this integral.

Note that this problem will not prevent us from doing the integral in (b) since $y = 0$ is not in the interval of integration.

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So what have we learned from this example?

First, in order to do a definite integral the first thing that we need to do is the indefinite integral. So we aren't going to get out of doing indefinite integrals, they will be in every integral that we'll be doing in the rest of this course so make sure that you're getting good at computing them.

Second, we need to be on the lookout for functions that aren't continuous at any point between the limits of integration. Also, it's important to note that this will only be a problem if the point(s) of discontinuity occur between the limits of integration or at the limits themselves. If the point of discontinuity occurs outside of the limits of integration the integral can still be evaluated.

In the following sets of examples we won't make too much of an issue with continuity problems, or lack of continuity problems, unless it affects the evaluation of the integral. Do not let this convince you that you don't need to worry about this idea. It arises often enough that it can cause real problems if you aren't on the lookout for it.

Finally, note the difference between indefinite and definite integrals. Indefinite integrals are functions while definite integrals are numbers.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$ [Solution]

(b) $\int_{4}^{0} \sqrt{t}\left(t-2\right)dt$ [Solution]

(c) $\displaystyle\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw$ [Solution]

(d) $\int_{25}^{-10} dR$ [Solution]

Solution

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

There isn't a lot to this one other than simply doing the work.

$$\begin{aligned}\int_{-3}^{1} 6x^2 - 5x + 2\,dx &= \left.\left(2x^3 - \frac{5}{2}x^2 + 2x\right)\right|_{-3}^{1} \\ &= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right) \\ &= 84\end{aligned}$$

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(b) $\int_{4}^{0} \sqrt{t}\left(t-2\right)dt$

Recall that we can't integrate products as a product of integrals and so we first need to multiply the integrand out before integrating, just as we did in the indefinite integral case.

$$\int_4^0 \sqrt{t}\,(t-2)\,dt = \int_4^0 t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\,dt$$
$$= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right)\Bigg|_4^0$$
$$= 0 - \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(2)^{\frac{3}{2}}\right)$$
$$= -\frac{32}{15}$$

In the evaluation process recall that,

$$(4)^{\frac{5}{2}} = \left((4)^{\frac{1}{2}}\right)^5 = (2)^5 = 32$$
$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = (2)^3 = 8$$

Also, don't get excited about the fact that the lower limit of integration is larger than the upper limit of integration. That will happen on occasion and there is absolutely nothing wrong with this.

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(c) $\displaystyle\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw$

First, notice that we will have a division by zero issue at $w=0$, but since this isn't in the interval of integration we won't have to worry about it.

Next again recall that we can't integrate quotients as a quotient of integrals and so the first step that we'll need to do is break up the quotient so we can integrate the function.

$$\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw = \int_1^2 2w^3 - \frac{1}{w} + 3w^{-2}\,dw$$
$$= \left(\frac{1}{2}w^4 - \ln|w| - \frac{3}{w}\right)\Bigg|_1^2$$
$$= \left(8 - \ln 2 - \frac{3}{2}\right) - \left(\frac{1}{2} - \ln 1 - 3\right)$$
$$= 9 - \ln 2$$

Don't get excited about answers that don't come down to a simple integer or fraction. Often times they won't. Also don't forget that $\ln(1) = 0$.

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(d) $\int_{25}^{-10} dR$

This one is actually pretty easy. Recall that we're just integrating 1!.

$$\begin{aligned}\int_{25}^{-10} dR &= R\Big|_{25}^{-10} \\ &= -10 - 25 \\ &= -35\end{aligned}$$

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The last set of examples dealt exclusively with integrating powers of *x*. Let's work a couple of examples that involve other functions.

Example 3 Evaluate each of the following.

(a) $\int_{0}^{1} 4x - 6\sqrt[3]{x^2}\, dx$ [Solution]

(b) $\int_{0}^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\, d\theta$ [Solution]

(c) $\int_{\pi/6}^{\pi/4} 5 - 2\sec z \tan z\, dz$ [Solution]

(d) $\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\, dz$ [Solution]

(e) $\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\, dt$ [Solution]

Solution

(a) $\int_{0}^{1} 4x - 6\sqrt[3]{x^2}\, dx$.

This one is here mostly here to contrast with the next example.

$$\begin{aligned}\int_{0}^{1} 4x - 6\sqrt[3]{x^2}\, dx &= \int_{0}^{1} 4x - 6x^{\frac{2}{3}}\, dx \\ &= \left(2x^2 - \frac{18}{5}x^{\frac{5}{3}}\right)\Bigg|_{0}^{1} \\ &= 2 - \frac{18}{5} - (0) \\ &= -\frac{8}{5}\end{aligned}$$

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(b) $\int_{0}^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\, d\theta$

Be careful with signs with this one. Recall from the indefinite integral sections that it's easy to mess up the signs when integrating sine and cosine.

$$\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\, d\theta = \left(-2\cos\theta - 5\sin\theta\right)\Big|_0^{\pi/3}$$
$$= -2\cos\left(\frac{\pi}{3}\right) - 5\sin\left(\frac{\pi}{3}\right) - \left(-2\cos 0 - 5\sin 0\right)$$
$$= -1 - \frac{5\sqrt{3}}{2} + 2$$
$$= 1 - \frac{5\sqrt{3}}{2}$$

Compare this answer to the previous answer, especially the evaluation at zero. It's very easy to get into the habit of just writing down zero when evaluating a function at zero. This is especially a problem when many of the functions that we integrate involve only x's raised to positive integers and in these cases evaluate is zero of course. After evaluating many of these kinds of definite integrals it's easy to get into the habit of just writing down zero when you evaluate at zero. However, there are many functions out there that aren't zero when evaluated at zero so be careful.

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(c) $\int_{\pi/6}^{\pi/4} 5 - 2\sec z\tan z\, dz$

Not much to do other than do the integral.

$$\int_{\pi/6}^{\pi/4} 5 - 2\sec z\tan z\, dz = \left(5z - 2\sec z\right)\Big|_{\pi/6}^{\pi/4}$$
$$= 5\left(\frac{\pi}{4}\right) - 2\sec\left(\frac{\pi}{4}\right) - \left(5\left(\frac{\pi}{6}\right) - 2\sec\left(\frac{\pi}{6}\right)\right)$$
$$= \frac{5\pi}{12} - 2\sqrt{2} + \frac{4}{\sqrt{3}}$$

For the evaluation, recall that

$$\sec z = \frac{1}{\cos z}$$

and so if we can evaluate cosine at these angles we can evaluate secant at these angles.

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(d) $\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\, dz$

In order to do this one will need to rewrite both of the terms in the integral a little as follows,

$$\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\, dz = \int_{-20}^{-1} 3\mathbf{e}^{z} - \frac{1}{3}\frac{1}{z}\, dz$$

For the first term recall we used the following fact about exponents.

$$x^{-a} = \frac{1}{x^a} \qquad\qquad \frac{1}{x^{-a}} = x^a$$

In the second term, taking the 3 out of the denominator will just make integrating that term easier.

Now the integral.

$$\begin{aligned}\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz &= \left(3\mathbf{e}^{z} - \frac{1}{3}\ln|z|\right)\Bigg|_{-20}^{-1} \\ &= 3\mathbf{e}^{-1} - \frac{1}{3}\ln|-1| - \left(3\mathbf{e}^{-20} - \frac{1}{3}\ln|-20|\right) \\ &= 3\mathbf{e}^{-1} - 3\mathbf{e}^{-20} + \frac{1}{3}\ln|20|\end{aligned}$$

Just leave the answer like this. It's messy, but it's also exact.

Note that the absolute value bars on the logarithm are required here. Without them we couldn't have done the evaluation.

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(e) $\displaystyle\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\,dt$

This integral can't be done. There is division by zero in the third term at $t = 0$ and $t = 0$ lies in the interval of integration. The fact that the first two terms can be integrated doesn't matter. If even one term in the integral can't be integrated then the whole integral can't be done.

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So, we've computed a fair number of definite integrals at this point. Remember that the vast majority of the work in computing them is first finding the indefinite integral. Once we've found that the rest is just some number crunching.

There are a couple of particularly tricky definite integrals that we need to take a look at next. Actually they are only tricky until you see how to do them, so don't get too excited about them. The first one involves integrating a piecewise function.

Example 4 Given,

$$f(x) = \begin{cases} 6 & \text{if } x > 1 \\ 3x^2 & \text{if } x \le 1 \end{cases}$$

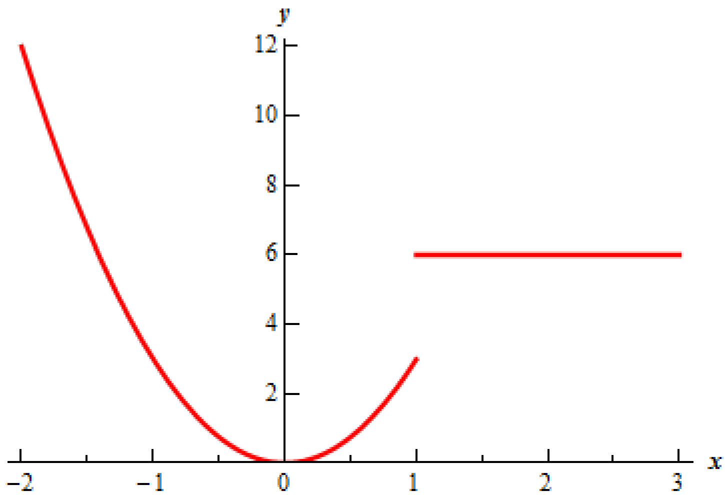
Evaluate each of the following integrals.

(a) $\displaystyle\int_{10}^{22} f(x)\,dx$ [Solution]

(b) $\displaystyle\int_{-2}^{3} f(x)\,dx$ [Solution]

Solution

Let's first start with a graph of this function.



The graph reveals a problem. This function is not continuous at $x = 1$ and we're going to have to watch out for that.

(a) $\int_{10}^{22} f(x)\,dx$

For this integral notice that $x = 1$ is not in the interval of integration and so that is something that we'll not need to worry about in this part.

Also note the limits for the integral lie entirely in the range for the first function. What this means for us is that when we do the integral all we need to do is plug in the first function into the integral.

Here is the integral.

$$\begin{aligned}\int_{10}^{22} f(x)\,dx &= \int_{10}^{22} 6\,dx \\ &= 6x\Big|_{10}^{22} \\ &= 132 - 60 \\ &= 72\end{aligned}$$

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(b) $\int_{-2}^{3} f(x)\,dx$

In this part $x = 1$ is between the limits of integration. This means that the integrand is no longer continuous in the interval of integration and that is a show stopper as far we're concerned. As noted above we simply can't integrate functions that aren't continuous in the interval of integration.

Also, even if the function was continuous at $x = 1$ we would still have the problem that the function is actually two different equations depending where we are in the interval of integration.

Let's first address the problem of the function not beginning continuous at $x = 1$. As we'll see, in this case, if we can find a way around this problem the second problem will also get taken care of at the same time.

In the previous examples where we had functions that weren't continuous we had division by zero and no matter how hard we try we can't get rid of that problem. Division by zero is a real problem and we can't really avoid it. In this case the discontinuity does not stem from problems with the function not existing at $x = 1$. Instead the function is not continuous because it takes on different values on either sides of $x = 1$. We can "remove" this problem by recalling Property 5 from the previous section. This property tells us that we can write the integral as follows,

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

On each of these intervals the function is continuous. In fact we can say more. In the first integral we will have *x* between -2 and 1 and this means that we can use the second equation for $f(x)$ and likewise for the second integral *x* will be between 1 and 3 and so we can use the first function for $f(x)$. The integral in this case is then,

$$\begin{aligned}
\int_{-2}^{3} f(x)\,dx &= \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx \\
&= \int_{-2}^{1} 3x^2\,dx + \int_{1}^{3} 6\,dx \\
&= x^3\Big|_{-2}^{1} + 6x\Big|_{1}^{3} \\
&= 1 - (-8) + (18 - 6) \\
&= 21
\end{aligned}$$

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So, to integrate a piecewise function, all we need to do is break up the integral at the break point(s) that happen to occur in the interval of integration and then integrate each piece.

Next we need to look at is how to integrate an absolute value function.

Example 5 Evaluate the following integral.

$$\int_{0}^{3} |3t - 5|\,dt$$

Solution

Recall that the point behind indefinite integration (which we'll need to do in this problem) is to determine what function we differentiated to get the integrand. To this point we've not seen any functions that will differentiate to get an absolute value nor will we ever see a function that will differentiate to get an absolute value.

The only way that we can do this problem is to get rid of the absolute value. To do this we need to recall the definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Once we remember that we can define absolute value as a piecewise function we can use the work from Example 4 as a guide for doing this integral.

What we need to do is determine where the quantity on the inside of the absolute value bars is negative and where it is positive. It looks like if $t > \frac{5}{3}$ the quantity inside the absolute value is positive and if $t < \frac{5}{3}$ the quantity inside the absolute value is negative.

Next, note that $t = \frac{5}{3}$ is in the interval of integration and so, if we break up the integral at this point we get,

$$\int_0^3 |3t-5|\,dt = \int_0^{\frac{5}{3}} |3t-5|\,dt + \int_{\frac{5}{3}}^3 |3t-5|\,dt$$

Now, in the first integrals we have $t < \frac{5}{3}$ and so $3t-5<0$ in this interval of integration. That means we can drop the absolute value bars if we put in a minus sign. Likewise in the second integral we have $t > \frac{5}{3}$ which means that in this interval of integration we have $3t-5>0$ and so we can just drop the absolute value bars in this integral.

After getting rid of the absolute value bars in each integral we can do each integral. So, doing the integration gives,

$$\begin{aligned}
\int_0^3 |3t-5|\,dt &= \int_0^{\frac{5}{3}} -(3t-5)\,dt + \int_{\frac{5}{3}}^3 3t-5\,dt \\
&= \int_0^{\frac{5}{3}} -3t+5\,dt + \int_{\frac{5}{3}}^3 3t-5\,dt \\
&= \left(-\frac{3}{2}t^2+5t\right)\Bigg|_0^{\frac{5}{3}} + \left(\frac{3}{2}t^2-5t\right)\Bigg|_{\frac{5}{3}}^3 \\
&= -\frac{3}{2}\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - (0) + \left(\frac{3}{2}(3)^2 - 5(3) - \left(\frac{3}{2}\left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right)\right)\right) \\
&= \frac{25}{6} + \frac{8}{3} \\
&= \frac{41}{6}
\end{aligned}$$

Integrating absolute value functions isn't too bad. It's a little more work than the "standard" definite integral, but it's not really all that much more work. First, determine where the quantity inside the absolute value bars is negative and where it is positive. When we've determined that point all we need to do is break up the integral so that in each range of limits the quantity inside the absolute value bars is always positive or always negative. Once this is done we can drop the absolute value bars (adding negative signs when the quantity is negative) and then we can do the integral as we've always done.

Substitution Rule for Definite Integrals

We now need to go back and revisit the substitution rule as it applies to definite integrals. At some level there really isn't a lot to do in this section. Recall that the first step in doing a definite integral is to compute the indefinite integral and that hasn't changed. We will still compute the indefinite integral first. This means that we already know how to do these. We use the substitution rule to find the indefinite integral and then do the evaluation.

There are however, two ways to deal with the evaluation step. One of the ways of doing the evaluation is the probably the most obvious at this point, but also has a point in the process where we can get in trouble if we aren't paying attention.

Let's work an example illustrating both ways of doing the evaluation step.

Example 1 Evaluate the following definite integral.

$$\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt$$

Solution

Let's start off looking at the first way of dealing with the evaluation step. We'll need to be careful with this method as there is a point in the process where if we aren't paying attention we'll get the wrong answer.

Solution 1 :

We'll first need to compute the indefinite integral using the substitution rule. Note however, that we will constantly remind ourselves that this is a definite integral by putting the limits on the integral at each step. Without the limits it's easy to forget that we had a definite integral when we've gotten the indefinite integral computed.

In this case the substitution is,

$$u = 1-4t^3 \qquad du = -12t^2 dt \qquad \Rightarrow \qquad t^2 dt = -\frac{1}{12} du$$

Plugging this into the integral gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt &= -\frac{1}{6}\int_{-2}^{0} u^{\frac{1}{2}}\, du \\ &= -\frac{1}{9} u^{\frac{3}{2}}\Big|_{-2}^{0}\end{aligned}$$

Notice that we didn't do the evaluation yet. This is where the potential problem arises with this solution method. The limits given here are from the original integral and hence are values of *t*. We have *u*'s in our solution. We can't plug values of *t* in for *u*.

Therefore, we will have to go back to *t*'s before we do the substitution. This is the standard step in the substitution process, but it is often forgotten when doing definite integrals. Note as well that in this case, if we don't go back to t's we will have a small problem in that one of the evaluations will end up giving us a complex number.

So, finishing this problem gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{9}\left(1-4t^3\right)^{\frac{3}{2}}\Big|_{-2}^{0} \\ &= -\frac{1}{9}-\left(-\frac{1}{9}\left(33\right)^{\frac{3}{2}}\right) \\ &= \frac{1}{9}\left(33\sqrt{33}-1\right)\end{aligned}$$

So, that was the first solution method. Let's take a look at the second method.

Solution 2 :

Note that this solution method isn't really all that different from the first method. In this method we are going to remember that when doing a substitution we want to eliminate all the t's in the integral and write everything in terms of u.

When we say all here we really mean all. In other words, remember that the limits on the integral are also values of t and we're going to convert the limits into u values. Converting the limits is pretty simple since our substitution will tell us how to relate t and u so all we need to do is plug in the original t limits into the substitution and we'll get the new u limits.

Here is the substitution (it's the same as the first method) as well as the limit conversions.

$$u = 1-4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$

$$t = -2 \qquad \Rightarrow \qquad u = 1-4\left(-2\right)^3 = 33$$

$$t = 0 \qquad \Rightarrow \qquad u = 1-4\left(0\right)^3 = 1$$

The integral is now,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}}\,du \\ &= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}\end{aligned}$$

As with the first method let's pause here a moment to remind us what we're doing. In this case, we've converted the limits to u's and we've also got our integral in terms of u's and so here we can just plug the limits directly into our integral. Note that in this case we won't plug our substitution back in. Doing this here would cause problems as we would have t's in the integral and our limits would be u's. Here's the rest of this problem.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt = -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}$$

$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) = \frac{1}{9}\left(33\sqrt{33}-1\right)$$

We got exactly the same answer and this time didn't have to worry about going back to t's in our answer.

So, we've seen two solution techniques for computing definite integrals that require the substitution rule. Both are valid solution methods and each have their uses. We will be using the second exclusively however since it makes the evaluation step a little easier.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw$ [Solution]

(b) $\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\,dx$ [Solution]

(c) $\int_{0}^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\,dy$ [Solution]

(d) $\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz$ [Solution]

Solution

Since we've done quite a few substitution rule integrals to this time we aren't going to put a lot of effort into explaining the substitution part of things here.

