## *Proof of Trig Limits*

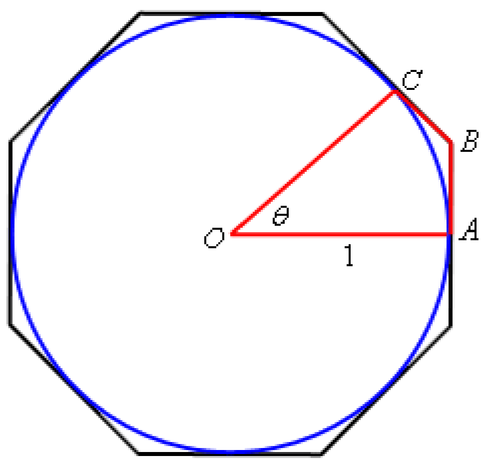
In this section we're going to provide the proof of the two limits that are used in the derivation of the derivative of sine and cosine in the Derivatives of Trig Functions section of the Derivatives chapter.

**Proof of :** $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

This proof of this limit uses the Squeeze Theorem. However, getting things set up to use the Squeeze Theorem can be a somewhat complex geometric argument that can be difficult to follow so we'll try to take it fairly slow.

Let's start by assuming that $0 \le \theta \le \frac{\pi}{2}$. Since we are proving a limit that has $\theta \to 0$ it's okay to assume that $\theta$ is not too large (*i.e.* $\theta \le \frac{\pi}{2}$). Also, by assuming that $\theta$ is positive we're actually going to first prove that the above limit is true if it is the right-hand limit. As you'll see if we can prove this then the proof of the limit will be easy.

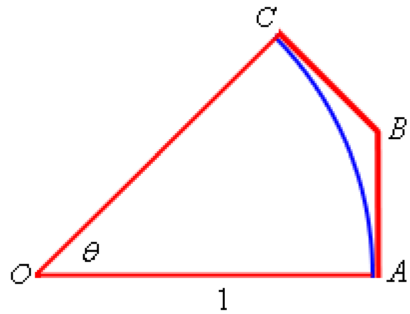
So, now that we've got our assumption on $\theta$ taken care of let's start off with the unit circle circumscribed by an octagon with a small slice marked out as shown below.




Points *A* and *C* are the midpoints of their respective sides on the octagon and are in fact tangent to the circle at that point. We'll call the point where these two sides meet *B*.

From this figure we can see that the circumference of the circle is less than the length of the octagon. This also means that if we look at the slice of the figure marked out above then the length of the portion of the circle included in the slice must be less than the length of the portion of the octagon included in the slice.

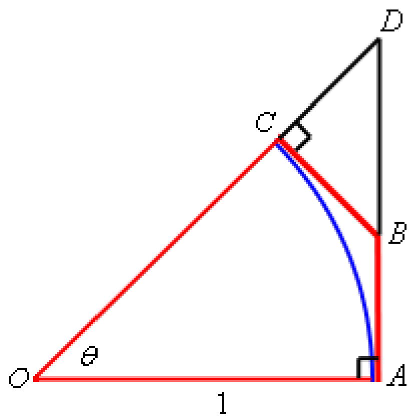
Because we're going to be doing most of our work on just the slice of the figure let's strip that out and look at just it. Here is a sketch of just the slice.




Now denote the portion of the circle by $\text{arc } AC$ and the lengths of the two portion of the octagon shown by $|AB|$ and $|BC|$. Then by the observation about lengths we made above we must have,

$$\text{arc } AC < |AB| + |BC| \tag{4}$$

Next, extend the lines *AB* and *OC* as shown below and call the point that they meet *D*. The triangle now formed by *AOD* is a right triangle. All this is shown in the figure below.




The triangle *BCD* is a right triangle with hypotenuse *BD* and so we know $|BC| < |BD|$. Also notice that $|AB| + |BD| = |AD|$. If we use these two facts in (1) we get,

$$\begin{aligned} \text{arc } AC &< |AB| + |BC| \\ &< |AB| + |BD| \\ &= |AD| \end{aligned} \tag{5}$$

Next, as noted already the triangle *AOD* is a right triangle and so we can use a little right triangle trigonometry to write $|AD| = |AO| \tan\theta$. Also note that $|AO| = 1$ since it is nothing more than the radius of the unit circle. Using this information in (2) gives,

$$\begin{aligned} \text{arc } AC &< |AD| \\ &< |AO| \tan\theta \\ &= \tan\theta \end{aligned} \tag{6}$$

The next thing that we need to recall is that the length of a portion of a circle is given by the radius of the circle times the angle that traces out the portion of the circle we're trying to measure. For our portion this means that,

$$\text{arc } AC = |AO|\,\theta = \theta$$

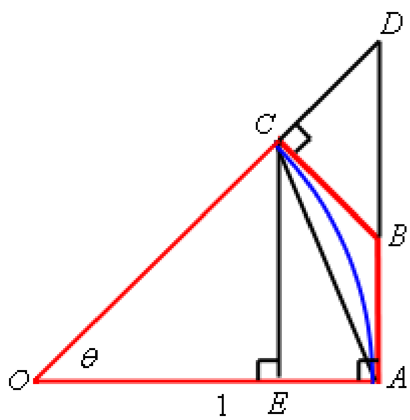
So, putting this into (3) we see that,

$$\theta = \text{arc } AC < \tan\theta = \frac{\sin\theta}{\cos\theta}$$

or, if we do a little rearranging we get,

$$\cos\theta < \frac{\sin\theta}{\theta} \tag{7}$$

We'll be coming back to (4) in a bit. Let's now add in a couple more lines into our figure above. Let's connect *A* and *C* with a line and drop a line straight down from *C* until it intersects *AO* at a right angle and let's call the intersection point *E*. This is all show in the figure below.




Okay, the first thing to notice here is that,

$$|CE| < |AC| < \text{arc } AC \tag{8}$$

Also note that triangle *EOC* is a right triangle with a hypotenuse of $|CO| = 1$. Using some right triangle trig we can see that,

$$|CE| = |CO|\sin\theta = \sin\theta$$

Plugging this into (5) and recalling that $\text{arc } AC = \theta$ we get,

$$\sin\theta = |CE| < \text{arc } AC = \theta$$

and with a little rewriting we get,

$$\frac{\sin\theta}{\theta} < 1 \tag{9}$$

Okay, we're almost done here. Putting (4) and (6) together we see that,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

provided $0 \le \theta \le \frac{\pi}{2}$. Let's also note that,

$$\lim_{\theta\to 0}\cos\theta = 1 \qquad\qquad \lim_{\theta\to 0} 1 = 1$$

We are now set up to use the Squeeze Theorem. The only issue that we need to worry about is that we are staying to the right of $\theta = 0$ in our assumptions and so the best that the Squeeze Theorem will tell us is,

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

So, we know that the limit is true if we are only working with a right-hand limit. However we know that $\sin\theta$ is an odd function and so,

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

In other words, if we approach zero from the left (*i.e.* negative $\theta$'s) then we'll get the same values in the function as if we'd approached zero from the right (*i.e.* positive $\theta$'s) and so,

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1$$

We have now shown that the two one-sided limits are the same and so we must also have,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

■

That was a somewhat long proof and if you're not really good at geometric arguments it can be kind of daunting and confusing. Nicely, the second limit is very simple to prove, provided you've already proved the first limit.

**Proof of :** $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$

We'll start by doing the following,

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0}\frac{(\cos\theta - 1)(\cos\theta + 1)}{\theta(\cos\theta + 1)} = \lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} \qquad (10)$$

Now, let's recall that,

$$\cos^2\theta + \sin^2\theta = 1 \qquad \Rightarrow \qquad \cos^2\theta - 1 = -\sin^2\theta$$

Using this in (7) gives us,

$$\begin{aligned}\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} &= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta\left(\cos\theta + 1\right)}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{-\sin\theta}{\cos\theta + 1}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\ \lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta + 1}\end{aligned}$$

At this point, because we just proved the first limit and the second can be taken directly we're pretty much done. All we need to do is take the limits.

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\ \lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta + 1} = \left(1\right)\left(0\right) = 0$$

■

## *Proofs of Derivative Applications Facts/Formulas*

In this section we'll be proving some of the facts and/or theorems from the Applications of Derivatives chapter. Not all of the facts and/or theorems will be proved here.

**Fermat's Theorem**

If $f(x)$ has a relative extrema at $x = c$ and $f'(c)$ exists then $x = c$ is a critical point of $f(x)$. In fact, it will be a critical point such that $f'(c) = 0$.

**Proof**

This is a fairly simple proof. We'll assume that $f(x)$ has a relative maximum to do the proof. The proof for a relative minimum is nearly identical. So, if we assume that we have a relative maximum at $x = c$ then we know that $f(c) \geq f(x)$ for all *x* that are sufficiently close to $x = c$. In particular for all *h* that are sufficiently close to zero (positive or negative) we must have,

$$f(c) \geq f(c+h)$$

or, with a little rewrite we must have,

$$f(c+h) - f(c) \leq 0 \qquad (1)$$

Now, at this point assume that $h > 0$ and divide both sides of (1) by *h*. This gives,

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Because we're assuming that $h > 0$ we can now take the right-hand limit of both sides of this.

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0$$

We are also assuming that $f'(c)$ exists and recall that if a normal limit exists then it must be equal to both one-sided limits. We can then say that,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

If we put this together we have now shown that $f'(c) \leq 0$.

Okay, now let's turn things around and assume that $h < 0$ and divide both sides of (1) by *h*. This gives,

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

Remember that because we're assuming $h < 0$ we'll need to switch the inequality when we divide by a negative number. We can now do a similar argument as above to get that,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq \lim_{h \to 0^-} 0 = 0$$

The difference here is that this time we're going to be looking at the left-hand limit since we're assuming that $h < 0$. This argument shows that $f'(c) \geq 0$.

We've now shown that $f'(c) \leq 0$ and $f'(c) \geq 0$. Then only way both of these can be true at the same time is to have $f'(c) = 0$ and this in turn means that $x = c$ must be a critical point.

As noted above, if we assume that $f(x)$ has a relative minimum then the proof is nearly identical and so isn't shown here. The main differences are simply some inequalities need to be switched.