(a) $\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw$

The substitution and converted limits are,

$$u = 2w + w^2 \qquad du = (2+2w)\,dw \qquad \Rightarrow \qquad (1+w)\,dw = \frac{1}{2}du$$

$$w = -1 \quad \Rightarrow \quad u = -1 \qquad\qquad w = 5 \quad \Rightarrow \quad u = 35$$

Sometimes a limit will remain the same after the substitution. Don't get excited when it happens and don't expect it to happen all the time.

Here is the integral,

$$\int_{-1}^{5}(1+w)\left(2w+w^2\right)^5 dw = \frac{1}{2}\int_{-1}^{35} u^5\,du$$
$$= \frac{1}{12}u^6\Big|_{-1}^{35} = 153188802$$

Don't get excited about large numbers for answers here. Sometime they are. That's life.

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(b) $\displaystyle\int_{-2}^{-6}\frac{4}{(1+2x)^3}-\frac{5}{1+2x}\,dx$

Here is the substitution and converted limits for this problem,

$$u = 1+2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$
$$x=-2 \quad \Rightarrow \quad u=-3 \qquad\qquad x=-6 \quad \Rightarrow \quad u=-11$$

The integral is then,

$$\int_{-2}^{-6}\frac{4}{(1+2x)^3}-\frac{5}{1+2x}\,dx = \frac{1}{2}\int_{-3}^{-11}4u^{-3}-\frac{5}{u}\,du$$
$$= \frac{1}{2}\left(-2u^{-2}-5\ln|u|\right)\Big|_{-3}^{-11}$$
$$= \frac{1}{2}\left(-\frac{2}{121}-5\ln 11\right)-\frac{1}{2}\left(-\frac{2}{9}-5\ln 3\right)$$
$$= \frac{112}{1089}-\frac{5}{2}\ln 11+\frac{5}{2}\ln 3$$

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(c) $\displaystyle\int_0^{\frac{1}{2}}\mathbf{e}^y+2\cos(\pi y)\,dy$

This integral needs to be split into two integrals since the first term doesn't require a substitution and the second does.

$$\int_0^{\frac{1}{2}}\mathbf{e}^y+2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}}\mathbf{e}^y\,dy+\int_0^{\frac{1}{2}}2\cos(\pi y)\,dy$$

Here is the substitution and converted limits for the second term.

$$u=\pi y \qquad du=\pi dy \qquad \Rightarrow \qquad dy=\frac{1}{\pi}du$$
$$y=0 \quad \Rightarrow \quad u=0 \qquad\qquad y=\frac{1}{2} \quad \Rightarrow \quad u=\frac{\pi}{2}$$

Here is the integral.

$$\int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}} \mathbf{e}^y\,dy + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\,du$$

$$= \mathbf{e}^y\Big|_0^{\frac{1}{2}} + \frac{2}{\pi}\sin u\Big|_0^{\frac{\pi}{2}}$$

$$= \mathbf{e}^{\frac{1}{2}} - \mathbf{e}^0 + \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{\pi}\sin 0$$

$$= \mathbf{e}^{\frac{1}{2}} - 1 + \frac{2}{\pi}$$

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(d) $\displaystyle\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz$

This integral will require two substitutions. So first split up the integral so we can do a substitution on each term.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = \int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right)dz - \int_{\frac{\pi}{3}}^{0} 5\cos(\pi - z)\,dz$$

There are the two substitutions for these integrals.

$$u = \frac{z}{2} \qquad du = \frac{1}{2}dz \qquad \Rightarrow \qquad dz = 2du$$

$$z = \frac{\pi}{3} \quad \Rightarrow \quad u = \frac{\pi}{6} \qquad\qquad z = 0 \quad \Rightarrow \quad u = 0$$

$$v = \pi - z \qquad dv = -dz \qquad \Rightarrow \qquad dz = -dv$$

$$z = \frac{\pi}{3} \quad \Rightarrow \quad v = \frac{2\pi}{3} \qquad\qquad z = 0 \quad \Rightarrow \quad v = \pi$$

Here is the integral for this problem.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = 6\int_{\frac{\pi}{6}}^{0} \sin(u)\,du + 5\int_{\frac{2\pi}{3}}^{\pi} \cos(v)\,dv$$

$$= -6\cos(u)\Big|_{\frac{\pi}{6}}^{0} + 5\sin(v)\Big|_{\frac{2\pi}{3}}^{\pi}$$

$$= 3\sqrt{3} - 6 + \left(-\frac{5\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} - 6$$

[Return to Problems]

The next set of examples is designed to make sure that we don't forget about a very important point about definite integrals.

Example 3 Evaluate each of the following.

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2-8t^2}\,dt$ [Solution]

(b) $\displaystyle\int_{3}^{5} \frac{4t}{2-8t^2}\,dt$ [Solution]

Solution

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2-8t^2}\,dt$

Be careful with this integral. The denominator is zero at $t = \pm\frac{1}{2}$ and both of these are in the interval of integration. Therefore, this integrand is not continuous in the interval and so the integral can't be done.

Be careful with definite integrals and be on the lookout for division by zero problems. In the previous section they were easy to spot since all the division by zero problems that we had there were at zero. Once we move into substitution problems however they will not always be so easy to spot so make sure that you first take a quick look at the integrand and see if there are any continuity problems with the integrand and if they occur in the interval of integration.

[Return to Problems]

(b) $\displaystyle\int_{3}^{5} \frac{4t}{2-8t^2}\,dt$

Now, in this case the integral can be done because the two points of discontinuity, $t = \pm\frac{1}{2}$, are both outside of the interval of integration. The substitution and converted limits in this case are,

$$u = 2-8t^2 \qquad du = -16t\,dt \qquad \Rightarrow \qquad dz = -\frac{1}{16}\,dt$$

$$t = 3 \quad \Rightarrow \quad u = -70 \qquad\qquad t = 5 \quad \Rightarrow \quad u = -198$$

The integral is then,

$$\begin{aligned}\int_{3}^{5} \frac{4t}{2-8t^2}\,dt &= -\frac{4}{16}\int_{-70}^{-198} \frac{1}{u}\,du \\ &= -\frac{1}{4}\ln\left|u\right|\Big|_{-70}^{-198} \\ &= -\frac{1}{4}\left(\ln\left(198\right) - \ln\left(70\right)\right)\end{aligned}$$

[Return to Problems]

Let's work another set of examples. These are a little tougher (at least in appearance) than the previous sets.

Example 4 Evaluate each of the following.

(a) $\displaystyle\int_0^{\ln(1-\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx$ [Solution]

(b) $\displaystyle\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\, dt$ [Solution]

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\, dP$ [Solution]

(d) $\displaystyle\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos\left(\sin(x)\right) dx$ [Solution]

(e) $\displaystyle\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\, dw$ [Solution]

Solution

(a) $\displaystyle\int_0^{\ln(1-\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx$

The limits are a little unusual in this case, but that will happen sometimes so don't get too excited about it. Here is the substitution.

$$u = 1-\mathbf{e}^x \qquad\qquad du = -\mathbf{e}^x dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1-\mathbf{e}^0 = 1-1 = 0$$

$$x = \ln(1-\pi) \qquad \Rightarrow \qquad u = 1-\mathbf{e}^{\ln(1-\pi)} = 1-(1-\pi) = \pi$$

The integral is then,

$$\begin{aligned}\int_0^{\ln(1-\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx &= -\int_0^{\pi} \cos u\, du \\ &= -\sin(u)\Big|_0^{\pi} \\ &= -\left(\sin\pi - \sin 0\right) = 0\end{aligned}$$

[Return to Problems]

(b) $\displaystyle\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\, dt$

Here is the substitution and converted limits for this problem.

$$u = \ln t \qquad du = \frac{1}{t}dt$$

$$t = \mathbf{e}^2 \quad \Rightarrow \quad u = \ln \mathbf{e}^2 = 2 \qquad\qquad t = \mathbf{e}^6 \quad \Rightarrow \quad u = \ln \mathbf{e}^6 = 6$$

The integral is,

$$\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\,dt = \int_2^6 u^4\,du$$

$$= \left.\frac{1}{5}u^5\right|_2^6$$

$$= \frac{7744}{5}$$

[Return to Problems]

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP$

Here is the substitution and converted limits and don't get too excited about the substitution. It's a little messy in the case, but that can happen on occasion.

$$u = 2+\sec(3P) \qquad du = 3\sec(3P)\tan(3P)\,dP \quad \Rightarrow \quad \sec(3P)\tan(3P)\,dP = \frac{1}{3}du$$

$$P = \frac{\pi}{12} \qquad \Rightarrow \qquad u = 2+\sec\left(\frac{\pi}{4}\right) = 2+\sqrt{2}$$

$$P = \frac{\pi}{9} \qquad \Rightarrow \qquad u = 2+\sec\left(\frac{\pi}{3}\right) = 4$$

Here is the integral,

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP = \frac{1}{3}\int_{2+\sqrt{2}}^{4} u^{-\frac{1}{3}}\,du$$

$$= \left.\frac{1}{2}u^{\frac{2}{3}}\right|_{2+\sqrt{2}}^{4}$$

$$= \frac{1}{2}\left(4^{\frac{3}{2}} - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right)$$

$$= \frac{1}{2}\left(8 - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right)$$

So, not only was the substitution messy, but we also a messy answer, but again that's life on occasion.

[Return to Problems]

(d) $\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$

This problem not as bad as it looks. Here is the substitution and converted limits.

$$u = \sin x \qquad du = \cos x\,dx$$

$$x = \frac{\pi}{2} \quad \Rightarrow \quad u = \sin\frac{\pi}{2} = 1 \qquad\qquad x = -\pi \quad \Rightarrow \quad u = \sin(-\pi) = 0$$

The cosine in the very front of the integrand will get substituted away in the differential and so this integrand actually simplifies down significantly. Here is the integral.

$$\begin{aligned} \int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx &= \int_0^1 \cos u\,du \\ &= \sin(u)\Big|_0^1 \\ &= \sin(1) - \sin(0) \\ &= \sin(1) \end{aligned}$$

Don't get excited about these kinds of answers. On occasion we will end up with trig function evaluations like this.

[Return to Problems]

(e) $\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw$

This is also a tricky substitution (at least until you see it). Here it is,

$$u = \frac{2}{w} \qquad du = -\frac{2}{w^2}\,dw \qquad \Rightarrow \qquad \frac{1}{w^2}\,dw = -\frac{1}{2}\,du$$

$$w = 2 \quad \Rightarrow \quad u = 1 \qquad\qquad w = \frac{1}{50} \quad \Rightarrow \quad u = 100$$

Here is the integral.

$$\begin{aligned} \int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw &= -\frac{1}{2}\int_{100}^{1} \mathbf{e}^u\,du \\ &= -\frac{1}{2}\mathbf{e}^u\Big|_{100}^{1} \\ &= -\frac{1}{2}\left(\mathbf{e}^1 - \mathbf{e}^{100}\right) \end{aligned}$$

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In this last set of examples we saw some tricky substitutions and messy limits, but these are a fact of life with some substitution problems and so we need to be prepared for dealing with them when the happen.

Even and Odd Functions

This is the last topic that we need to discuss in this chapter. It is probably better suited in the previous section, but that section has already gotten fairly large so I decided to put it here.

First, recall that an even function is any function which satisfies,

$$f(-x) = f(x)$$

Typical examples of even functions are,

$$f(x) = x^2 \qquad\qquad f(x) = \cos(x)$$

An odd function is any function which satisfies,

$$f(-x) = -f(x)$$

The typical examples of odd functions are,

$$f(x) = x^3 \qquad\qquad f(x) = \sin(x)$$

There are a couple of nice facts about integrating even and odd functions over the interval [-*a*,*a*]. If *f(x)* is an even function then,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Likewise, if *f(x)* is an odd function then,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Note that in order to use these facts the limit of integration must be the same number, but opposite signs!

Example 5 Integrate each of the following.

(a) $\int_{-2}^{2} 4x^4 - x^2 + 1\,dx$ [Solution]

(b) $\int_{-10}^{10} x^5 + \sin(x)\,dx$ [Solution]

Solution

Neither of these are terribly difficult integrals, but we can use the facts on them anyway.

(a) $\int_{-2}^{2} 4x^4 - x^2 + 1\,dx$

In this case the integrand is even and the interval is correct so,

$$\begin{aligned}\int_{-2}^{2} 4x^4 - x^2 + 1\,dx &= 2\int_{0}^{2} 4x^4 - x^2 + 1\,dx \\ &= 2\left(\frac{4}{5}x^5 - \frac{1}{3}x^3 + x\right)\Bigg|_0^2 \\ &= \frac{748}{15}\end{aligned}$$

So, using the fact cut the evaluation in half (in essence since one of the new limits was zero).

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(b) $\int_{-10}^{10} x^5 + \sin\left(x\right) dx$

The integrand in this case is odd and the interval is in the correct form and so we don't even need to integrate. Just use the fact.

$$\int_{-10}^{10} x^5 + \sin\left(x\right) dx = 0$$

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Note that the limits of integration are important here. Take the last integral as an example. A small change to the limits will not give us zero.

$$\int_{-10}^{9} x^5 + \sin\left(x\right) dx = \cos\left(10\right) - \cos\left(9\right) - \frac{468559}{6} = -78093.09461$$

The moral here is to be careful and not misuse these facts.

Applications of Integrals

Introduction

In this last chapter of this course we will be taking a look at a couple of applications of integrals. There are many other applications, however many of them require integration techniques that are typically taught in Calculus II. We will therefore be focusing on applications that can be done only with knowledge taught in this course.

Because this chapter is focused on the applications of integrals it is assumed in all the examples that you are capable of doing the integrals. There will not be as much detail in the integration process in the examples in this chapter as there was in the examples in the previous chapter.

Here is a listing of applications covered in this chapter.

Average Function Value – We can use integrals to determine the average value of a function.

Area Between Two Curves – In this section we'll take a look at determining the area between two curves.

Volumes of Solids of Revolution / Method of Rings – This is the first of two sections devoted to find the volume of a solid of revolution. In this section we look that the method of rings/disks.

Volumes of Solids of Revolution / Method of Cylinders – This is the second section devoted to finding the volume of a solid of revolution. Here we will look at the method of cylinders.

Work – The final application we will look at is determining the amount of work required to move an object.

Average Function Value

The first application of integrals that we'll take a look at is the average value of a function. The following fact tells us how to compute this.

Average Function Value

The average value of a function $f(x)$ over the interval [*a*,*b*] is given by,

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

To see a justification of this formula see the Proof of Various Integral Properties section of the Extras chapter.

Let's work a couple of quick examples.

Example 1 Determine the average value of each of the following functions on the given interval.

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$ [Solution]

(b) $R(z) = \sin(2z)\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$ [Solution]

Solution

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$

There's really not a whole lot to do in this problem other than just use the formula.

$$\begin{aligned} f_{avg} &= \frac{1}{\frac{5}{2}-(-1)}\int_{-1}^{\frac{5}{2}} t^2 - 5t + 6\cos(\pi t)\,dt \\ &= \frac{2}{7}\left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + \frac{6}{\pi}\sin(\pi t)\right)\Bigg|_{-1}^{\frac{5}{2}} \\ &= \frac{12}{7\pi} - \frac{13}{6} \\ &= -1.620993 \end{aligned}$$

You caught the substitution needed for the third term right?

So, the average value of this function of the given interval is -1.620993.

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(b) $R(z) = \sin(2z)\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$

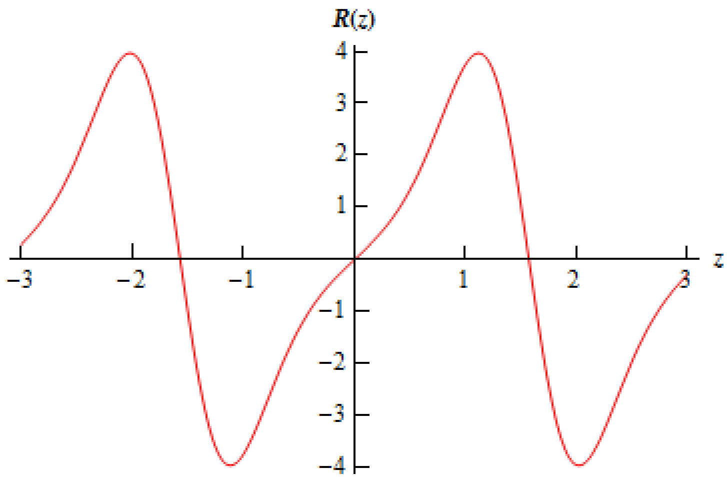
Again, not much to do here other than use the formula. Note that the integral will need the following substitution.

$$u = 1 - \cos(2z)$$

Here is the average value of this function,

$$\begin{aligned} R_{avg} &= \frac{1}{\pi - (-\pi)} \int_{\pi}^{-\pi} \sin(2z)\, \mathbf{e}^{1-\cos(2z)}\, dz \\ &= \frac{1}{2}\mathbf{e}^{1-\cos(2z)} \Big|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

So, in this case the average function value is zero. Do not get excited about getting zero here. It will happen on occasion. In fact, if you look at the graph of the function on this interval it's not too hard to see that this is the correct answer.



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There is also a theorem that is related to the average function value.

The Mean Value Theorem for Integrals

If $f(x)$ is a continuous function on [a,b] then there is a number c in [a,b] such that,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Note that this is very similar to the Mean Value Theorem that we saw in the Derivatives Applications chapter. See the Proof of Various Integral Properties section of the Extras chapter for the proof.

Note that one way to think of this theorem is the following. First rewrite the result as,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

and from this we can see that this theorem is telling us that there is a number $a < c < b$ such that $f_{avg} = f\left(c\right)$. Or, in other words, if $f\left(x\right)$ is a continuous function then somewhere in [a,b] the function will take on its average value.

Let's take a quick look at an example using this theorem.

Example 2 Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f\left(x\right) = x^2 + 3x + 2$ on the interval [1,4]

Solution

First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. So, let's do that.

$$\int_1^4 x^2 + 3x + 2\,dx = \left(c^2 + 3c + 2\right)\left(4 - 1\right)$$

$$\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\Bigg|_1^4 = 3\left(c^3 + 3c + 2\right)$$

$$\frac{99}{2} = 3c^3 + 9c + 6$$

$$0 = 3c^3 + 9c - \frac{87}{2}$$

This is a quadratic equation that we can solve. Using the quadratic formula we get the following two solutions,

$$c = \frac{-3 + \sqrt{67}}{2} = 2.593$$

$$c = \frac{-3 - \sqrt{67}}{2} = -5.593$$

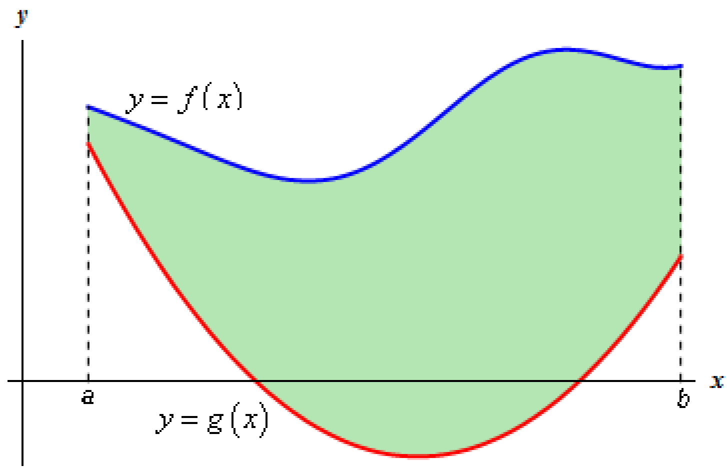
Clearly the second number is not in the interval and so that isn't the one that we're after. The first however is in the interval and so that's the number we want.