■

**Fact, The Shape of a Graph, Part I**

1. If $f'(x) > 0$ for every $x$ on some interval $I$, then $f(x)$ is increasing on the interval.
2. If $f'(x) < 0$ for every $x$ on some interval $I$, then $f(x)$ is decreasing on the interval.
3. If $f'(x) = 0$ for every $x$ on some interval $I$, then $f(x)$ is constant on the interval.

The proof of this fact uses the Mean Value Theorem which, if you're following along in my notes has actually not been covered yet. The Mean Value Theorem can be covered at any time and for whatever the reason I decided to put where it is. Before reading through the proof of this fact you should take a quick look at the Mean Value Theorem section. You really just need the conclusion of the Mean Value Theorem for this proof however.

**Proof of 1**

Let $x_1$ and $x_2$ be in $I$ and suppose that $x_1 < x_2$. Now, using the Mean Value Theorem on $[x_1, x_2]$ means there is a number $c$ such that $x_1 < c < x_2$ and,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Because $x_1 < c < x_2$ we know that $c$ must also be in $I$ and so we know that $f'(c) > 0$ we also know that $x_2 - x_1 > 0$. So, this means that we have,

$$f(x_2) - f(x_1) > 0$$

Rewriting this gives,

$$f(x_1) < f(x_2)$$

and so, by definition, since $x_1$ and $x_2$ were two arbitrary numbers in *I*, $f(x)$ must be increasing on *I*.

■

**Proof of 2**

This proof is nearly identical to the previous part.

Let $x_1$ and $x_2$ be in *I* and suppose that $x_1 < x_2$. Now, using the Mean Value Theorem on $[x_1, x_2]$ means there is a number *c* such that $x_1 < c < x_2$ and,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Because $x_1 < c < x_2$ we know that *c* must also be in *I* and so we know that $f'(c) < 0$ we also know that $x_2 - x_1 > 0$. So, this means that we have,

$$f(x_2) - f(x_1) < 0$$

Rewriting this gives,

$$f(x_1) > f(x_2)$$

and so, by definition, since $x_1$ and $x_2$ were two arbitrary numbers in *I*, $f(x)$ must be decreasing on *I*.

■

**Proof of 3**

Again, this proof is nearly identical to the previous two parts, but in this case is actually somewhat easier.

Let $x_1$ and $x_2$ be in *I*. Now, using the Mean Value Theorem on $[x_1, x_2]$ there is a number *c* such that *c* is between $x_1$ and $x_2$ and,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Note that for this part we didn't need to assume that $x_1 < x_2$ and so all we know is that *c* is between $x_1$ and $x_2$ and so, more importantly, *c* is also in *I*. and this means that $f'(c) = 0$. So, this means that we have,

$$f(x_2) - f(x_1) = 0$$

Rewriting this gives,

$$f(x_1) = f(x_2)$$

and so, since $x_1$ and $x_2$ were two arbitrary numbers in *I*, $f(x)$ must be constant on *I*.

■

**Fact, The Shape of a Graph, Part II**

Given the function $f(x)$ then,

1. If $f''(x) > 0$ for all *x* in some interval *I* then $f(x)$ is concave up on *I*.

2. If $f''(x) < 0$ for all *x* in some interval *I* then $f(x)$ is concave up on *I*.

The proof of this fact uses the Mean Value Theorem which, if you're following along in my notes has actually not been covered yet. The Mean Value Theorem can be covered at any time and for whatever the reason I decided to put it after the section this fact is in. Before reading through the proof of this fact you should take a quick look at the Mean Value Theorem section. You really just need the conclusion of the Mean Value Theorem for this proof however.

**Proof of 1**

Let *a* be any number in the interval *I*. The tangent line to $f(x)$ at $x = a$ is,

$$y = f(a) + f'(a)(x-a)$$

To show that $f(x)$ is concave up on *I* then we need to show that for any *x*, $x \neq a$, in *I* that,

$$f(x) > f(a) + f'(a)(x-a)$$

or in other words, the tangent line is always below the graph of $f(x)$ on *I*. Note that we require $x \neq a$ because at that point we know that $f(x) = f(a)$ since we are talking about the tangent line.

Let's start the proof off by first assuming that $x > a$. Using the Mean Value Theorem on $[a, x]$ means there is a number *c* such that $a < c < x$ and,

$$f(x) - f(a) = f'(c)(x-a)$$

With some rewriting this is,

$$f(x) = f(a) + f'(c)(x-a) \tag{2}$$

Next, let's use the fact that $f''(x) > 0$ for every *x* on *I*. This means that the first derivative, $f'(x)$, must be increasing (because its derivative, $f''(x)$, is positive). Now, we know from the Mean Value Theorem that $a < c$ and so because $f'(x)$ is increasing we must have,

$$f'(a) < f'(c) \tag{3}$$

Recall as well that we are assuming $x > a$ and so $x - a > 0$. If we now multiply (3) by $x - a$ (which is positive and so the inequality stays the same) we get,

$$f'(a)(x-a) < f'(c)(x-a)$$

Next, add $f(a)$ to both sides of this to get,

$$f(a) + f'(a)(x-a) < f(a) + f'(c)(x-a)$$

However, by (2), the right side of this is nothing more than $f(x)$ and so we have,

$$f(a) + f'(a)(x-a) < f(x)$$

but this is exactly what we wanted to show.