Note that it is possible for both numbers to be in the interval so don't expect only one to be in the interval.

Area Between Curves

In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

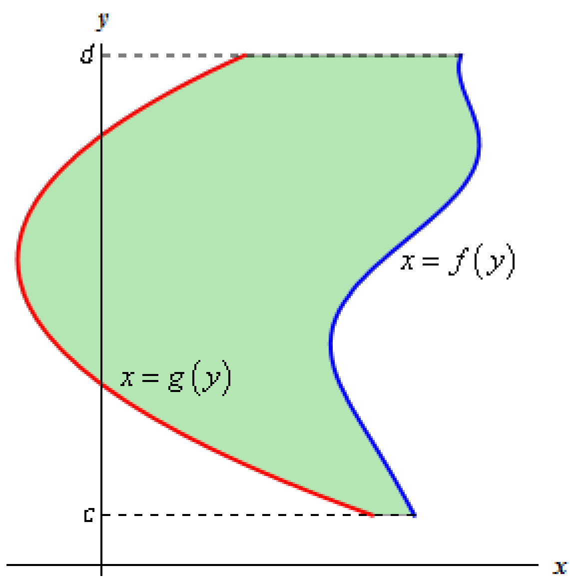
In the first case we are want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval [a,b]. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



In the Area and Volume Formulas section of the Extras chapter we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x)\,dx \tag{1}$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval [c,d] with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\,dy \tag{2}$$

Now (1) and (2) are perfectly serviceable formulas, however, it is sometimes easy to forget that these always require the first function to be the larger of the two functions. So, instead of these formulas we will instead use the following "word" formulas to make sure that we remember that the formulas area always the "larger" function minus the "smaller" function.

In the first case we will use,

$$A = \int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx, \qquad a \le x \le b \tag{3}$$

In the second case we will use,

$$A = \int_c^d \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right) dy, \qquad c \le y \le d \tag{4}$$

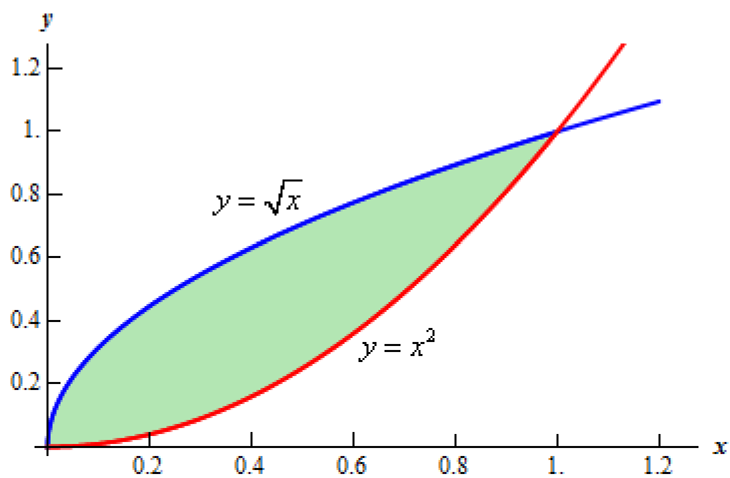
Using these formulas will always force us to think about what is going on with each problem and to make sure that we've got the correct order of functions when we go to use the formula.

Let's work an example.

Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution

First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Note that we don't take any part of the region to the right of the intersection point of these two graphs. In this region there is no boundary on the right side and so is not part of the enclosed area. Remember that one of the given functions must be on the each boundary of the enclosed region.

Also from this graph it's clear that the upper function will be dependent on the range of *x*'s that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the *x*'s. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x = 0$ and $x = 1$ so these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$\begin{aligned} A &= \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx \\ &= \int_0^1 \sqrt{x} - x^2 \, dx \\ &= \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right) \Bigg|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Before moving on to the next example, there are a couple of important things to note.

First, in almost all of these problems a graph is pretty much required. Often the bounding region, which will give the limits of integration, is difficult to determine without a graph.

Also, it can often be difficult to determine which of the functions is the upper function and with is the lower function without a graph. This is especially true in cases like the last example where the answer to that question actually depended upon the range of *x*'s that we were using.

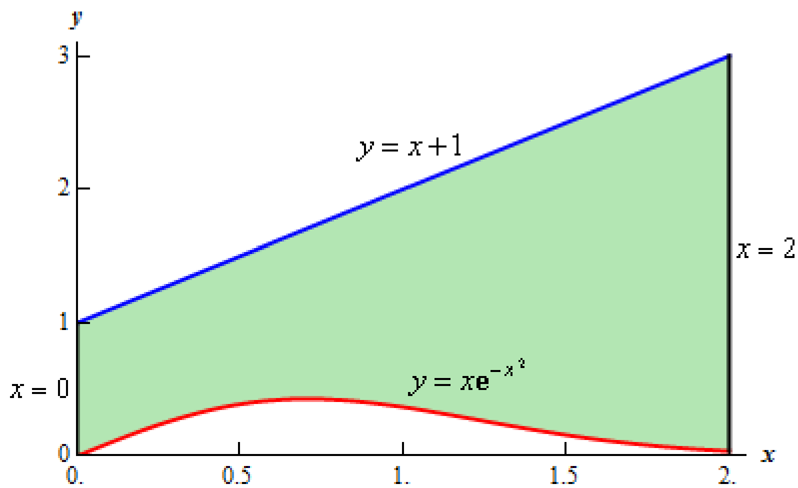
Finally, unlike the area under a curve that we looked at in the previous chapter the area between two curves will always be positive. If we get a negative number or zero we can be sure that we've made a mistake somewhere and will need to go back and find it.

Note as well that sometimes instead of saying region enclosed by we will say region bounded by. They mean the same thing.

Let's work some more examples.

Example 2 Determine the area of the region bounded by $y = x\mathbf{e}^{-x^2}$, $y = x+1$, $x = 2$, and the *y*-axis.

Solution
In this case the last two pieces of information, $x = 2$ and the *y*-axis, tell us the right and left boundaries of the region. Also, recall that the *y*-axis is given by the line $x = 0$. Here is the graph with the enclosed region shaded in.



Here, unlike the first example, the two curves don't meet. Instead we rely on two vertical lines to bound the left and right sides of the region as we noted above

Here is the integral that will give the area.

$$
\begin{aligned}
A &= \int_a^b \begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix} dx \\
&= \int_0^2 x + 1 - x\mathbf{e}^{-x^2}\,dx \\
&= \left(\frac{1}{21}x^2 + x + \frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_0^2 \\
&= \frac{7}{2} + \frac{\mathbf{e}^{-4}}{2} = 3.5092
\end{aligned}
$$

Example 3 Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

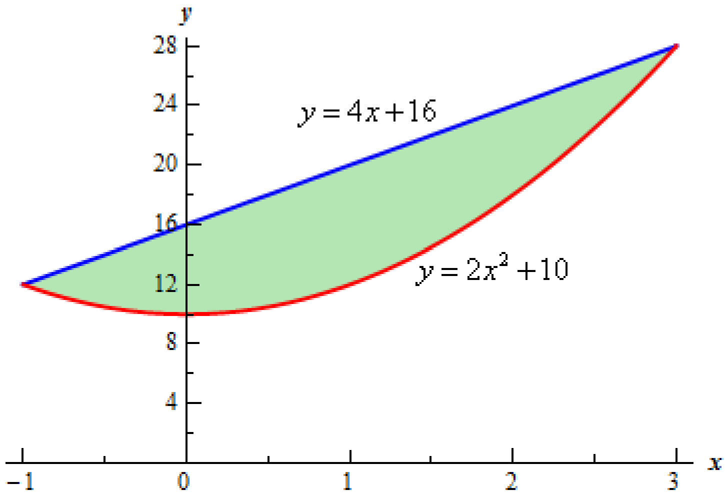
Solution
In this case the intersection points (which we'll need eventually) are not going to be easily identified from the graph so let's go ahead and get them now. Note that for most of these problems you'll not be able to accurately identify the intersection points from the graph and so you'll need to be able to determine them by hand. In this case we can get the intersection points

by setting the two equations equal.

$$\begin{aligned} 2x^2 + 10 &= 4x + 16 \\ 2x^2 - 4x - 6 &= 0 \\ 2(x+1)(x-3) &= 0 \end{aligned}$$

So it looks like the two curves will intersect at $x = -1$ and $x = 3$. If we need them we can get the *y* values corresponding to each of these by plugging the values back into either of the equations. We'll leave it to you to verify that the coordinates of the two intersection points on the graph are (-1,12) and (3,28).

Note as well that if you aren't good at graphing knowing the intersection points can help in at least getting the graph started. Here is a graph of the region.



With the graph we can now identify the upper and lower function and so we can now find the enclosed area.

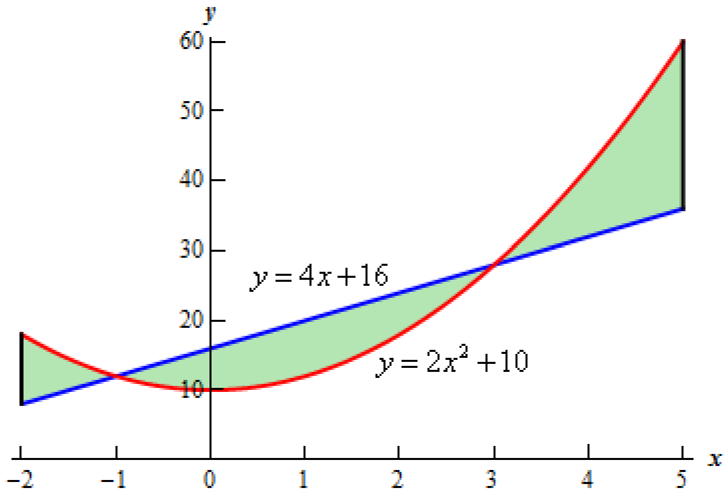
$$\begin{aligned} A &= \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx \\ &= \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) dx \\ &= \int_{-1}^{3} -2x^2 + 4x + 6\, dx \\ &= \left. \left(-\frac{2}{3}x^3 + 2x^2 + 6x \right) \right|_{-1}^{3} \\ &= \frac{64}{3} \end{aligned}$$

Be careful with parenthesis in these problems. One of the more common mistakes students make with these problems is to neglect parenthesis on the second term.

Example 4 Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$

Solution
So, the functions used in this problem are identical to the functions from the first problem. The difference is that we've extended the bounded region out from the intersection points. Since these are the same functions we used in the previous example we won't bother finding the intersection points again.

Here is a graph of this region.



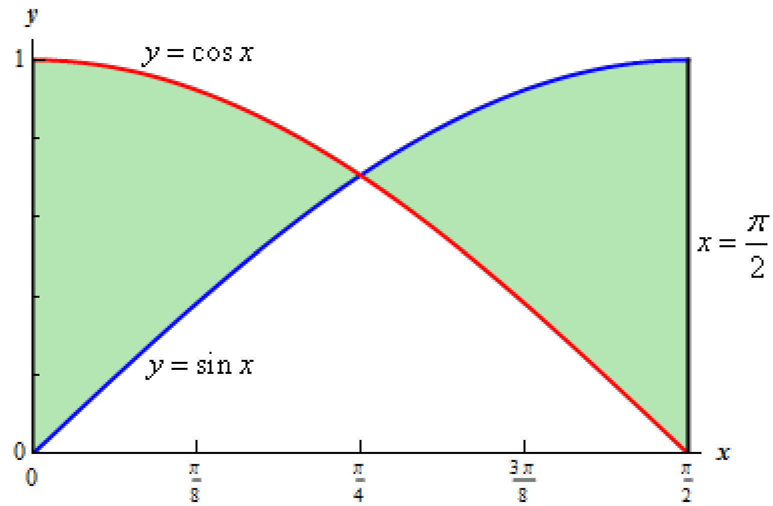
Okay, we have a small problem here. Our formula requires that one function always be the upper function and the other function always be the lower function and we clearly do not have that here. However, this actually isn't the problem that it might at first appear to be. There are three regions in which one function is always the upper function and the other is always the lower function. So, all that we need to do is find the area of each of the three regions, which we can do, and then add them all up.

Here is the area.

$$\begin{aligned} A &= \int_{-2}^{-1} 2x^2 + 10 - \left(4x + 16\right) dx + \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) dx + \int_{3}^{5} 2x^2 + 10 - \left(4x + 16\right) dx \\ &= \int_{-2}^{-1} 2x^2 - 4x - 6\, dx + \int_{-1}^{3} -2x^2 + 4x + 6\, dx + \int_{3}^{5} 2x^2 - 4x - 6\, dx \\ &= \left.\left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\right|_{-2}^{-1} + \left.\left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\right|_{-1}^{3} + \left.\left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\right|_{3}^{5} \\ &= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} \\ &= \frac{142}{3} \end{aligned}$$

Example 5 Determine the area of the region enclosed by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$, and the y-axis.

Solution
First let's get a graph of the region.



So, we have another situation where we will need to do two integrals to get the area. The intersection point will be where

$$\sin x = \cos x$$

in the interval. We'll leave it to you to verify that this will be $x = \frac{\pi}{4}$. The area is then,

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx \\ &= \left(\sin x + \cos x\right)\Big|_0^{\frac{\pi}{4}} + \left(-\cos x - \sin x\right)\Big|_{\pi/4}^{\pi/2} \\ &= \sqrt{2} - 1 + \left(\sqrt{2} - 1\right) \\ &= 2\sqrt{2} - 2 = 0.828427 \end{aligned}$$

We will need to be careful with this next example.

Example 6 Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

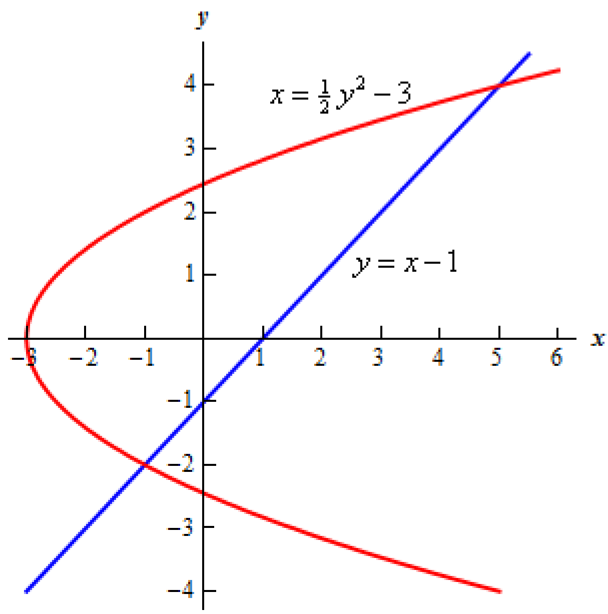
Solution
Don't let the first equation get you upset. We will have to deal with these kinds of equations occasionally so we'll need to get used to dealing with them.

As always, it will help if we have the intersection points for the two curves. In this case we'll get the intersection points by solving the second equation for *x* and the setting them equal. Here is that work,

$$\begin{aligned} y+1 &= \frac{1}{2}y^2 - 3 \\ 2y+2 &= y^2 - 6 \\ 0 &= y^2 - 2y - 8 \\ 0 &= (y-4)(y+2) \end{aligned}$$

So, it looks like the two curves will intersect at $y = -2$ and $y = 4$ or if we need the full coordinates they will be : (-1,-2) and (5,4).

Here is a sketch of the two curves.



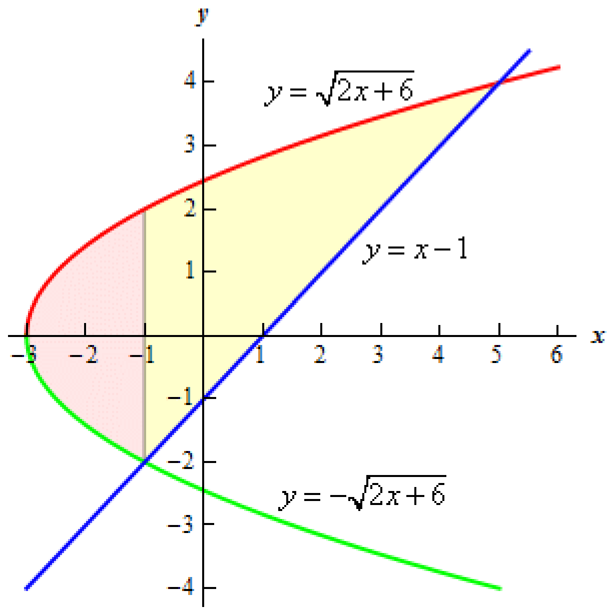
Now, we will have a serious problem at this point if we aren't careful. To this point we've been using an upper function and a lower function. To do that here notice that there are actually two portions of the region that will have different lower functions. In the range [-2,-1] the parabola is actually both the upper and the lower function.

To use the formula that we've been using to this point we need to solve the parabola for *y*. This gives,

$$y = \pm\sqrt{2x+6}$$

where the "+" gives the upper portion of the parabola and the "-" gives the lower portion.

Here is a sketch of the complete area with each region shaded that we'd need if we were going to use the first formula.



The integrals for the area would then be,

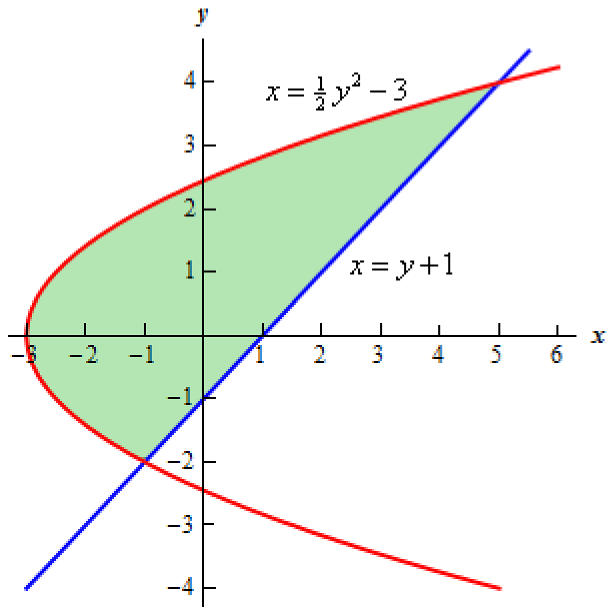
$$\begin{aligned}
A &= \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx + \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6} - x + 1\, dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6}\, dx + \int_{-1}^{5} -x + 1\, dx \\
&= \frac{2}{3} u^{\frac{3}{2}} \Big|_{0}^{4} + \frac{1}{3} u^{\frac{3}{2}} \Big|_{4}^{16} + \left(-\frac{1}{2} x^2 + x \right) \Big|_{-1}^{5} \\
&= 18
\end{aligned}$$

While these integrals aren't terribly difficult they are more difficult than they need to be.

Recall that there is another formula for determining the area. It is,

$$A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad\qquad c \le y \le d$$

and in our case we do have one function that is always on the left and the other is always on the right. So, in this case this is definitely the way to go. Note that we will need to rewrite the equation of the line since it will need to be in the form $x = f(y)$ but that is easy enough to do. Here is the graph for using this formula.



The area is,

$$A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy$$

$$= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy$$

$$= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy$$

$$= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Bigg|_{-2}^{4}$$

$$= 18$$

This is the same that we got using the first formula and this was definitely easier than the first method.

So, in this last example we've seen a case where we could use either formula to find the area. However, the second was definitely easier.

Students often come into a calculus class with the idea that the only easy way to work with functions is to use them in the form $y = f(x)$. However, as we've seen in this previous example there are definitely times when it will be easier to work with functions in the form $x = f(y)$. In fact, there are going to be occasions when this will be the only way in which a problem can be worked so make sure that you can deal with functions in this form.

Let's take a look at one more example to make sure we can deal with functions in this form.

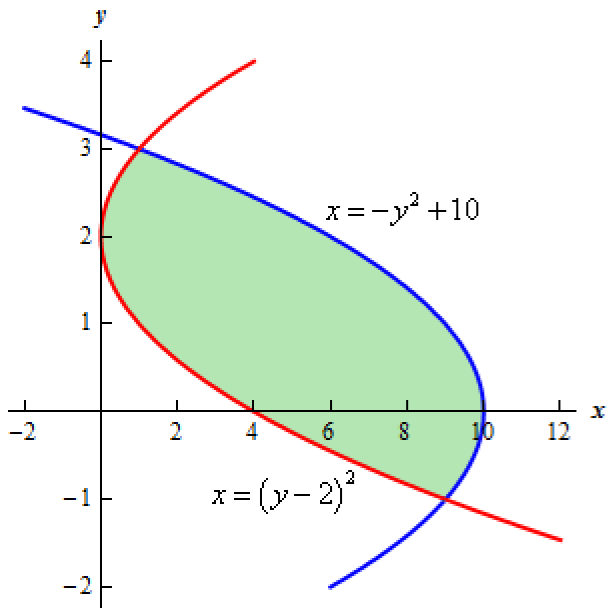
Example 7 Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.

Solution

First, we will need intersection points.

$$\begin{aligned} -y^2 + 10 &= (y-2)^2 \\ -y^2 + 10 &= y^2 - 4y + 4 \\ 0 &= 2y^2 - 4y - 6 \\ 0 &= 2(y+1)(y-3) \end{aligned}$$

The intersection points are $y = -1$ and $y = 3$. Here is a sketch of the region.



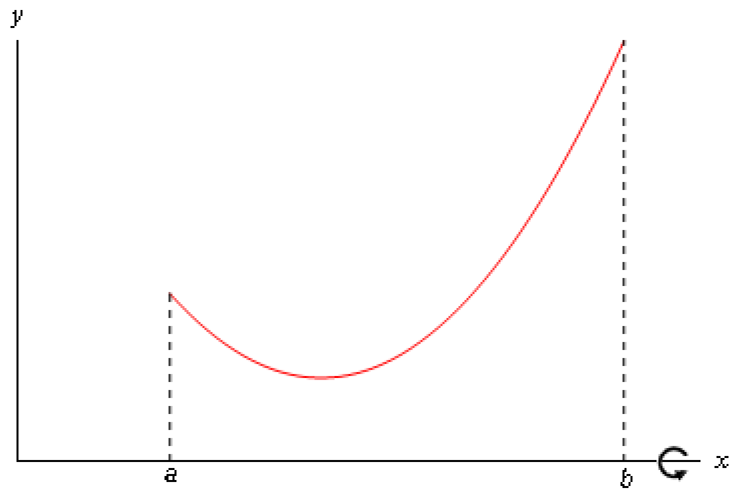
This is definitely a region where the second area formula will be easier. If we used the first formula there would be three different regions that we'd have to look at.

The area in this case is,

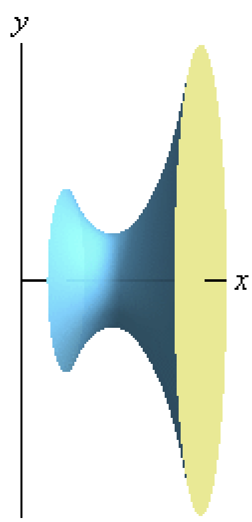
$$\begin{aligned} A &= \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy \\ &= \int_{-1}^{3} -y^2 + 10 - (y-2)^2 \, dy \\ &= \int_{-1}^{3} -2y^2 + 4y + 6 \, dy \\ &= \left(-\frac{2}{3}y^3 + 2y^2 + 6y \right)\Bigg|_{-1}^{3} = \frac{64}{3} \end{aligned}$$

Volumes of Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval [a,b].



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the *x*-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.



What we want to do over the course of the next two sections is to determine the volume of this object.

In the final the Area and Volume Formulas section of the Extras chapter we derived the following formulas for the volume of this solid.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

where, $A(x)$ and $A(y)$ is the cross-sectional area of the solid. There are many ways to get the cross-sectional area and we'll see two (or three depending on how you look at it) over the next two sections. Whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.

One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).

In the case that we get a solid disk the area is,

$$A = \pi(\text{radius})^2$$

where the radius will depend upon the function and the axis of rotation.

In the case that we get a ring the area is,

$$A = \pi\left(\begin{pmatrix}\text{outer}\\ \text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{radius}\end{pmatrix}^2\right)$$

where again both of the radii will depend on the functions given and the axis of rotation. Note as well that in the case of a solid disk we can think of the inner radius as zero and we'll arrive at the correct formula for a solid disk and so this is a much more general formula to use.

Also, in both cases, whether the area is a function of *x* or a function of *y* will depend upon the axis of rotation as we will see.

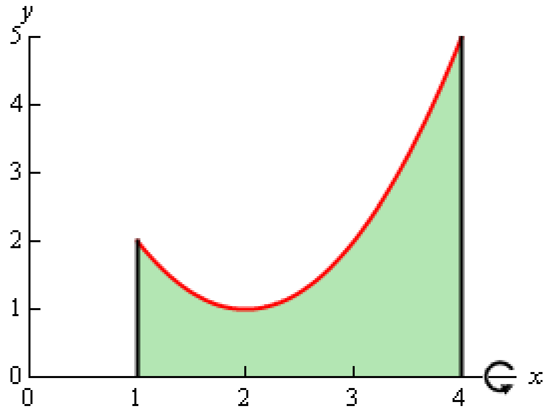
This method is often called the **method of disks** or the **method of rings**.

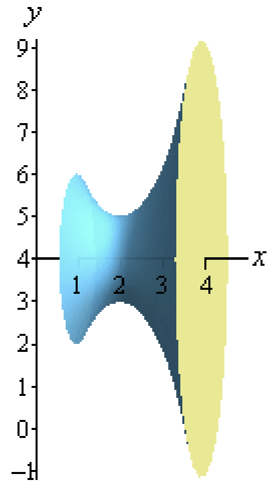
Let's do an example.

Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the *x*-axis about the *x*-axis.

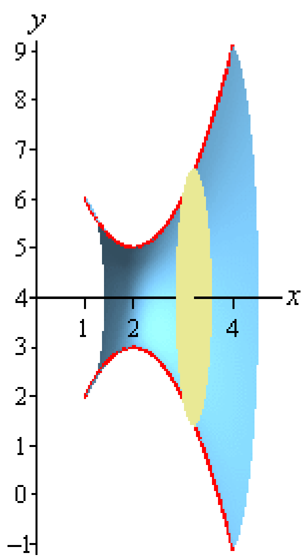
Solution

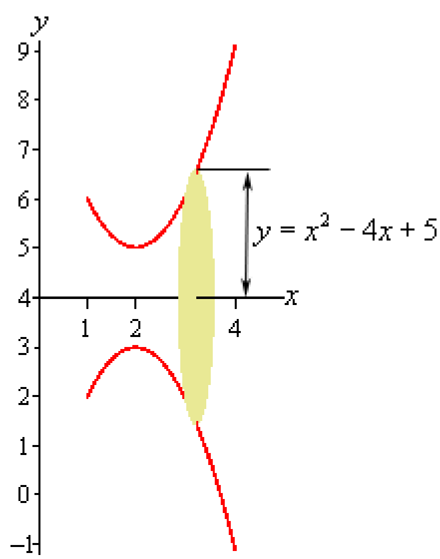
The first thing to do is get a sketch of the bounding region and the solid obtained by rotating the region about the *x*-axis. Here are both of these sketches.





Okay, to get a cross section we cut the solid at any *x*. Below are a couple of sketches showing a typical cross section. The sketch on the right shows a cut away of the object with a typical cross section without the caps. The sketch on the left shows just the curve we're rotating as well as it's mirror image along the bottom of the solid.





In this case the radius is simply the distance from the *x*-axis to the curve and this is nothing more than the function value at that particular *x* as shown above. The cross-sectional area is then,

$$A(x) = \pi\left(x^2 - 4x + 5\right)^2 = \pi\left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)$$

Next we need to determine the limits of integration. Working from left to right the first cross section will occur at $x = 1$ and the last cross section will occur at $x = 4$. These are the limits of integration.

The volume of this solid is then,

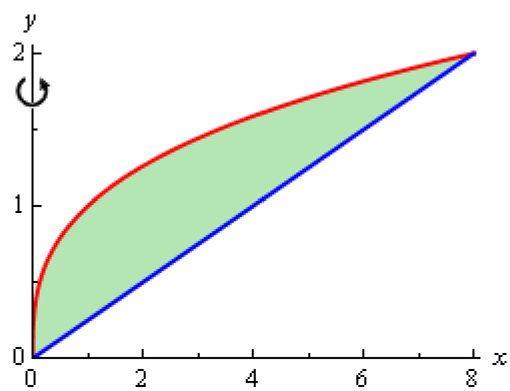
$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi\int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\,dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 \\ &= \frac{78\pi}{5} \end{aligned}$$

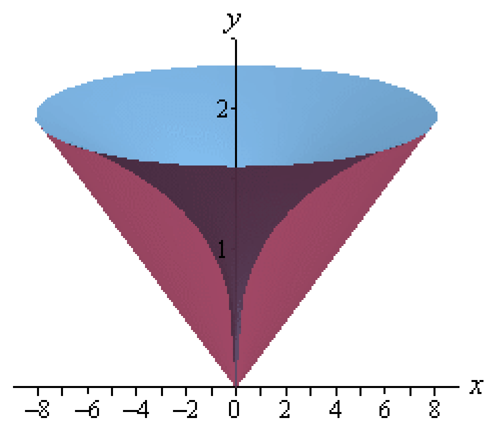
In the above example the object was a solid object, but the more interesting objects are those that are not solid so let's take a look at one of those.

Example 2 Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$ that lies in the first quadrant about the *y*-axis.

Solution

First, let's get a graph of the bounding region and a graph of the object. Remember that we only want the portion of the bounding region that lies in the first quadrant. There is a portion of the bounding region that is in the third quadrant as well, but we don't want that for this problem.





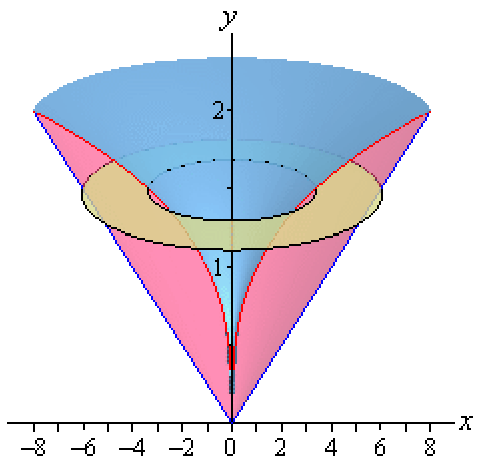
There are a couple of things to note with this problem. First, we are only looking for the volume of the "walls" of this solid, not the complete interior as we did in the last example.

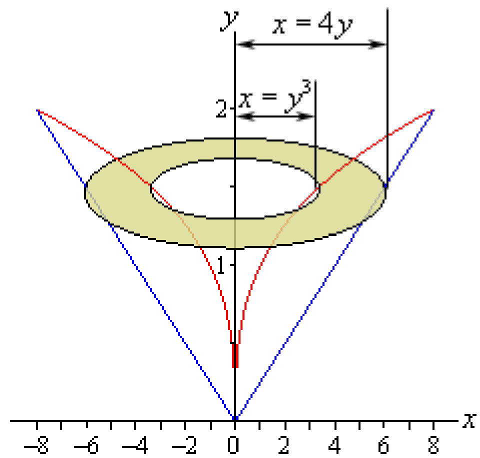
Next, we will get our cross section by cutting the object perpendicular to the axis of rotation. The cross section will be a ring (remember we are only looking at the walls) for this example and it will be horizontal at some *y*. This means that the inner and outer radius for the ring will be *x* values and so we will need to rewrite our functions into the form $x = f(y)$. Here are the functions written in the correct form for this example.

$$y = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$

$$y = \frac{x}{4} \qquad \Rightarrow \qquad x = 4y$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





The inner radius in this case is the distance from the *y*-axis to the inner curve while the outer radius is the distance from the *y*-axis to the outer curve. Both of these are then *x* distances and so are given by the equations of the curves as shown above.

The cross-sectional area is then,

$$A(y) = \pi\left((4y)^2 - (y^3)^2\right) = \pi\left(16y^2 - y^6\right)$$

Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y = 0$ and the last cross-section will occur at $y = 2$. These will be the limits of integration. The volume is then,

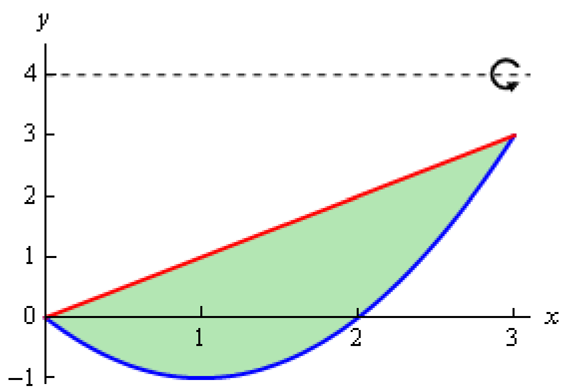
$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \pi\int_0^2 16y^2 - y^6\,dy \\ &= \pi\left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right)\Bigg|_0^2 \\ &= \frac{512\pi}{21} \end{aligned}$$

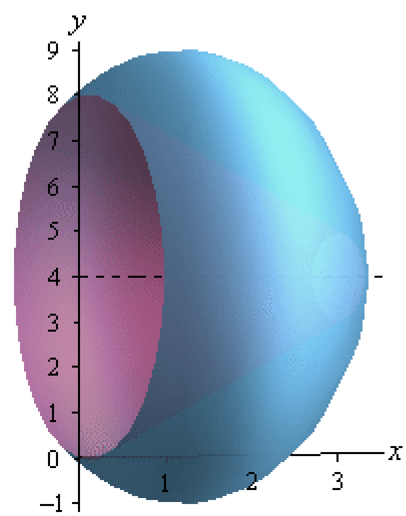
With these two examples out of the way we can now make a generalization about this method. If we rotate about a horizontal axis (the *x*-axis for example) then the cross sectional area will be a function of *x*. Likewise, if we rotate about a vertical axis (the *y*-axis for example) then the cross sectional area will be a function of *y*.

The remaining two examples in this section will make sure that we don't get too used to the idea of always rotating about the *x* or *y*-axis.

Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

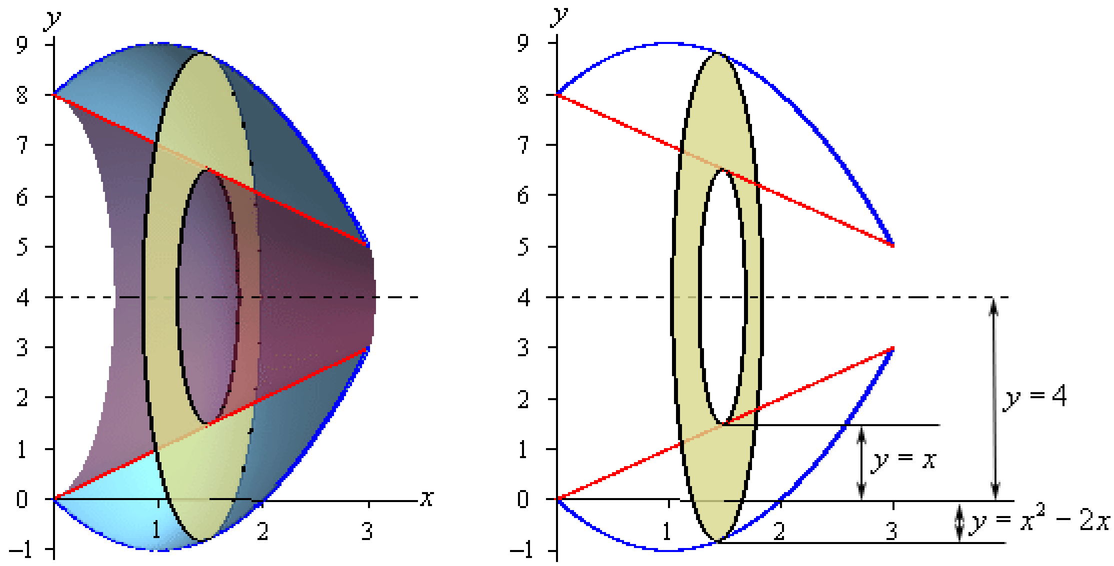
Solution
First let's get the bounding region and the solid graphed.





Again, we are going to be looking for the volume of the walls of this object. Also since we are rotating about a horizontal axis we know that the cross-sectional area will be a function of *x*.

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



Now, we're going to have to be careful here in determining the inner and outer radius as they aren't going to be quite as simple they were in the previous two examples.

Let's start with the inner radius as this one is a little clearer. First, the inner radius is NOT x. The distance from the x-axis to the inner edge of the ring is x, but we want the radius and that is the distance from the axis of rotation to the inner edge of the ring. So, we know that the distance from the axis of rotation to the x-axis is 4 and the distance from the x-axis to the inner ring is x. The inner radius must then be the difference between these two. Or,

$$\text{inner radius} = 4 - x$$

The outer radius works the same way. The outer radius is,

$$\text{outer radius} = 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$$

Note that give, the sketch above this may not look quite right, but it is. As sketched the outer edge is below the x-axis and at this point the value of the function will be negative and so when we do the subtraction in the formula for the outer radius we'll actually be subtracting off a negative number which has the net effect of adding this distance onto 4 and that gives the correct outer radius. Likewise, if the outer edge is above the x-axis, the function value will be positive and so we'll be doing an honest subtraction here and again we'll get the correct radius in this case.