So, provided $x > a$ the tangent line is in fact below the graph of $f(x)$.

We now need to assume $x < a$. Using the Mean Value Theorem on $[x, a]$ means there is a number *c* such that $x < c < a$ and,

$$f(a) - f(x) = f'(c)(a-x)$$

If we multiply both sides of this by –1 and then adding $f(a)$ to both sides and we again arise at (2).

Now, from the Mean Value Theorem we know that $c < a$ and because $f''(x) > 0$ for every *x* on *I* we know that the derivative is still increasing and so we have,

$$f'(c) < f'(a)$$

Let's now multiply this by $x - a$, which is now a negative number since $x < a$. This gives,

$$f'(c)(x-a) > f'(a)(x-a)$$

Notice that we had to switch the direction of the inequality since we were multiplying by a negative number. If we now add $f(a)$ to both sides of this and then substitute (2) into the results we arrive at,

$$\begin{aligned} f(a)+f'(c)(x-a) &> f(a)+f'(a)(x-a) \\ f(x) &> f(a)+f'(a)(x-a) \end{aligned}$$

So, again we've shown that the tangent line is always below the graph of $f(x)$.

We've now shown that if *x* is any number in *I*, with $x \neq a$ the tangent lines are always below the graph of $f(x)$ on *I* and so $f(x)$ is concave up on *I*.

■

**Proof of 2**

This proof is nearly identical to the proof of 1 and since that proof is fairly long we're going to just get things started and then leave the rest of it to you to go through.

Let *a* be any number in $I$. To show that $f(x)$ is concave down we need to show that for any *x* in *I*, $x \neq a$, that the tangent line is always above the graph of $f(x)$ or,

$$f(x) < f(a)+f'(a)(x-a)$$

From this point on the proof is almost identical to the proof of 1 except that you'll need to use the fact that the derivative in this case is decreasing since $f''(x)<0$. We'll leave it to you to fill in the details of this proof.

■

**Second Derivative Test**

Suppose that $x=c$ is a critical point of $f'(c)$ such that $f'(c)=0$ and that $f''(x)$ is continuous in a region around $x=c$. Then,

1. If $f''(c)<0$ then $x=c$ is a relative maximum.
2. If $f''(c)>0$ then $x=c$ is a relative minimum.
3. If $f''(c)=0$ then $x=c$ can be a relative maximum, relative minimum or neither.

The proof of this fact uses the Mean Value Theorem which, if you're following along in my notes has actually not been covered yet. The Mean Value Theorem can be covered at any time and for whatever the reason I decided to put it after the section this fact is in. Before reading through the proof of this fact you should take a quick look at the Mean Value Theorem section. You really just need the conclusion of the Mean Value Theorem for this proof however.

**Proof of 1**

First since we are assuming that $f''(x)$ is continuous in a region around $x = c$ then we can assume that in fact $f''(c) < 0$ is also true in some open region, say $(a,b)$ around $x = c$, *i.e.* $a < c < b$.

Now let *x* be any number such that $a < x < c$, we're going to use the Mean Value Theorem on $[x,c]$. However, instead of using it on the function itself we're going to use it on the first derivative. So, the Mean Value Theorem tells us that there is a number $x < d < c$ such that,

$$f'(c) - f'(x) = f''(d)(c - x)$$

Now, because $a < x < d < c$ we know that $f''(d) < 0$ and we also know that $c - x > 0$ so we then get that,

$$f'(c) - f'(x) < 0$$

However, we also assumed that $f'(c) = 0$ and so we have,

$$-f'(x) < 0 \qquad \Rightarrow \qquad f'(x) > 0$$

Or, in other words to the left of $x = c$ the function is increasing.

Let's now turn things around and let *x* be any number such that $c < x < b$ and use the Mean Value Theorem on $[c,x]$ and the first derivative. The Mean Value Theorem tells us that there is a number $c < d < x$ such that,

$$f'(x) - f'(c) = f''(d)(x - c)$$

Now, because $c < d < x < b$ we know that $f''(d) < 0$ and we also know that $x - c > 0$ so we then get that,

$$f'(x) - f'(c) < 0$$

Again use the fact that we also assumed that $f'(c) = 0$ to get,

$$f'(x) < 0$$

We now know that to the right of $x = c$ the function is decreasing.

So, to the left of $x = c$ the function is increasing and to the right of $x = c$ the function is decreasing so by the first derivative test this means that $x = c$ must be a relative maximum.

■

**Proof of 2**

This proof is nearly identical to the proof of 1 and since that proof is somewhat long we're going to leave the proof to you to do. In this case the only difference is that now we are going to assume that $f''(x) < 0$ and that will give us the opposite signs of the first derivative on either side of $x = c$ which gives us the conclusion we were after. We'll leave it to you to fill in all the details of this.

■

**Proof of 3**

There isn't really anything to prove here. All this statement says is that any of the three cases are possible and to "prove" this all one needs to do is provide an example of each of the three cases. This was done in The Shape of a Graph, Part II section where this test was presented so we'll leave it to you to go back to that section to see those graphs to verify that all three possibilities really can happen.