The cross-sectional area for this case is,

$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - \left(4 - x\right)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

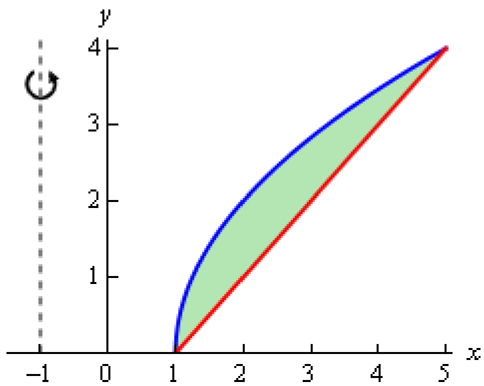
The first ring will occur at $x = 0$ and the last ring will occur at $x = 3$ and so these are our limits of integration. The volume is then,

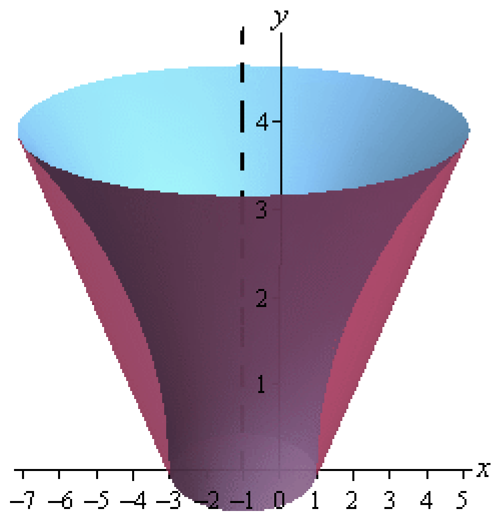
$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi\int_0^3 x^4 - 4x^3 - 5x^2 + 24x\,dx \\ &= \pi\left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2\right)\Bigg|_0^3 \\ &= \frac{153\pi}{5} \end{aligned}$$

Example 4 Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.

Solution

As with the previous examples, let's first graph the bounded region and the solid.

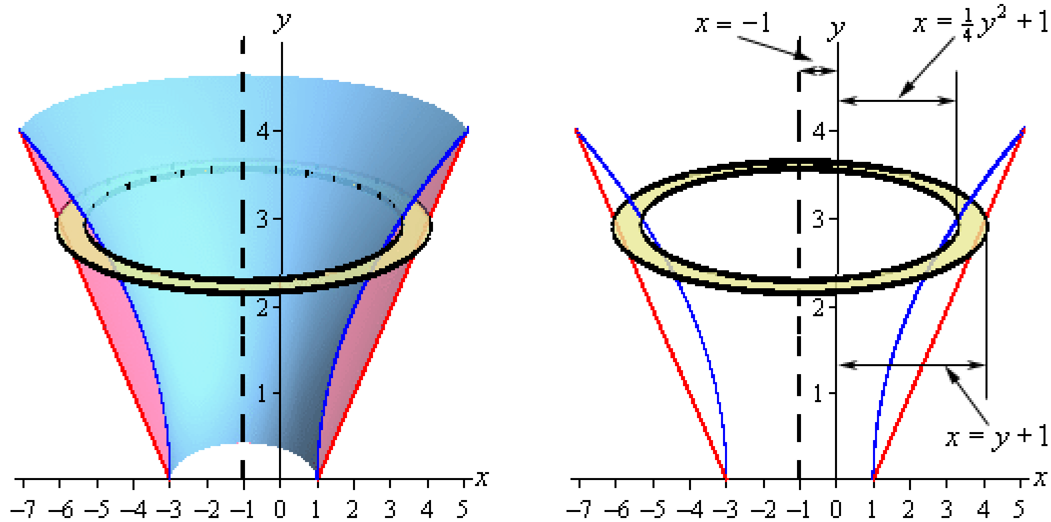




Now, let's notice that since we are rotating about a vertical axis and so the cross-sectional area will be a function of *y*. This also means that we are going to have to rewrite the functions to also get them in terms of *y*.

$$\begin{aligned} y &= 2\sqrt{x-1} & &\Rightarrow & x &= \frac{y^2}{4} + 1 \\ y &= x - 1 & &\Rightarrow & x &= y + 1 \end{aligned}$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



The inner and outer radius for this case is both similar and different from the previous example. This example is similar in the sense that the radii are not just the functions. In this example the functions are the distances from the *y*-axis to the edges of the rings. The center of the ring however is a distance of 1 from the *y*-axis. This means that the distance from the center to the edges is a distance from the axis of rotation to the *y*-axis (a distance of 1) and then from the *y*-axis to the edge of the rings.

So, the radii are then the functions plus 1 and that is what makes this example different from the previous example. Here we had to add the distance to the function value whereas in the previous example we needed to subtract the function from this distance. Note that without sketches the radii on these problems can be difficult to get.

So, in summary, we've got the following for the inner and outer radius for this example.

$$\text{outer radius} = y+1+1 = y+2$$
$$\text{inner radius} = \frac{y^2}{4}+1+1 = \frac{y^2}{4}+2$$

The cross-sectional area it then,

$$A(y) = \pi\left((y+2)^2 - \left(\frac{y^2}{4}+2\right)^2\right) = \pi\left(4y - \frac{y^4}{16}\right)$$

The first ring will occur at $y=0$ and the final ring will occur at $y=4$ and so these will be our limits of integration.

The volume is,

$$
\begin{aligned}
V &= \int_c^d A(y)\,dy \\
&= \pi \int_0^4 4y - \frac{y^4}{16}\,dy \\
&= \pi\left(2y^2 - \frac{1}{80}y^5\right)\Bigg|_0^4 \\
&= \frac{96\pi}{5}
\end{aligned}
$$

Volumes of Solids of Revolution / Method of Cylinders

In the previous section we started looking at finding volumes of solids of revolution. In that section we took cross sections that were rings or disks, found the cross-sectional area and then used the following formulas to find the volume of the solid.

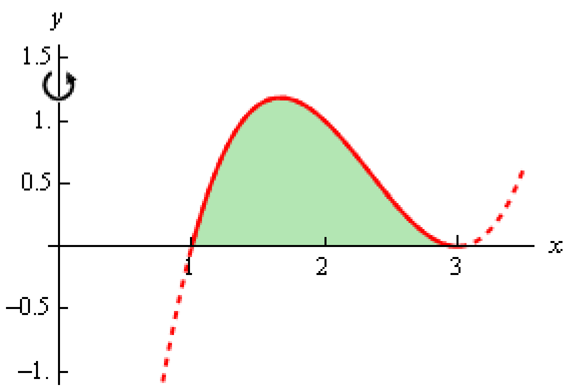
$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

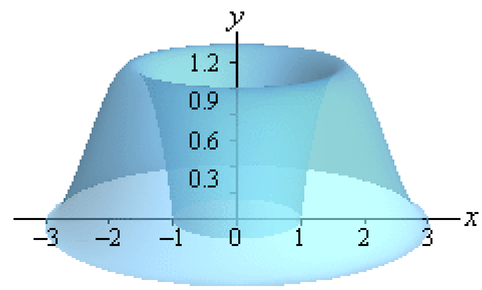
In the previous section we only used cross sections that where in the shape of a disk or a ring. This however does not always need to be the case. We can use any shape for the cross sections as long as it can be expanded or contracted to completely cover the solid we're looking at. This is a good thing because as our first example will show us we can't always use rings/disks.

Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = (x-1)(x-3)^2$ and the x-axis about the y-axis.

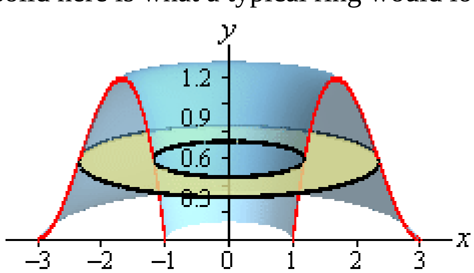
Solution

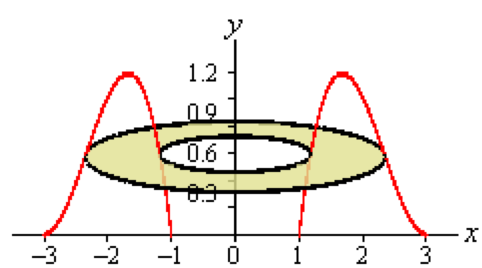
As we did in the previous section, let's first graph the bounded region and solid. Note that the bounded region here is the shaded portion shown. The curve is extended out a little past this for the purposes of illustrating what the curve looks like.





So, we've basically got something that's roughly doughnut shaped. If we were to use rings on this solid here is what a typical ring would look like.





This leads to several problems. First, both the inner and outer radius are defined by the same function. This, in itself, can be dealt with on occasion as we saw in a example in the Area

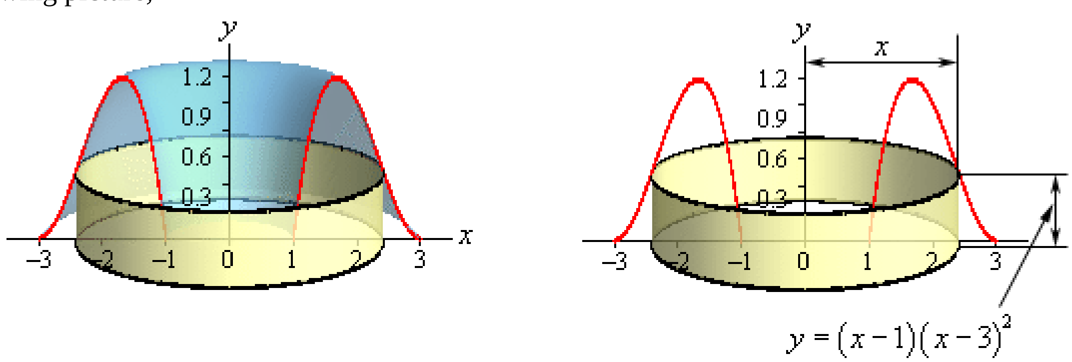
Between Curves section. However, this usually means more work than other methods so it's often not the best approach.

This leads to the second problem we got here. In order to use rings we would need to put this function into the form $x = f(y)$. That is NOT easy in general for a cubic polynomial and in other cases may not even be possible to do. Even when it is possible to do this the resulting equation is often significantly messier than the original which can also cause problems.

The last problem with rings in this case is not so much a problem as its just added work. If we were to use rings the limit would be *y* limits and this means that we will need to know how high the graph goes. To this point the limits of integration have always been intersection points that were fairly easy to find. However, in this case the highest point is not an intersection point, but instead a relative maximum. We spent several sections in the Applications of Derivatives chapter talking about how to find maximum values of functions. However, finding them can, on occasion, take some work.

So, we've seen three problems with rings in this case that will either increase our work load or outright prevent us from using rings.

What we need to do is to find a different way to cut the solid that will give us a cross-sectional area that we can work with. One way to do this is to think of our solid as a lump of cookie dough and instead of cutting it perpendicular to the axis of rotation we could instead center a cylindrical cookie cutter on the axis of rotation and push this down into the solid. Doing this would give the following picture,



Doing this gives us a cylinder or shell in the object and we can easily find its surface area. The surface area of this cylinder is,

$$
\begin{aligned}
A(x) &= 2\pi(\text{radius})(\text{height}) \\
&= 2\pi(x)\left((x-1)(x-3)^2\right) \\
&= 2\pi\left(x^4 - 7x^3 + 15x^2 - 9x\right)
\end{aligned}
$$

Notice as well that as we increase the radius of the cylinder we will completely cover the solid and so we can use this in our formula to find the volume of this solid. All we need are limits of integration. The first cylinder will cut into the solid at $x = 1$ and as we increase *x* to $x = 3$ we will completely cover both sides of the solid since expanding the cylinder in one direction will automatically expand it in the other direction as well.

The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi \int_1^3 x^4 - 7x^3 + 15x^2 - 9x\,dx \\ &= 2\pi \left(\frac{1}{5}x^5 - \frac{7}{4}x^4 + 5x^3 - \frac{9}{2}x^2 \right) \Bigg|_1^3 \\ &= \frac{24\pi}{5} \end{aligned}$$

The method used in the last example is called the **method of cylinders** or **method of shells**. The formula for the area in all cases will be,

$$A = 2\pi \left(\text{radius}\right)\left(\text{height}\right)$$

There are a couple of important differences between this method and the method of rings/disks that we should note before moving on. First, rotation about a vertical axis will give an area that is a function of x and rotation about a horizontal axis will give an area that is a function of y. This is exactly opposite of the method of rings/disks.

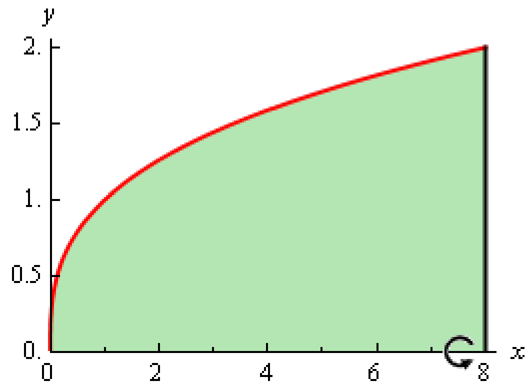
Second, we don't take the complete range of x or y for the limits of integration as we did in the previous section. Instead we take a range of x or y that will cover one side of the solid. As we noted in the first example if we expand out the radius to cover one side we will automatically expand in the other direction as well to cover the other side.

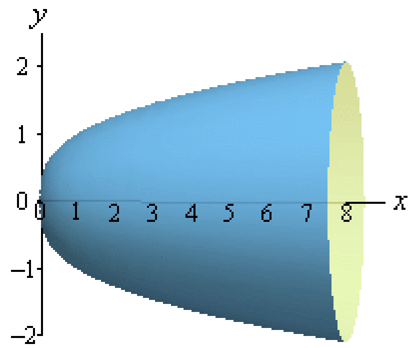
Let's take a look at some another example.

Example 2 Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $x = 8$ and the x-axis about the x-axis.

Solution

First let's get a graph of the bounded region and the solid.

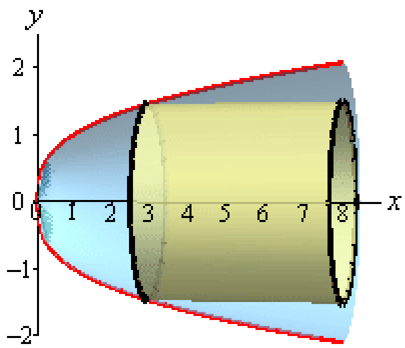


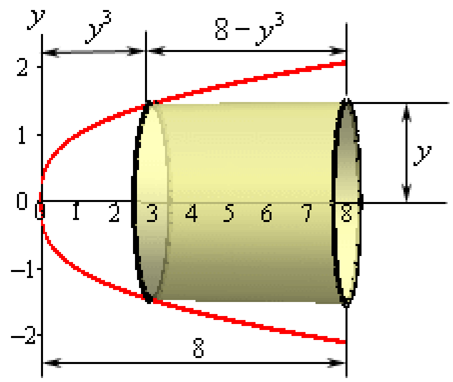


Okay, we are rotating about a horizontal axis. This means that the area will be a function of *y* and so our equation will also need to be wrote in $x = f(y)$ form.

$$y = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$

As we did in the ring/disk section let's take a couple of looks at a typical cylinder. The sketch on the left shows a typical cylinder with the back half of the object also in the sketch to give the right sketch some context. The sketch on the right contains a typical cylinder and only the curves that define the edge of the solid.





In this case the width of the cylinder is not the function value as it was in the previous example. In this case the function value is the distance between the edge of the cylinder and the *y*-axis. We the distance from the edge out to the line $x = 8$ and so the width is then $8 - y^3$. The cross sectional area in this case is,

$$\begin{aligned} A(y) &= 2\pi(\text{radius})(\text{width}) \\ &= 2\pi(y)(8 - y^3) \\ &= 2\pi(8y - y^4) \end{aligned}$$

The first cylinder will cut into the solid at $y = 0$ and the final cylinder will cut in at $y = 2$ and so these are our limits of integration.

The volume of this solid is,

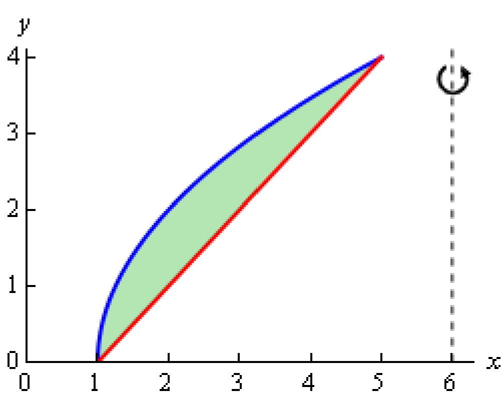
$$
\begin{aligned}
V &= \int_c^d A(y)\,dy \\
&= 2\pi\int_0^2 8y - y^4\,dy \\
&= 2\pi\left(4y^2 - \frac{1}{5}y^5\right)\Bigg|_0^2 \\
&= \frac{96\pi}{5}
\end{aligned}
$$

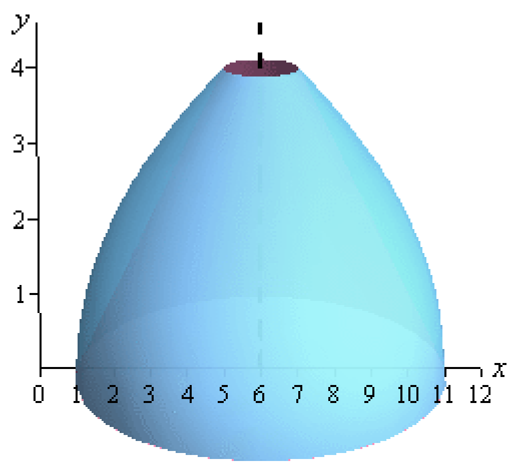
The remaining examples in this section will have axis of rotation about axis other than the *x* and *y*-axis. As with the method of rings/disks we will need to be a little careful with these.

Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = 6$.