■

**Rolle's Theorem**

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a,b]$.
2. $f(x)$ is differentiable on the open interval $(a,b)$.
3. $f(a) = f(b)$

Then there is a number *c* such that $a < c < b$ and $f'(c) = 0$. Or, in other words $f(x)$ has a critical point in $(a,b)$.

**Proof**

We'll need to do this with 3 cases.

**Case 1 :** $f(x) = k$ on $[a,b]$ where *k* is a constant.

In this case $f'(x) = 0$ for all *x* in $[a,b]$ and so we can take *c* to be any number in $[a,b]$.

**Case 2 :** There is some number $d$ in ($a$,$b$) such that $f(d) > f(a)$.

Because $f(x)$ is continuous on [$a$,$b$] by the Extreme Value Theorem we know that $f(x)$ will have a maximum somewhere in [$a$,$b$]. Also, because $f(a) = f(b)$ and $f(d) > f(a)$ we know that in fact the maximum value will have to occur at some $c$ that is in the open interval ($a$,$b$), or $a < c < b$. Because $c$ occurs in the interior of the interval this means that $f(x)$ will actually have a relative maximum at $x = c$ and by the second hypothesis above we also know that $f'(c)$ exists. Finally, by Fermat's Theorem we then know that in fact $x = c$ must be a critical point and because we know that $f'(c)$ exists we must have $f'(c) = 0$ (as opposed to $f'(c)$ not existing…).

**Case 3 :** There is some number $d$ in ($a$,$b$) such that $f(d) < f(a)$.

This is nearly identical to Case 2 so we won't put in quite as much detail. By the Extreme Value Theorem $f(x)$ will have minimum in [$a$,$b$] and because $f(a) = f(b)$ and $f(d) < f(a)$ we know that the minimum must occur at $x = c$ where $a < c < b$. Finally, by Fermat's Theorem we know that $f'(c) = 0$.

■

**The Mean Value Theorem**

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval [$a$,$b$].
2. $f(x)$ is differentiable on the open interval ($a$,$b$).

Then there is a number $c$ such that $a < c < b$ and

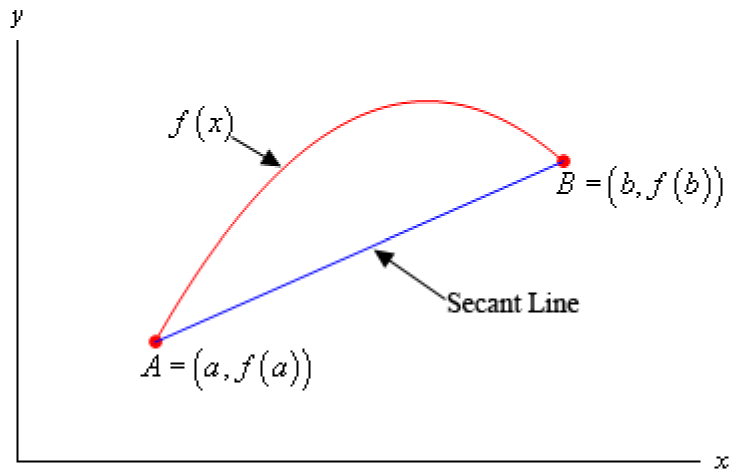
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

**Proof**

For illustration purposes let's suppose that the graph of $f(x)$ is,




Note of course that it may not look like this, but we just need a quick sketch to make it easier to see what we're talking about here.

The first thing that we need is the equation of the secant line that goes through the two points *A* and *B* as shown above. This is,

$$y = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$$

Let's now define a new function, $g(x)$, as to be the difference between $f(x)$ and the equation of the secant line or,

$$g(x) = f(x) - \left( f(a) + \frac{f(b) - f(a)}{b-a}(x-a) \right) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a}(x-a)$$

Next, let's notice that because $g(x)$ is the sum of $f(x)$, which is assumed to be continuous on [*a*,*b*], and a linear polynomial, which we know to be continuous everywhere, we know that $g(x)$ must also be continuous on [*a*,*b*].

Also, we can see that $g(x)$ must be differentiable on (*a*,*b*) because it is the sum of $f(x)$, which is assumed to be differentiable on (*a*,*b*), and a linear polynomial, which we know to be differentiable.

We could also have just computed the derivative as follows,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

at which point we can see that it exists on ($a$,$b$) because we assumed that $f'(x)$ exists on ($a$,$b$) and the last term is just a constant.

Finally, we have,

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b-a}(a-a) = f(a) - f(a) = 0$$

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b-a}(b-a) = f(b) - f(a) - \left(f(b) - f(a)\right) = 0$$

In other words, $g(x)$ satisfies the three conditions of Rolle's Theorem and so we know that there must be a number $c$ such that $a < c < b$ and that,

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} \qquad \Rightarrow \qquad f'(c) = \frac{f(b) - f(a)}{b-a}$$

■

## *Proof of Various Integral Facts/Formulas/Properties*

In this section we've got the proof of several of the properties we saw in the Integrals Chapter as well as a couple from the Applications of Integrals Chapter.

**Proof of :** $\int k\, f(x)\,dx = k\int f(x)\,dx$ where *k* is any number.

This is a very simple proof. Suppose that $F(x)$ is an anti-derivative of $f(x)$, *i.e.* $F'(x) = f(x)$. Then by the basic properties of derivatives we also have that,

$$\left(k\,F(x)\right)' = k\,F'(x) = k\,f(x)$$

and so $k\,F(x)$ is an anti-derivative of $k\,f(x)$, *i.e.* $\left(k\,F(x)\right)' = k\,f(x)$. In other words,

$$\int k\,f(x)\,dx = k\,F(x) + c = k\int f(x)\,dx$$

■

**Proof of :** $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

This is also a very simple proof Suppose that $F(x)$ is an anti-derivative of $f(x)$ and that $G(x)$ is an anti-derivative of $g(x)$. So we have that $F'(x) = f(x)$ and $G'(x) = g(x)$. Basic properties of derivatives we also tell us that

$$\left(F(x) \pm G(x)\right)' = F'(x) \pm G'(x) = f(x) \pm g(x)$$

and so $F(x)+G(x)$ is an anti-derivative of $f(x)+g(x)$ and $F(x)-G(x)$ is an anti-derivative of $f(x)-g(x)$. In other words,

$$\int f(x) \pm g(x)\,dx = F(x) \pm G(x) + c = \int f(x)\,dx \pm \int g(x)\,dx$$

■

**Proof of :** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

From the definition of the definite integral we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

and we also have,

$$\int_b^a f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{a-b}{n}$$

Therefore,

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{-(a-b)}{n}\\ &= \lim_{n\to\infty}\left(-\sum_{i=1}^{n} f(x_i^*)\frac{a-b}{n}\right)\\ &= -\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{a-b}{n} = -\int_b^a f(x)\,dx\end{aligned}$$

■

**Proof of :** $\int_a^a f(x)\,dx = 0$

From the definition of the definite integral we have,

$$\begin{aligned}\int_a^a f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{a-a}{n} = 0\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)(0)\\ &= \lim_{n\to\infty} 0\\ &= 0\end{aligned}$$

■

**Proof of :** $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

From the definition of the definite integral we have,

$$\begin{aligned}\int_a^b c\,f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} c\,f(x_i^*)\Delta x \\ &= \lim_{n\to\infty} c\sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= c\int_a^b f(x)\,dx\end{aligned}$$

Remember that we can pull constants out of summations and out of limits.

■

**Proof of :** $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

First we'll prove the formula for "+". From the definition of the definite integral we have,

$$\begin{aligned}\int_a^b f(x)+g(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(f(x_i^*)+g(x_i^*)\right)\Delta x \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x + \sum_{i=1}^{n} g(x_i^*)\Delta x\right) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x + \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i^*)\Delta x \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

To prove the formula for "-" we can either redo the above work with a minus sign instead of a plus sign or we can use the fact that we now know this is true with a plus and using the properties proved above as follows.

$$\begin{aligned}\int_a^b f(x)-g(x)\,dx &= \int_a^b f(x)+\left(-g(x)\right)dx \\ &= \int_a^b f(x)\,dx + \int_a^b \left(-g(x)\right)dx \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\end{aligned}$$

■

**Proof of :** $\int_a^b c\,dx = c(b-a)$, $c$ is any number.

If we define $f(x) = c$ then from the definition of the definite integral we have,

$$\begin{aligned}\int_a^b c\,dx &= \int_a^b f(x)\,dx \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad\qquad \Delta x = \frac{b-a}{n} \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} c\right)\frac{b-a}{n} \\ &= \lim_{n\to\infty}(cn)\frac{b-a}{n} \\ &= \lim_{n\to\infty} c(b-a) \\ &= c(b-a)\end{aligned}$$

■

**Proof of :** If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$.

From the definition of the definite integral we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad\qquad \Delta x = \frac{b-a}{n}$$

Now, by assumption $f(x) \ge 0$ and we also have $\Delta x > 0$ and so we know that

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \ge 0$$

So, from the basic properties of limits we then have,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \ge \lim_{n\to\infty} 0 = 0$$

But the left side is exactly the definition of the integral and so we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \ge 0$$

■

**Proof of :** If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Since we have $f(x) \ge g(x)$ then we know that $f(x) - g(x) \ge 0$ on $a \le x \le b$ and so by Property 8 proved above we know that,

$$\int_a^b f(x) - g(x)\,dx \ge 0$$

We also know from Property 4 that,

$$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

So, we then have,

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \ge 0$$

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

■

**Proof of :** If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

Give $m \le f(x) \le M$ we can use Property 9 on each inequality to write,

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

Then by Property 7 on the left and right integral to get,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

■

**Proof of :** $\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$

First let's note that we can say the following about the function and the absolute value,

$$-\left|f(x)\right| \le f(x) \le \left|f(x)\right|$$

If we now use Property 9 on each inequality we get,

$$\int_a^b -\left|f(x)\right| dx \le \int_a^b f(x)\,dx \le \int_a^b \left|f(x)\right| dx$$

We know that we can factor the minus sign out of the left integral to get,

$$-\int_a^b \left|f(x)\right| dx \le \int_a^b f(x)\,dx \le \int_a^b \left|f(x)\right| dx$$

Finally, recall that if $\left|p\right| \le b$ then $-b \le p \le b$ and of course this works in reverse as well so we then must have,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right| dx$$