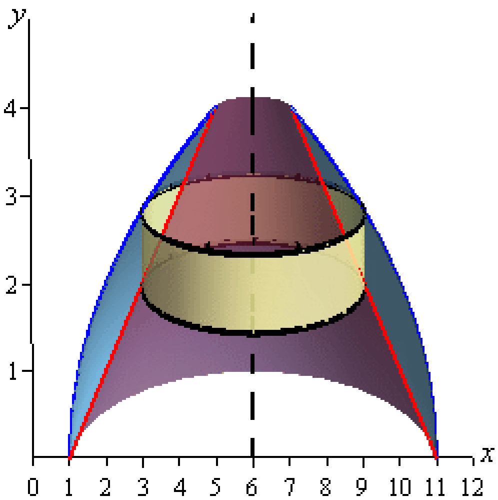
Solution

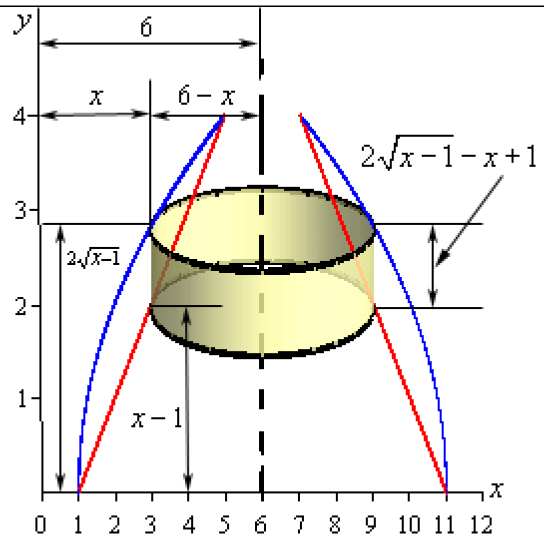
Here's a graph of the bounded region and solid.





Here are our sketches of a typical cylinder. Again, the sketch on the left is here to provide some context for the sketch on the right.





Okay, there is a lot going on in the sketch to the left. First notice that the radius is not just an *x* or *y* as it was in the previous two cases. In this case *x* is the distance from the *x*-axis to the edge of the cylinder and we need the distance from the axis of rotation to the edge of the cylinder. That means that the radius of this cylinder is $6 - x$.
Secondly, the height of the cylinder is the difference of the two functions in this case.

The cross sectional area is then,

$$\begin{aligned} A(x) &= 2\pi(\text{radius})(\text{height}) \\ &= 2\pi(6-x)\left(2\sqrt{x-1}-x+1\right) \\ &= 2\pi\left(x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1}\right) \end{aligned}$$

Now the first cylinder will cut into the solid at $x = 1$ and the final cylinder will cut into the solid at $x = 5$x=5 so there are our limits.

Here is the volume.

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi\int_1^5 x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1}\,dx \\ &= 2\pi\left(\frac{1}{3}x^3-\frac{7}{2}x^2+6x+8(x-1)^{\frac{3}{2}}-\frac{4}{3}(x-1)^{\frac{3}{2}}-\frac{4}{5}(x-1)^{\frac{5}{2}}\right)\Bigg|_1^5 \\ &= 2\pi\left(\frac{136}{15}\right) \\ &= \frac{272\pi}{15} \end{aligned}$$

The integration of the last term is a little tricky so let's do that here. It will use the substitution,

$$u = x - 1 \qquad du = dx \qquad x = u + 1$$

$$\begin{aligned} \int 2x\sqrt{x-1}\,dx &= 2\int (u+1)u^{\frac{1}{2}}\,du \\ &= 2\int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du \\ &= 2\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + c \\ &= \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + c \end{aligned}$$

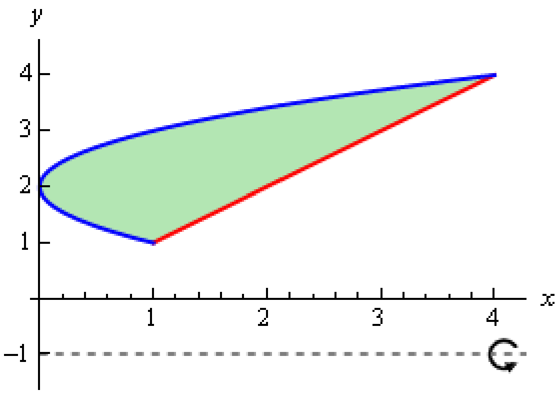
We saw one of these kinds of substitutions back in the substitution section.

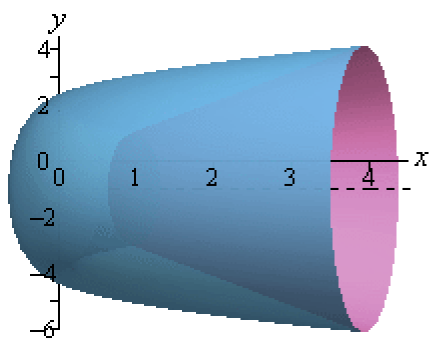
Example 4 Determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$.

Solution
We should first get the intersection points there.

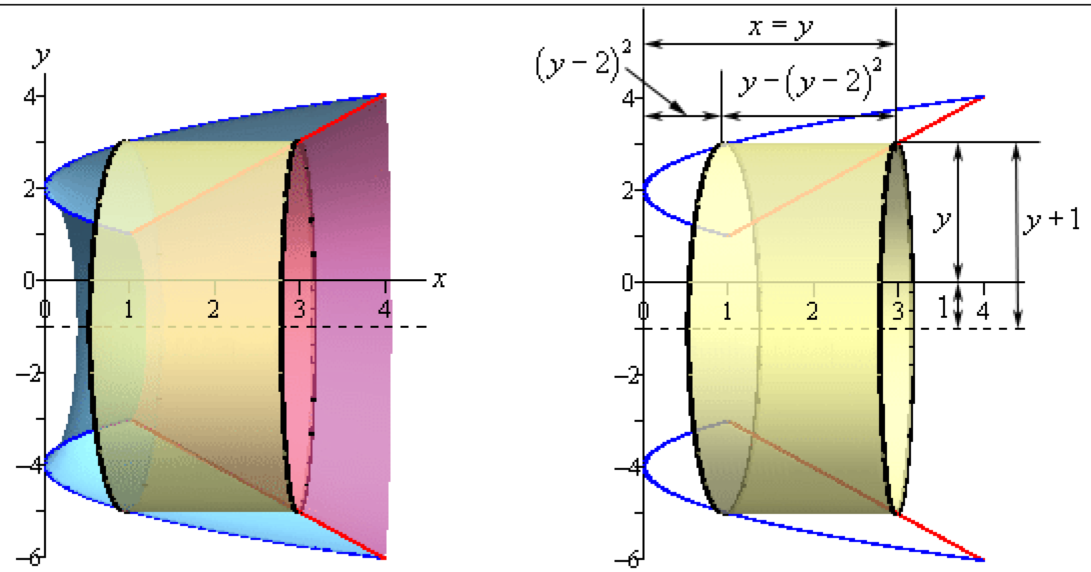
$$\begin{aligned} y &= (y-2)^2 \\ y &= y^2 - 4y + 4 \\ 0 &= y^2 - 5y + 4 \\ 0 &= (y-4)(y-1) \end{aligned}$$

So, the two curves will intersect at $y = 1$ and $y = 4$. Here is a sketch of the bounded region and the solid.





Here are our sketches of a typical cylinder. Tthe sketch on the left is here to provide some context for the sketch on the right.



Here's the cross sectional area for this cylinder.

$$\begin{aligned} A(y) &= 2\pi(\text{radius})(\text{width}) \\ &= 2\pi(y+1)\left(y-(y-2)^2\right) \\ &= 2\pi\left(-y^3+4y^2+y-4\right) \end{aligned}$$

The first cylinder will cut into the solid at $y=1$ and the final cylinder will cut in at $y=4$. The volume is then,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= 2\pi\int_1^4 -y^3+4y^2+y-4\,dy \\ &= 2\pi\left(-\frac{1}{4}y^4+\frac{4}{3}y^3+\frac{1}{2}y^2-4y\right)\Bigg|_1^4 \\ &= \frac{63\pi}{2} \end{aligned}$$

Work

This is the final application of integral that we'll be looking at in this course. In this section we will be looking at the amount of work that is done by a force in moving an object.

In a first course in Physics you typically look at the work that a constant force, *F*, does when moving an object over a distance of *d*. In these cases the work is,

$$W = Fd$$

However, most forces are not constant and will depend upon where exactly the force is acting. So, let's suppose that the force at any *x* is given by *F(x)*. Then the work done by the force in moving an object from $x = a$ to $x = b$ is given by,

$$W = \int_a^b F(x)\,dx$$

To see a justification of this formula see the Proof of Various Integral Properties section of the Extras chapter.

Notice that if the force constant we get the correct formula for a constant force.

$$\begin{aligned} W &= \int_a^b F\,dx \\ &= Fx\big|_a^b \\ &= F(b-a) \end{aligned}$$

where *b*-*a* is simply the distance moved, or *d*.

So, let's take a look at a couple of examples of non-constant forces.

Example 1 A spring has a natural length of 20 cm. A 40 N force is required to stretch (and hold the spring) to a length of 30 cm. How much work is done in stretching the spring from 35 cm to 38 cm?

Solution

This example will require Hooke's Law to determine the force. Hooke's Law tells us that the force required to stretch a spring a distance of *x* meters from its natural length is,

$$F(x) = k\,x$$

where $k > 0$ is called the spring constant.

The first thing that we need to do is determine the spring constant for this spring. We can do that using the initial information. A force of 40 N is required to stretch the spring 30cm-20cm = 10cm = 0.10m from its natural length. Using Hooke's Law we have,

$$40 = 0.10k \qquad \Rightarrow \qquad k = 400$$

So, according to Hooke's Law the force required to hold this spring *x* meters from its natural length is,

$$F(x) = 400x$$

We want to know the work required to stretch the spring from 35cm to 38cm. First we need to convert these into distances from the natural length in meters. Doing that gives us *x*'s of 0.15m and 0.18m.

The work is then,

$$\begin{aligned} W &= \int_{0.15}^{0.18} 400x\,dx \\ &= 200x^2 \Big|_{0.15}^{0.18} \\ &= 1.98\,\mathrm{J} \end{aligned}$$

Example 2 We have a cable that weighs 2 lbs/ft attached to a bucket filled with coal that weighs 800 lbs. The bucket is initially at the bottom of a 500 ft mine shaft. Answer each of the following about this.

(a) Determine the amount of work required to lift the bucket to the midpoint of the shaft.

(b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft.

(c) Determine the amount of work required to lift the bucket all the way up the shaft.

Solution

Before answering either part we first need to determine the force. In this case the force will be the weight of the bucket and cable at any point in the shaft.

To determine a formula for this we will first need to set a convention for *x*. For this problem we will set *x* to be the amount of cable that has been pulled up. So at the bottom of the shaft $x = 0$, at the midpoint of the shaft $x = 250$ and at the top of the shaft $x = 500$. Also at any point in the shaft there is $500 - x$ feet of cable still in the shaft.

The force then for any *x* is then nothing more than the weight of the cable and bucket at that point. This is,

$$\begin{aligned} F(x) &= \text{weight of cable} + \text{ weight of bucket/coal} \\ &= 2(500 - x) + 800 \\ &= 1800 - 2x \end{aligned}$$

We can now answer the questions.

(a) In this case we want to know the work required to move the cable and bucket/coal from $x = 0$ to $x = 250$. The work required is,

$$\begin{aligned} W &= \int_0^{250} F(x)\,dx \\ &= \int_0^{250} 1800 - 2x\,dx \\ &= \left(1800x - x^2\right)\Big|_0^{250} \\ &= 387500 \end{aligned}$$

(b) In this case we want to move the cable and bucket/coal from $x = 250$ to $x = 500$. The work required is,

$$\begin{aligned} W &= \int_{250}^{500} F(x)\,dx \\ &= \int_{250}^{500} 1800 - 2x\,dx \\ &= \left(1800x - x^2\right)\Big|_{250}^{500} \\ &= 262500 \end{aligned}$$

(c) In this case the work is,

$$\begin{aligned} W &= \int_0^{500} F(x)\,dx \\ &= \int_0^{500} 1800 - 2x\,dx \\ &= \left(1800x - x^2\right)\Big|_0^{500} \\ &= 650000 \end{aligned}$$

Note that we could have instead just added the results from the first two parts and we would have gotten the same answer to the third part.

Example 3 A 20 ft cable weighs 80 lbs and hangs from the ceiling of a building without touching the floor. Determine the work that must be done to lift the bottom end of the chain all the way up until it touches the ceiling.

Solution

First we need to determine the weight per foot of the cable. This is easy enough to get,

$$\frac{80 \text{ lbs}}{20 \text{ ft}} = 4 \text{ lb/ft}$$

Next, let *x* be the distance from the ceiling to any point on the cable. Using this convention we can see that the portion of the cable in the range $10 < x \le 20$ will actually be lifted. The portion of the cable in the range $0 \le x \le 10$ will not be lifted at all since once the bottom of the cable has been lifted up to the ceiling the cable will be doubled up and each portion will have a length of 10 ft. So, the upper 10 foot portion of the cable will never be lifted while the lower 10 ft portion will be lifted.

Now, the very bottom of the cable, $x = 20$, will be lifted 10 feet to get to the midpoint and then a further 10 feet to get to the ceiling. A point 2 feet from the bottom of the cable, $x = 18$ will lift 8 feet to get to the midpoint and then a further 8 feet until it reaches its final position (if it is 2 feet from the bottom then its final position will be 2 feet from the ceiling). Continuing on in this fashion we can see that for any point on the lower half of the cable, *i.e.* $10 \le x \le 20$ it will be lifted a total of $2\left(x-10\right)$.

As with the previous example the force required to lift any point of the cable in this range is simply the distance that point will be lifted times the weight/foot of the cable. So, the force is then,

$$\begin{aligned} F\left(x\right) &= \left(\text{distance lifted}\right)\left(\text{weight per foot of cable}\right) \\ &= 2\left(x-10\right)\left(4\right) \\ &= 8\left(x-10\right) \end{aligned}$$

The work required is now,

$$\begin{aligned} W &= \int_{10}^{20} 8\left(x-10\right)dx \\ &= \left.\left(4x^2 - 80x\right)\right|_{10}^{20} \\ &= 400\,\text{J} \end{aligned}$$

Extras

Introduction

In this chapter material that didn't fit into other sections for a variety of reasons. Also, in order to not obscure the mechanics of actually working problems, most of the proofs of various facts and formulas are in this chapter as opposed to being in the section with the fact/formula.

This chapter contains those topics.

Proof of Various Limit Properties – In we prove several of the limit properties and facts that were given in various sections of the Limits chapter.

Proof of Various Derivative Facts/Formulas/Properties – In this section we give the proof for several of the rules/formulas/properties of derivatives that we saw in Derivatives Chapter. Included are multiple proofs of the Power Rule, Product Rule, Quotient Rule and Chain Rule.

Proof of Trig Limits – Here we give proofs for the two limits that are needed to find the derivative of the sine and cosine functions.

Proofs of Derivative Applications Facts/Formulas – We'll give proofs of many of the facts that we saw in the Applications of Derivatives chapter.

Proof of Various Integral Facts/Formulas/Properties – Here we will give the proofs of some of the facts and formulas from the Integral Chapter as well as a couple from the Applications of Integrals chapter.

Area and Volume Formulas – Here is the derivation of the formulas for finding area between two curves and finding the volume of a solid of revolution.

Types of Infinity – This is a discussion on the types of infinity and how these affect certain limits.

Summation Notation – Here is a quick review of summation notation.

Constant of Integration – This is a discussion on a couple of subtleties involving constants of integration that many students don't think about.

Proof of Various Limit Properties

In this section we are going to prove some of the basic properties and facts about limits that we saw in the Limits chapter. Before proceeding with any of the proofs we should note that many of the proofs use the precise definition of the limit and it is assumed that not only have you read that section but that you have a fairly good feel for doing that kind of proof. If you're not very comfortable using the definition of the limit to prove limits you'll find many of the proofs in this section difficult to follow.

The proofs that we'll be doing here will not be quite as detailed as those in the precise definition of the limit section. The "proofs" that we did in that section first did some work to get a guess for the δ and then we verified the guess. The reality is that often the work to get the guess is not shown and the guess for δ is just written down and then verified. For the proofs in this section where a δ is actually chosen we'll do it that way. To make matters worse, in some of the proofs in this section work very differently from those that were in the limit definition section.

So, with that out of the way, let's get to the proofs.

Limit Properties

In the Limit Properties section we gave several properties of limits. We'll prove most of them here. First, let's recall the properties here so we have them in front of us. We'll also be making a small change to the notation to make the proofs go a little easier. Here are the properties for reference purposes.

Assume that $\lim\limits_{x\to a} f(x) = K$ and $\lim\limits_{x\to a} g(x) = L$ exist and that c is any constant. Then,

1. $\lim\limits_{x\to a}\left[cf(x)\right] = c\lim\limits_{x\to a} f(x) = cK$

2. $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = K \pm L$

3. $\lim\limits_{x\to a}\left[f(x)g(x)\right] = \lim\limits_{x\to a} f(x)\ \lim\limits_{x\to a} g(x) = KL$

4. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{K}{L}$, provided $L = \lim\limits_{x\to a} g(x) \neq 0$

5. $\lim\limits_{x\to a}\left[f(x)\right]^n = \left[\lim\limits_{x\to a} f(x)\right]^n = K^n$, where n is any real number

6. $\lim\limits_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$

Note that we added values (*K*, *L*, *etc*.) to each of the limits to make the proofs much easier. In these proofs we'll be using the fact that we know $\lim_{x \to a} f(x) = K$ and $\lim_{x \to a} g(x) = L$ we'll use the definition of the limit to make a statement about $\left| f(x) - K \right|$ and $\left| g(x) - L \right|$ which will then be used to prove what we actually want to prove. When you see these statements do not worry too much about why we chose them as we did. The reason will become apparent once the proof is done.

Also, we're not going to be doing the proofs in the order they are written above. Some of the proofs will be easier if we've got some of the others proved first.

Proof of 7

This is a very simple proof. To make the notation a little clearer let's define the function $f(x) = c$ then what we're being asked to prove is that $\lim_{x \to a} f(x) = c$. So let's do that.

Let $\varepsilon > 0$ and we need to show that we can find a $\delta > 0$ so that

$$\left| f(x) - c \right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta$$

The left inequality is trivially satisfied for any *x* however because we defined $\lim_{x \to a} f(x) = c$. **S**o simply choose $\delta > 0$ to be any number you want (you generally can't do this with these proofs). Then,

$$\left| f(x) - c \right| = \left| c - c \right| = 0 < \varepsilon$$

■

Proof of 1

There are several ways to prove this part. If you accept **3** And **7** then all you need to do is let $g(x) = c$ and then this is a direct result of **3** and **7.** However, we'd like to do a more rigorous mathematical proof. So here is that proof.

First, note that if $c = 0$ then $cf(x) = 0$ and so,

$$\lim_{x \to a} \left[0 f(x) \right] = \lim_{x \to a} 0 = 0 = 0 f(x)$$

The limit evaluation is a special case of **7** (with $c = 0$) which we just proved Therefore we know **1** is true for $c = 0$ and so we can assume that $c \neq 0$ for the remainder of this proof.

Let $\varepsilon > 0$ then because $\lim_{x \to a} f(x) = K$ by the definition of the limit there is a $\delta_1 > 0$ such that,

$$\left| f(x) - K \right| < \frac{\varepsilon}{|c|} \qquad \text{whenever} \qquad 0 < |x - a| < \delta_1$$

Now choose $\delta = \delta_1$ and we need to show that

$$\left| cf(x) - cK \right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

and we'll be done. So, assume that $0 < |x - a| < \delta$ and then,

$$\left| cf(x) - cK \right| = |c| \left| f(x) - K \right| < |c| \frac{\varepsilon}{|c|} = \varepsilon$$

■

Proof of 2

Note that we'll need something called the **triangle inequality** in this proof. The triangle inequality states that,

$$|a + b| \le |a| + |b|$$

Here's the actual proof.

We'll be doing this proof in two parts. First let's prove $\lim_{x \to a} \left[f(x) + g(x) \right] = K + L$.

Let $\varepsilon > 0$ then because $\lim_{x \to a} f(x) = K$ and $\lim_{x \to a} g(x) = L$ there is a $\delta_1 > 0$ and a $\delta_2 > 0$ such that,

$$\left| f(x) - K \right| < \frac{\varepsilon}{2} \qquad \text{whenever} \qquad 0 < |x - a| < \delta_1$$

$$\left| g(x) - L \right| < \frac{\varepsilon}{2} \qquad \text{whenever} \qquad 0 < |x - a| < \delta_2$$

Now choose $\delta = \min\left\{ \delta_1, \delta_2 \right\}$. Then we need to show that

$$\left| f(x) + g(x) - (K + L) \right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

Assume that $0 < |x - a| < \delta$. We then have,

$$\begin{aligned}\left|f(x)+g(x)-(K+L)\right| &= \left|\left(f(x)-K\right)+\left(g(x)-L\right)\right| \\ &\le \left|f(x)-K\right|+\left|g(x)-L\right| \qquad \text{by the triangle inequality} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

In the third step we used the fact that, by our choice of δ, we also have $0<|x-a|<\delta_1$ and $0<|x-a|<\delta_2$ and so we can use the initial statements in our proof.

Next, we need to prove $\lim\limits_{x\to a}\left[f(x)-g(x)\right]=K-L$. We could do a similar proof as we did above for the sum of two functions. However, we might as well take advantage of the fact that we've proven this for a sum and that we've also proven **1.**

$$\begin{aligned}\lim_{x\to a}\left[f(x)-g(x)\right] &= \lim_{x\to a}\left[f(x)+(-1)g(x)\right] \\ &= \lim_{x\to a} f(x)+\lim_{x\to a}(-1)g(x) \qquad \text{by first part of } \mathbf{2}. \\ &= \lim_{x\to a} f(x)+(-1)\lim_{x\to a} g(x) \qquad \text{by } \mathbf{1}. \\ &= K+(-1)L \\ &= K-L\end{aligned}$$

■

Proof of 3

This one is a little tricky. First, let's note that because $\lim\limits_{x\to a} f(x)=K$ and $\lim\limits_{x\to a} g(x)=L$ we can use **2** and **7** to prove the following two limits.

$$\lim_{x\to a}\left[f(x)-K\right]=\lim_{x\to a} f(x)-\lim_{x\to a} K=K-K=0$$
$$\lim_{x\to a}\left[g(x)-L\right]=\lim_{x\to a} g(x)-\lim_{x\to a} L=L-L=0$$

Now, let $\varepsilon>0$. Then there is a $\delta_1>0$ and a $\delta_2>0$ such that,

$$\left|\left(f(x)-K\right)-0\right|<\sqrt{\varepsilon} \qquad \text{whenever} \qquad 0<|x-a|<\delta_1$$
$$\left|\left(g(x)-L\right)-0\right|<\sqrt{\varepsilon} \qquad \text{whenever} \qquad 0<|x-a|<\delta_2$$

Choose $\delta=\min\left\{\delta_1,\delta_2\right\}$. If $0<|x-a|<\delta$ we then get,

$$\begin{aligned}\left|\left[f(x)-K\right]\left[g(x)-L\right]-0\right| &= \left|f(x)-K\right|\left|g(x)-L\right| \\ &< \sqrt{\varepsilon}\sqrt{\varepsilon} \\ &= \varepsilon\end{aligned}$$

So, we've managed to prove that,

$$\lim_{x\to a}\left[f(x)-K\right]\left[g(x)-L\right]=0$$

This apparently has nothing to do with what we actually want to prove, but as you'll see in a bit it is needed.

Before launching into the actual proof of **3** let's do a little Algebra. First, expand the following product.

$$\left[f(x)-K\right]\left[g(x)-L\right]=f(x)g(x)-Lf(x)-Kg(x)+KL$$

Rearranging this gives the following way to write the product of the two functions.

$$f(x)g(x)=\left[f(x)-K\right]\left[g(x)-L\right]+Lf(x)+Kg(x)-KL$$

With this we can now proceed with the proof of **3**.

$$\begin{aligned}\lim_{x\to a}f(x)g(x) &= \lim_{x\to a}\left[f(x)-K\right]\left[g(x)-L\right]+Lf(x)+Kg(x)-KL \\ &= \lim_{x\to a}\left[f(x)-K\right]\left[g(x)-L\right]+\lim_{x\to a}Lf(x)+\lim_{x\to a}Kg(x)-\lim_{x\to a}KL \\ &= 0+\lim_{x\to a}Lf(x)+\lim_{x\to a}Kg(x)-\lim_{x\to a}KL \\ &= LK+KL-KL \\ &= LK\end{aligned}$$

Fairly simple proof really, once you see all the steps that you have to take before you even start. The second step made multiple uses of property **2.** In the third step we used the limit we initially proved. In the fourth step we used properties **1** and **7.** Finally, we just did some simplification.

■

Proof of 4

This one is also a little tricky. First, we'll start of by proving,

$$\lim_{x\to a}\frac{1}{g(x)}=\frac{1}{L}$$

Let $\varepsilon>0$. We'll not need this right away, but these proofs always start off with this statement. Now, because $\lim_{x\to a}g(x)=L$ there is a $\delta_1>0$ such that,

$$\left|g(x)-L\right|<\frac{|L|}{2} \qquad \text{whenever} \qquad 0<|x-a|<\delta_1$$

Now, assuming that $0 < |x-a| < \delta_1$ we have,

$$\begin{aligned} |L| &= |L - g(x) + g(x)| && \text{just adding zero to } L \\ &< |L - g(x)| + |g(x)| && \text{using the triangle inequality} \\ &= |g(x) - L| + |g(x)| && |L - g(x)| = |g(x) - L| \\ &< \frac{|L|}{2} + |g(x)| && \text{assuming that } 0 < |x-a| < \delta_1 \end{aligned}$$

Rearranging this gives,

$$|L| < \frac{|L|}{2} + |g(x)| \qquad \Rightarrow \qquad \frac{|L|}{2} < |g(x)| \qquad \Rightarrow \qquad \frac{1}{|g(x)|} < \frac{2}{|L|}$$

Now, there is also a $\delta_2 > 0$ such that,

$$|g(x) - L| < \frac{|L|^2}{2}\varepsilon \qquad \text{whenever} \qquad 0 < |x-a| < \delta_2$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$ we have,

$$\begin{aligned} \left|\frac{1}{g(x)} - \frac{1}{L}\right| &= \left|\frac{L - g(x)}{Lg(x)}\right| && \text{common denominators} \\ &= \frac{1}{|Lg(x)|}|L - g(x)| && \text{doing a little rewriting} \\ &= \frac{1}{|L|}\frac{1}{|g(x)|}|g(x) - L| && \text{doing a little more rewriting} \\ &< \frac{1}{|L|}\frac{2}{|L|}|g(x) - L| && \text{assuming that } 0 < |x-a| < \delta \le \delta_1 \\ &< \frac{2}{|L|^2}\frac{|L|^2}{2}\varepsilon && \text{assuming that } 0 < |x-a| < \delta \le \delta_2 \\ &= \varepsilon \end{aligned}$$

Now that we've proven $\lim\limits_{x \to a} \frac{1}{g(x)} = \frac{1}{L}$ the more general fact is easy.

$$\begin{aligned} \lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] &= \lim_{x \to a}\left[f(x)\frac{1}{g(x)}\right] \\ &= \lim_{x \to a} f(x) \lim_{x \to a}\frac{1}{g(x)} && \text{using property } \mathbf{3}. \\ &= K\frac{1}{L} = \frac{K}{L} \end{aligned}$$

■

Proof of 5. for *n* an integer

As noted we're only going to prove **5** for integer exponents. This will also involve proof by induction so if you aren't familiar with induction proofs you can skip this proof.

So, we're going to prove,

$$\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n = K^n, \qquad n \geq 2,\ n \text{ is an integer.}$$

For $n = 2$ we have nothing more than a special case of property **3.**

$$\lim_{x\to a}\left[f(x)\right]^2 = \lim_{x\to a} f(x)\,f(x) = \lim_{x\to a} f(x)\lim_{x\to a} f(x) = KK = K^2$$

So, **5** is proven for $n = 2$. Now assume that **5** is true for $n-1$, or $\lim\limits_{x\to a}\left[f(x)\right]^{n-1} - K^{n-1}$. Then, again using property **3** we have,

$$\begin{aligned}\lim_{x\to a}\left[f(x)\right]^n &= \lim_{x\to a}\left(\left[f(x)\right]^{n-1} f(x)\right)\\ &= \lim_{x\to a}\left[f(x)\right]^{n-1}\lim_{x\to a} f(x)\\ &= K^{n-1}K\\ &= K^n\end{aligned}$$

■

Proof of 6

As pointed out in the Limit Properties section this is nothing more than a special case of the full version of **5** and the proof is given there and so is the proof is not give here.

Proof of 8

This is a simple proof. If we define $f(x) = x$ to make the notation a little easier, we're being asked to prove that $\lim\limits_{x\to a} f(x) = a$.

Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then, if $0 < |x-a| < \delta = \varepsilon$ we have,

$$|f(x) - a| = |x-a| < \delta = \varepsilon$$

So, we've proved that $\lim\limits_{x\to a} x = a$**.**

■

Proof of 9

This is just a special case of property **5** with $f(x) = x$ and so we won't prove it here.

Fact 1, Limits At Infinity, Part 1

1. If r is a positive rational number and c is any real number then,

$$\lim_{x \to \infty} \frac{c}{x^r} = 0$$

2. If r is a positive rational number, c is any real number and x^r is defined for $x < 0$ then,

$$\lim_{x \to -\infty} \frac{c}{x^r} = 0$$

Proof of 1

This is actually a fairly simple proof but we'll need to do three separate cases.

Case 1 : Assume that $c > 0$. Next, let $\varepsilon > 0$ and define $M = \sqrt[r]{\frac{c}{\varepsilon}}$. Note that because c and ε are both positive we know that this root will exist. Now, assume that we have $x > M = \sqrt[r]{\frac{c}{\varepsilon}}$. Give this assumption we have,

$$\begin{aligned} x &> \sqrt[r]{\frac{c}{\varepsilon}} & & \\ x^r &> \frac{c}{\varepsilon} & & \text{get rid of the root} \\ \frac{c}{x^r} &< \varepsilon & & \text{rearrange things a little} \\ \left| \frac{c}{x^r} - 0 \right| &< \varepsilon & & \text{everything is positive so we can add absolute value bars} \end{aligned}$$

So, provided $c > 0$ we've proven that $\lim_{x \to \infty} \frac{c}{x^r} = 0$.

Case 2 : Assume that $c = 0$. Here all we need to do is the following,

$$\lim_{x \to \infty} \frac{c}{x^r} = \lim_{x \to \infty} \frac{0}{x^r} = \lim_{x \to \infty} 0 = 0$$

Case 3 : Finally, assume that $c < 0$. In this case we can then write $c = -k$ where $k > 0$. Then using Case 1 and the fact that we can factor constants out of a limit we get,

$$\lim_{x\to\infty}\frac{c}{x^r} = \lim_{x\to\infty}\frac{-k}{x^r} = -\lim_{x\to\infty}\frac{k}{x^r} = -0 = 0$$

■

Proof of 2

This is very similar to the proof of **1** so we'll just do the first case (as it's the hardest) and leave the other two cases up to you to prove.

Case 1 : Assume that $c > 0$. Next, let $\varepsilon > 0$ and define $N = -\sqrt[r]{\frac{c}{\varepsilon}}$. Note that because *c* and ε are both positive we know that this root will exist. Now, assume that we have $x < N = -\sqrt[r]{\frac{c}{\varepsilon}}$.

Note that this assumption also tells us that *x* will be negative. Give this assumption we have,

$$x < -\sqrt[r]{\frac{c}{\varepsilon}}$$

$$|x| > \left|\sqrt[r]{\frac{c}{\varepsilon}}\right| \qquad \text{take absolute value of both sides}$$

$$|x^r| > \left|\frac{c}{\varepsilon}\right| \qquad \text{get rid of the root}$$

$$\left|\frac{c}{x^r}\right| < |\varepsilon| = \varepsilon \qquad \text{rearrange things a little and use the fact that } \varepsilon > 0$$

$$\left|\frac{c}{x^r} - 0\right| < \varepsilon \qquad \text{rewrite things a little}$$

So, provided $c > 0$ we've proven that $\lim_{x\to\infty}\frac{c}{x^r} = 0$. Note that the main difference here is that we need to take the absolute value first to deal with the minus sign. Because we both sides are negative we know that when we take the absolute value of both sides the direction of the inequality will have to switch as well.

Case 2, Case 3 : As noted above these are identical to the proof of the corresponding cases in the first proof and so are omitted here.

■

Fact 2, Limits At Infinity, Part I

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree *n* (*i.e.* $a_n \neq 0$) then,

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n \qquad \lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$$

Proof of $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$

We're going to prove this in an identical fashion to the problems that we worked in this section involving polynomials. We'll first factor out x^n from the polynomial and then make a giant use of **Fact 1** (which we just proved above) and the basic properties of limits.

$$\begin{aligned}
\lim_{x\to\infty} p(x) &= \lim_{x\to\infty} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
&= \lim_{x\to\infty} x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right) \\
&= \lim_{x\to\infty} x^n \left(\lim_{x\to\infty} a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right) \\
&= \lim_{x\to\infty} x^n \left(a_n + 0 + \cdots + 0 + 0 \right) \\
&= a_n \lim_{x\to\infty} x^n \\
&= \lim_{x\to\infty} a_n x^n
\end{aligned}$$

■

Proof of $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$

The proof of this part is literally identical to the proof of the first part, with the exception that all ∞'s are changed to $=\infty$, and so is omitted here.

Fact 2, Continuity

If $f(x)$ is continuous at $x = b$ and $\lim_{x\to a} g(x) = b$ then,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$

Proof

Let $\varepsilon > 0$ then we need to show that there is a $\delta > 0$ such that,

$$\left| f\left(g\left(x\right)\right) - f\left(b\right) \right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta$$

Let's start with the fact that $f\left(x\right)$ is continuous at $x = b$. Recall that this means that $\lim\limits_{x \to b} f\left(x\right) = f\left(b\right)$ and so there must be a $\delta_1 > 0$ so that,

$$\left| f\left(x\right) - f\left(b\right) \right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left| x - b \right| < \delta_1$$

Now, let's recall that $\lim\limits_{x \to a} g\left(x\right) = b$. This means that there must be a $\delta > 0$ so that,

$$\left| g\left(x\right) - b \right| < \delta_1 \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta$$

But all this means that we're done.

Let's summarize up. First assume that $0 < \left| x - a \right| < \delta$. This then tells us that,

$$\left| g\left(x\right) - b \right| < \delta_1$$

But, we also know that if $0 < \left| x - b \right| < \delta_1$ then we must also have $\left| f\left(x\right) - f\left(b\right) \right| < \varepsilon$. What this is telling us is that if a number is within a distance of δ_1 of *b* then we can plug that number into $f\left(x\right)$ and we'll be within a distance of ε of $f\left(b\right)$.

So, $\left| g\left(x\right) - b \right| < \delta_1$ is telling us that $g\left(x\right)$ is within a distance of δ_1 of *b* and so if we plug it into $f\left(x\right)$ we'll get,

$$\left| f\left(g\left(x\right)\right) - f\left(b\right) \right| < \varepsilon$$

and this is exactly what we wanted to show.

■

Proof of Various Derivative Facts/Formulas/Properties

In this section we're going to prove many of the various derivative facts, formulas and/or properties that we encountered in the early part of the Derivatives chapter. Not all of them will be proved here and some will only be proved for special cases, but at least you'll see that some of them aren't just pulled out of the air.

<u>Theorem, from Definition of Derivative</u>

If $f\left(x\right)$ is differentiable at $x=a$ then $f\left(x\right)$ is continuous at $x=a$.

Proof

Because $f\left(x\right)$ is differentiable at $x=a$ we know that

$$f'\left(a\right)=\lim_{x\to a}\frac{f\left(x\right)-f\left(a\right)}{x-a}$$

exists. We'll need this in a bit.

If we next assume that $x\neq a$ we can write the following,

$$f\left(x\right)-f\left(a\right)=\frac{f\left(x\right)-f\left(a\right)}{x-a}\left(x-a\right)$$

Then basic properties of limits tells us that we have,

$$\begin{aligned}\lim_{x\to a}\left(f\left(x\right)-f\left(a\right)\right)&=\lim_{x\to a}\left[\frac{f\left(x\right)-f\left(a\right)}{x-a}\left(x-a\right)\right]\\&=\lim_{x\to a}\frac{f\left(x\right)-f\left(a\right)}{x-a}\ \lim_{x\to a}\left(x-a\right)\end{aligned}$$

The first limit on the right is just $f'\left(a\right)$ as we noted above and the second limit is clearly zero and so,

$$\lim_{x\to a}\left(f\left(x\right)-f\left(a\right)\right)=f'\left(a\right)\cdot 0=0$$

Okay, we've managed to prove that $\lim_{x\to a}\left(f\left(x\right)-f\left(a\right)\right)=0$. But just how does this help us to prove that $f\left(x\right)$ is continuous at $x=a$?

Let's start with the following.

$$\lim_{x\to a}f\left(x\right)=\lim_{x\to a}\left[f\left(x\right)+f\left(a\right)-f\left(a\right)\right]$$

Note that we've just added in zero on the right side. A little rewriting and the use of limit properties gives,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left[f(a)+f(x)-f(a)\right]\\ &= \lim_{x\to a} f(a)+\lim_{x\to a}\left[f(x)-f(a)\right]\end{aligned}$$

Now, we just proved above that $\lim_{x\to a}\left(f(x)-f(a)\right)=0$ and because $f(a)$ is a constant we also know that $\lim_{x\to a} f(a)=f(a)$ and so this becomes,

$$\lim_{x\to a} f(x)=\lim_{x\to a} f(a)+0=f(a)$$

Or, in other words, $\lim_{x\to a} f(x)=f(a)$ but this is exactly what it means for $f(x)$ is continuous at $x=a$ and so we're done.