■

**Fundamental Theorem of Calculus, Part I**

If $f(x)$ is continuous on [*a*,*b*] then,

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on [*a*,*b*] and it is differentiable on $(a,b)$ and that,

$$g'(x) = f(x)$$

**Proof**

Suppose that *x* and $x+h$ are in $(a,b)$. We then have,

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

Now, using Property 5 of the Integral Properties we can rewrite the first integral and then do a little simplification as follows.

$$\begin{aligned} g(x+h) - g(x) &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

Finally assume that $h \ne 0$ and we get,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \qquad (1)$$

Let's now assume that $h > 0$ and since we are still assuming that $x + h$ are in $(a,b)$ we know that $f(x)$ is continuous on $[x, x+h]$ and so be the Extreme Value Theorem we know that there are numbers *c* and *d* in $[x, x+h]$ so that $f(c) = m$ is the absolute minimum of $f(x)$ in $[x, x+h]$ and that $f(d) = M$ is the absolute maximum of $f(x)$ in $[x, x+h]$.

So, by Property 10 of the Integral Properties we then know that we have,

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

Or,

$$f(c)h \le \int_x^{x+h} f(t)\,dt \le f(d)h$$

Now divide both sides of this by h to get,

$$f(c) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(d)$$

and then use (1) to get,

$$f(c) \le \frac{g(x+h) - g(x)}{h} \le f(d) \qquad (2)$$

Next, if $h < 0$ we can go through the same argument above except we'll be working on $[x+h, x]$ to arrive at exactly the same inequality above. In other words, (2) is true provided $h \ne 0$.

Now, if we take $h \to 0$ we also have $c \to x$ and $d \to x$ because both *c* and *d* are between *x* and $x + h$. This means that we have the following two limits.

$$\lim_{h \to 0} f(c) = \lim_{c \to x} f(c) = f(x) \qquad\qquad \lim_{h \to 0} f(d) = \lim_{d \to x} f(d) = f(x)$$

The Squeeze Theorem then tells us that,

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x) \qquad (3)$$

but the left side of this is exactly the definition of the derivative of $g(x)$ and so we get that,

$$g'(x) = f(x)$$

So, we've shown that $g(x)$ is differentiable on $(a,b)$.

Now, the Theorem at the end of the Definition of the Derivative section tells us that $g(x)$ is also continuous on $(a,b)$. Finally, if we take $x=a$ or $x=b$ we can go through a similar argument we used to get (3) using one-sided limits to get the same result and so the theorem at the end of the Definition of the Derivative section will also tell us that $g(x)$ is continuous at $x=a$ or $x=b$ and so in fact $g(x)$ is also continuous on $[a,b]$.

■

**Fundamental Theorem of Calculus, Part II**

Suppose $f(x)$ is a continuous function on [$a$,$b$] and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$$

**Proof**

First let $g(x)=\int_a^x f(t)\,dt$ and then we know from Part I of the Fundamental Theorem of Calculus that $g'(x)=f(x)$ and so $g(x)$ is an anti-derivative of $f(x)$ on [$a$,$b$]. Further suppose that $F(x)$ is any anti-derivative of $f(x)$ on [$a$,$b$] that we want to chose. So, this means that we must have,

$$g'(x)=F'(x)$$

Then, by Fact 2 in the Mean Value Theorem section we know that $g(x)$ and $F(x)$ can differ by no more than an additive constant on $(a,b)$. In other words for $a<x<b$ we have,

$$F(x)=g(x)+c$$

Now because $g(x)$ and $F(x)$ are continuous on [$a$,$b$], if we take the limit of this as $x\to a^+$ and $x\to b^-$ we can see that this also holds if $x=a$ and $x=b$.

So, for $a\le x\le b$ we know that $F(x)=g(x)+c$. Let's use this and the definition of $g(x)$ to do the following.

$$\begin{aligned} F(b)-F(a) &= \left(g(b)+c\right)-\left(g(a)+c\right) \\ &= g(b)-g(a) \\ &= \int_a^b f(t)\,dt + \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt + 0 \\ &= \int_a^b f(x)\,dx \end{aligned}$$

Note that in the last step we used the fact that the variable used in the integral does not matter and so we could change the *t*'s to *x*'s.

■

**Average Function Value**

The average value of a function $f(x)$ over the interval [*a*,*b*] is given by,

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

**Proof**

We know that the average value of *n* numbers is simply the sum of all the numbers divided by *n* so let's start off with that. Let's take the interval [*a*,*b*] and divide it into *n* subintervals each of length,

$$\Delta x = \frac{b-a}{n}$$

Now from each of these intervals choose the points $x_1^*, x_2^*, \ldots, x_n^*$ and note that it doesn't really matter how we choose each of these numbers as long as they come from the appropriate interval. We can then compute the average of the function values $f\left(x_1^*\right), f\left(x_2^*\right), \ldots, f\left(x_n^*\right)$ by computing,

$$\frac{f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)}{n} \qquad (4)$$

Now, from our definition of $\Delta x$ we can get the following formula for *n*.

$$n = \frac{b-a}{\Delta x}$$

and we can plug this into (4) to get,

$$\frac{f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)}{\frac{b-a}{\Delta x}}=\frac{\left[f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)\right]\Delta x}{b-a}$$

$$=\frac{1}{b-a}\left[f\left(x_1^*\right)\Delta x+f\left(x_2^*\right)\Delta x+\cdots+f\left(x_n^*\right)\Delta x\right]$$

$$=\frac{1}{b-a}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

Let's now increase *n*. Doing this will mean that we're taking the average of more and more function values in the interval and so the larger we chose *n* the better this will approximate the average value of the function.