■

Proof of Sum/Difference of Two Functions : $\left(f(x)\pm g(x)\right)' = f'(x)\pm g'(x)$

This is easy enough to prove using the definition of the derivative. We'll start with the sum of two functions. First plug the sum into the definition of the derivative and rewrite the numerator a little.

$$\begin{aligned}\left(f(x)+g(x)\right)' &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-\left(f(x)+g(x)\right)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\end{aligned}$$

Now, break up the fraction into two pieces and recall that the limit of a sum is the sum of the limits. Using this fact we see that we end up with the definition of the derivative for each of the two functions.

$$\begin{aligned}\left(f(x)+g(x)\right)' &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\end{aligned}$$

The proof of the difference of two functions in nearly identical so we'll give it here without any explanation.

$$\begin{aligned}\left(f(x)-g(x)\right)' &= \lim_{h\to 0}\frac{f(x+h)-g(x+h)-\left(f(x)-g(x)\right)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)-\left(g(x+h)-g(x)\right)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-\frac{g(x+h)-g(x)}{h}\\ &= f'(x)-g'(x)\end{aligned}$$

■

Proof of Constant Times a Function : $\left(cf(x)\right)' = cf'(x)$

This is property is very easy to prove using the definition provided you recall that we can factor a constant out of a limit. Here's the work for this property.

$$\left(cf(x)\right)' = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x)$$

■

Proof of the Derivative of a Constant : $\frac{d}{dx}(c) = 0$

This is very easy to prove using the definition of the derivative so define $f(x) = c$ and the use the definition of the derivative.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

■

Power Rule : $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

There are actually three proofs that we can give here and we're going to go through all three here so you can see all of them. However, having said that, for the first two we will need to restrict n to be a positive integer. At the time that the Power Rule was introduced only enough information has been given to allow the proof for only integers. So, the first two proofs are really to be read at that point.

The third proof will work for any real number *n*. However, it does assume that you've read most of the Derivatives chapter and so should only be read after you've gone through the whole chapter.

Proof 1

In this case as noted above we need to assume that *n* is a positive integer. We'll use the definition of the derivative and the Binomial Theorem in this theorem. The Binomial Theorem tells us that,

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n \\ &= a^n + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \cdots + n a b^{n-1} + b^n\end{aligned}$$

where,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are called the binomial coefficients and $n! = n(n-1)(n-2)\cdots(2)(1)$ is the factorial.

So, let's go through the details of this proof. First, plug $f(x) = x^n$ into the definition of the derivative and use the Binomial Theorem to expand out the first term.

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{\left(x^n + n x^{n-1} h + \frac{n(n-1)}{2!} x^{n-2} h^2 + \cdots + n x h^{n-1} + h^n\right) - x^n}{h}\end{aligned}$$

Now, notice that we can cancel an x^n and then each term in the numerator will have an *h* in them that can be factored out and then canceled against the *h* in the numerator. At this point we can evaluate the limit.

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{n x^{n-1} h + \frac{n(n-1)}{2!} x^{n-2} h^2 + \cdots + n x h^{n-1} + h^n}{h} \\ &= \lim_{h\to 0} n x^{n-1} + \frac{n(n-1)}{2!} x^{n-2} h + \cdots + n x h^{n-2} + h^{n-1} \\ &= n x^{n-1}\end{aligned}$$

■

Proof 2

For this proof we'll again need to restrict *n* to be a positive integer. In this case if we define $f(x) = x^n$ we know from the alternate limit form of the definition of the derivative that the derivative $f'(a)$ is given by,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

Now we have the following formula,

$$x^n - a^n = (x - a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-3}x^2 + a^{n-2}x + a^{n-1}\right)$$

You can verify this if you'd like by simply multiplying the two factors together. Also, notice that there are a total of *n* terms in the second factor (this will be important in a bit).

If we plug this into the formula for the derivative we see that we can cancel the $x - a$ and then compute the limit.

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{(x-a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-3}x^2 + a^{n-2}x + a^{n-1}\right)}{x - a} \\ &= \lim_{x \to a} x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-3}x^2 + a^{n-2}x + a^{n-1} \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \cdots + a^{n-3}a^2 + a^{n-2}a + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

After combining the exponents in each term we can see that we get the same term. So, then recalling that there are *n* terms in second factor we can see that we get what we claimed it would be.

To completely finish this off we simply replace the *a* with an *x* to get,

$$f'(x) = nx^{n-1}$$

■

Proof 3

In this proof we no longer need to restrict *n* to be a positive integer. It can now be any real number. However, this proof also assumes that you've read all the way through the Derivative chapter. In particular it needs both Implicit Differentiation and Logarithmic Differentiation. If you've not read, and understand, these sections then this proof will not make any sense to you.

So, to get set up for logarithmic differentiation let's first define $y = x^n$ then take the log of both sides, simplify the right side using logarithm properties and then differentiate using implicit differentiation.

$$\ln y = \ln x^n$$
$$\ln y = n \ln x$$
$$\frac{y'}{y} = n\frac{1}{x}$$

Finally, all we need to do is solve for y' and then substitute in for *y*.

$$y' = y\frac{n}{x} - x^n\left(\frac{n}{x}\right) = nx^{n-1}$$

■

Before moving onto the next proof, let's notice that in all three proofs we did require that the exponent, *n*, be a number (integer in the first two, any real number in the third). In the first proof we couldn't have used the Binomial Theorem if the exponent wasn't a positive integer. In the second proof we couldn't have factored $x^n - a^n$ if the exponent hadn't been a positive integer. Finally, in the third proof we would have gotten a much different derivative if *n* had not been a constant.

This is important because people will often misuse the power rule and use it even when the exponent is not a number and/or the base is not a variable.

Product Rule : $(f\,g)' = f'\,g + f\,g'$

As with the Power Rule above, the Product Rule can be proved either by using the definition of the derivative or it can be proved using Logarithmic Differentiation. We'll show both proofs here.

Proof 1

This proof can be a little tricky when you first see it so let's be a little careful here. We'll first use the definition of the derivative on the product.

$$(f\,g)' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

On the surface this appears to do nothing for us. We'll first need to manipulate things a little to get the proof going. What we'll do is subtract out and add in $f(x+h)g(x)$ to the numerator. Note that we're really just adding in a zero here since these two terms will cancel. This will give us,

$$(f\,g)' = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

Notice that we added the two terms into the middle of the numerator. As written we can break up the limit into two pieces. From the first piece we can factor a $f(x+h)$ out and we can factor a $g(x)$ out of the second piece. Doing this gives,

$$\begin{aligned}(f\,g)' &= \lim_{h\to 0} \frac{f(x+h)\big(g(x+h)-g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h)-f(x)\big)}{h}\\ &= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h}\end{aligned}$$

At this point we can use limit properties to write,

$$(f\,g)' = \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right) + \left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$$

The individual limits in here are,

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x) \qquad\qquad \lim_{h\to 0} g(x) = g(x)$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) \qquad\qquad \lim_{h\to 0} f(x+h) = f(x)$$

The two limits on the left are nothing more than the definition the derivative for $g(x)$ and $f(x)$ respectively. The upper limit on the right seems a little tricky, but remember that the limit of a constant is just the constant. In this case since the limit is only concerned with allowing *h* to go to zero. The key here is to recognize that changing *h* will not change *x* and so as far as this limit is concerned $g(x)$ is a constant. Note that the function is probably not a constant, however as far as the limit is concerned the function can be treated as a constant. We get the lower limit on the right we get simply by plugging $h = 0$ into the function

Plugging all these into the last step gives us,

$$(f\,g)' = f(x)\,g'(x) + g(x)\,f'(x)$$

■

Proof 2

This is a much quicker proof but does presuppose that you've read and understood the Implicit Differentiation and Logarithmic Differentiation sections. If you haven't then this proof will not make a lot of sense to you.

First write call the product y and take the log of both sides and use a property of logarithms on the right side.

$$y = f(x)g(x)$$
$$\ln(y) = \ln\left(f(x)g(x)\right) = \ln f(x) + \ln g(x)$$

Next, we take the derivative of both sides and solve for y'.

$$\frac{y'}{y} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \qquad \Rightarrow \qquad y' = y\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right)$$

Finally, all we need to do is plug in for y and then multiply this through the parenthesis and we get the Product Rule.

$$y = f(x)g(x)\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right) \qquad \Rightarrow \qquad (fg)' = g(x)f'(x) + f(x)g'(x)$$

■

Quotient Rule : $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Again, we can do this using the definition of the derivative or with Logarithmic Definition.

Proof 1

First plug the quotient into the definition of the derivative and rewrite the quotient a little.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\,\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

To make our life a little easier we moved the h in the denominator of the first step out to the front as a $\frac{1}{h}$. We also wrote the numerator as a single rational expression. This step is required to make this proof work.

Now, for the next step will need to subtract out and add in $f(x)g(x)$ to the numerator.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{h}\,\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)}$$

The next step is to rewrite things a little,

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\,\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h}$$

Note that all we did was interchange the two denominators. Since we are multiplying the fractions we can do this.

Next, the larger fraction can be broken up as follows.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left(\frac{f(x+h)g(x)-f(x)g(x)}{h}+\frac{f(x)g(x)-f(x)g(x+h)}{h}\right)$$

In the first fraction we will factor a $g(x)$ out and in the second we will factor a $-f(x)$ out. This gives,

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left(g(x)\frac{f(x+h)-f(x)}{h}-f(x)\frac{g(x+h)-g(x)}{h}\right)$$

We can now use the basic properties of limits to write this as,

$$\left(\frac{f}{g}\right)' = \frac{1}{\lim\limits_{h\to 0}g(x+h)\lim\limits_{h\to 0}g(x)}\left(\left(\lim_{h\to 0}g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)-\left(\lim_{h\to 0}f(x)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\right)$$

The individual limits are,

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=g'(x) \qquad \lim_{h\to 0}g(x+h)=g(x) \qquad \lim_{h\to 0}g(x)=g(x)$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x) \qquad \lim_{h\to 0}f(x)=f(x)$$

The first two limits in each row are nothing more than the definition the derivative for $g(x)$ and

$f(x)$ respectively. The middle limit in the top row we get simply by plugging in $h=0$. The final limit in each row may seem a little tricky. Recall that the limit of a constant is just the constant. Well since the limit is only concerned with allowing *h* to go to zero as far as its concerned $g(x)$ and $f(x)$ are constants since changing *h* will not change *x*. Note that the function is probably not a constant, however as far as the limit is concerned the function can be treated as a constant.

Plugging in the limits and doing some rearranging gives,

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \frac{1}{g(x)g(x)}\left(g(x)f'(x)-f(x)g'(x)\right)\\ &= \frac{f'g-fg'}{g^2}\end{aligned}$$

There's the quotient rule.

■

Proof 2

Now let's do the proof using Logarithmic Differentiation. We'll first call the quotient *y*, take the log of both sides and use a property of logs on the right side.

$$\begin{aligned}y &= \frac{f(x)}{g(x)}\\ \ln y = \ln\left(\frac{f(x)}{g(x)}\right) &= \ln f(x) - \ln g(x)\end{aligned}$$

Next, we take the derivative of both sides and solve for y'.

$$\frac{y'}{y} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \qquad \Rightarrow \qquad y' = y\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right)$$

Next, plug in *y* and do some simplification to get the quotient rule.

$$\begin{aligned}y' &= \frac{f(x)}{g(x)}\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right)\\ &= \frac{f'(x)}{g(x)} - \frac{g'(x)f(x)}{\left(g(x)\right)^2}\\ &= \frac{f'(x)g(x)}{\left(g(x)\right)^2} - \frac{f(x)g'(x)}{\left(g(x)\right)^2} = \frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}\end{aligned}$$

■

Chain Rule

If $f(x)$ and $g(x)$ are both differentiable functions and we define $F(x) = (f \circ g)(x)$ then the derivative of *F(x)* is $F'(x) = f'(g(x))\ g'(x)$.

Proof

We'll start off the proof by defining $u = g(x)$ and noticing that in terms of this definition what we're being asked to prove is,

$$\frac{d}{dx}\left[f(u)\right] = f'(u)\frac{du}{dx}$$

Let's take a look at the derivative of $u(x)$ (again, remember we've defined $u = g(x)$ and so *u* really is a function of *x*) which we know exists because we are assuming that $g(x)$ is differentiable. By definition we have,

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

Note as well that,

$$\lim_{h \to 0}\left(\frac{u(x+h) - u(x)}{h} - u'(x)\right) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} - \lim_{h \to 0} u'(x) = u'(x) - u'(x) = 0$$

Now, define,

$$v(h) = \begin{cases} \dfrac{u(x+h) - u(x)}{h} - u'(x) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

and notice that $\lim_{h \to 0} v(h) = 0 = v(0)$ and so $v(h)$ is continuous at $h = 0$

Now if we assume that $h \neq 0$ we can rewrite the definition of $v(h)$ to get,

$$u(x+h) = u(x) + h\left(v(h) + u'(x)\right) \tag{1}$$

Now, notice that (1) is in fact valid even if we let $h = 0$ and so is valid for any value of *h*.

Next, since we also know that $f(x)$ is differentiable we can do something similar. However, we're going to use a different set of letters/variables here for reasons that will be apparent in a bit.

So, define,

$$w(k) = \begin{cases} \dfrac{f(z+h) - f(z)}{k} - f'(z) & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

we can go through a similar argument that we did above so show that $w(k)$ is continuous at $k = 0$ and that,

$$f(z+k) = f(z) + k\big(w(k) + f'(z)\big) \tag{2}$$

Do not get excited about the different letters here all we did was use *k* instead of *h* and let $x = z$. Nothing fancy here, but the change of letters will be useful down the road.

Okay, to this point it doesn't look like we've really done anything that gets us even close to proving the chain rule. The work above will turn out to be very important in our proof however so let's get going on the proof.

What we need to do here is use the definition of the derivative and evaluate the following limit.

$$\frac{d}{dx}\Big[f\big[u(x)\big]\Big] = \lim_{h \to 0} \frac{f\big[u(x+h)\big] - f\big[u(x)\big]}{h} \tag{3}$$

Note that even though the notation is more than a little messy if we use $u(x)$ instead of *u* we need to remind ourselves here that *u* really is a function of *x*.

Let's now use (1) to rewrite the $u(x+h)$ and yes the notation is going to be unpleasant but we're going to have to deal with it. By using (1), the numerator in the limit above becomes,

$$f\big[u(x+h)\big] - f\big[u(x)\big] = f\Big[u(x) + h\big(v(h) + u'(x)\big)\Big] - f\big[u(x)\big]$$

If we then define $z = u(x)$ and $k = h\big(v(h) + u'(x)\big)$ we can use (2) to further write this as,

$$\begin{aligned} f\big[u(x+h)\big] - f\big[u(x)\big] &= f\Big[u(x) + h\big(v(h) + u'(x)\big)\Big] - f\big[u(x)\big] \\ &= f\big[u(x)\big] + h\big(v(h) + u'(x)\big)\Big(w(k) + f'\big[u(x)\big]\Big) - f\big[u(x)\big] \\ &= h\big(v(h) + u'(x)\big)\Big(w(k) + f'\big[u(x)\big]\Big) \end{aligned}$$

Notice that we were able to cancel a $f\big[u(x)\big]$ to simplify things up a little. Also, note that the $w(k)$ was intentionally left that way to keep the mess to a minimum here, just remember that $k = h\big(v(h) + u'(x)\big)$ here as that will be important here in a bit. Let's now go back and remember that all this was the numerator of our limit, (3). Plugging this into (3) gives,

$$\begin{aligned}\frac{d}{dx}\left[f\left[u(x)\right]\right] &= \lim_{h\to 0}\frac{h\left(v(h)+u'(x)\right)\left(w(k)+f'\left[u(x)\right]\right)}{h} \\ &= \lim_{h\to 0}\left(v(h)+u'(x)\right)\left(w(k)+f'\left[u(x)\right]\right)\end{aligned}$$

Notice that the *h*'s canceled out. Next, recall that $k = h\left(v(h)+u'(x)\right)$ and so,

$$\lim_{h\to 0} k = \lim_{h\to 0} h\left(v(h)+u'(x)\right) = 0$$

But, if $\lim_{h\to 0} k = 0$, as we've defined *k* anyway, then by the definition of *w* and the fact that we know $w(k)$ is continuous at $k = 0$ we also know that,

$$\lim_{h\to 0} w(k) = w\left(\lim_{h\to 0} k\right) = w(0) = 0$$

Also, recall that $\lim_{h\to 0} v(h) = 0$. Using all of these facts our limit becomes,

$$\begin{aligned}\frac{d}{dx}\left[f\left[u(x)\right]\right] &= \lim_{h\to 0}\left(v(h)+u'(x)\right)\left(w(k)+f'\left[u(x)\right]\right) \\ &= u'(x)\, f'\left[u(x)\right] \\ &= f'\left[u(x)\right]\frac{du}{dx}\end{aligned}$$

This is exactly what we needed to prove and so we're done.

■

Proof of Trig Limits

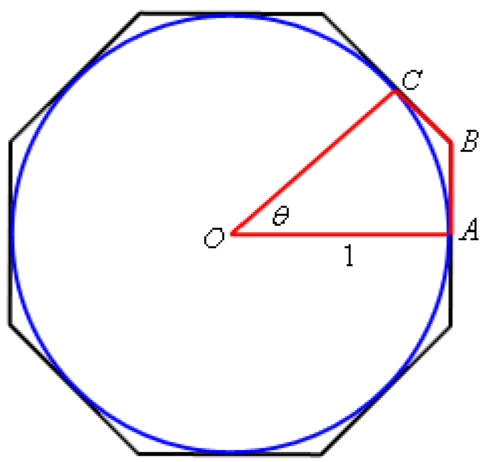
In this section we're going to provide the proof of the two limits that are used in the derivation of the derivative of sine and cosine in the Derivatives of Trig Functions section of the Derivatives chapter.

Proof of : $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

This proof of this limit uses the Squeeze Theorem. However, getting things set up to use the Squeeze Theorem can be a somewhat complex geometric argument that can be difficult to follow so we'll try to take it fairly slow.

Let's start by assuming that $0 \le \theta \le \frac{\pi}{2}$. Since we are proving a limit that has $\theta \to 0$ it's okay to assume that θ is not too large (*i.e.* $\theta \le \frac{\pi}{2}$). Also, by assuming that θ is positive we're actually going to first prove that the above limit is true if it is the right-hand limit. As you'll see if we can prove this then the proof of the limit will be easy.

So, now that we've got our assumption on θ taken care of let's start off with the unit circle circumscribed by an octagon with a small slice marked out as shown below.



Points *A* and *C* are the midpoints of their respective sides on the octagon and are in fact tangent to the circle at that point. We'll call the point where these two sides meet *B*.

From this figure we can see that the circumference of the circle is less than the length of the octagon. This also means that if we look at the slice of the figure marked out above then the length of the portion of the circle included in the slice must be less than the length of the portion of the octagon included in the slice.

Because we're going to be doing most of our work on just the slice of the figure let's strip that out and look at just it. Here is a sketch of just the slice.