If we then take the limit as *n* goes to infinity we should get the average function value. Or,

$$f_{avg}=\lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x=\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

We can factor the $\frac{1}{b-a}$ out of the limit as we've done and now the limit of the sum should look familiar as that is the definition of the definite integral. So, putting in definite integral we get the formula that we were after.

$$f_{avg}=\frac{1}{b-a}\int_a^b f\left(x\right)dx$$

■

## The Mean Value Theorem for Integrals

If $f\left(x\right)$ is a continuous function on [*a*,*b*] then there is a number *c* in [*a*,*b*] such that,

$$\int_a^b f\left(x\right)dx=f\left(c\right)\left(b-a\right)$$

**Proof**

Let's start off by defining,

$$F\left(x\right)=\int_a^x f\left(t\right)dt$$

Since $f\left(x\right)$ is continuous we know from the Fundamental Theorem of Calculus, Part I that $F\left(x\right)$ is continuous on [*a*,*b*], differentiable on (*a*,*b*) and that $F'\left(x\right)=f\left(x\right)$.

Now, from the Mean Value Theorem we know that there is a number *c* such that $a < c < b$ and that,

$$F(b) - F(a) = F'(c)(b - a)$$

However we know that $F'(c) = f(c)$ and,

$$F(b) = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx \qquad\qquad F(a) = \int_a^a f(t)\,dt = 0$$

So, we then have,

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

■

**Work**

The work done by the force $F(x)$ (assuming that $F(x)$ is continuous) over the range $a \le x \le b$ is,

$$W = \int_a^b F(x)\,dx$$

**Proof**

Let's start off by dividing the range $a \le x \le b$ into *n* subintervals of width $\Delta x$ and from each of these intervals choose the points $x_1^*, x_2^*, \ldots, x_n^*$.

Now, if *n* is large and because $F(x)$ is continuous we can assume that $F(x)$ won't vary by much over each interval and so in the $i^{\text{th}}$ interval we can assume that the force is approximately constant with a value of $F(x) \approx F(x_i^*)$. The work on each interval is then approximately,

$$W_i \approx F(x_i^*)\Delta x$$

The total work over $a \le x \le b$ is then approximately,

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=0}^{n} F(x_i^*)\Delta x$$

Finally, if we take the limit of this as *n* goes to infinity we'll get the exact work done. So,

$$W = \lim_{n\to\infty} \sum_{i=0}^{n} F(x_i^*)\Delta x$$

This is, however, nothing more than the definition of the definite integral and so the work done by the force $F\left(x\right)$ over $a \le x \le b$ is,

$$W = \int_{a}^{b} F\left(x\right) dx$$

■

## *Area and Volume Formulas*

In this section we will derive the formulas used to get the area between two curves and the volume of a solid of revolution.

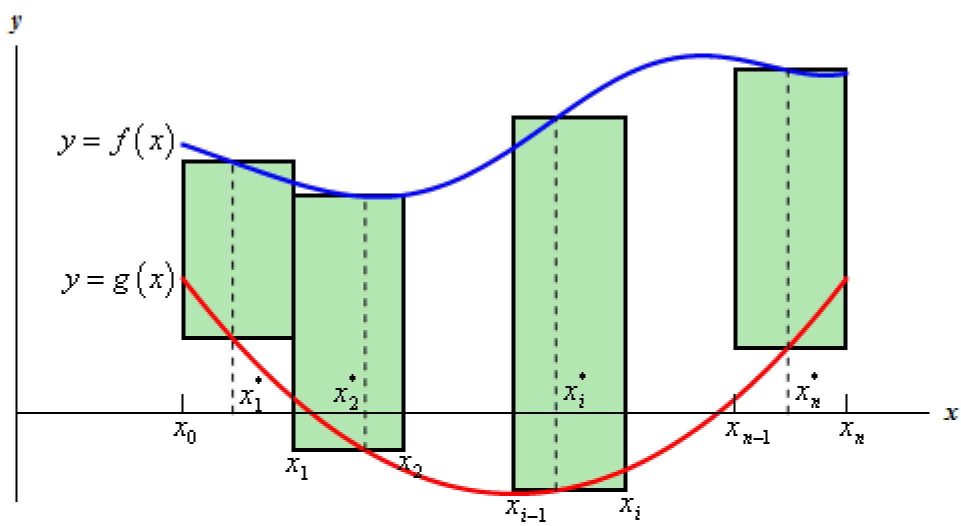
**Area Between Two Curves**

We will start with the formula for determining the area between $y = f(x)$ and $y = g(x)$ on the interval [*a*,*b*]. We will also assume that $f(x) \geq g(x)$ on [*a*,*b*].

We will now proceed much as we did when we looked that the Area Problem in the Integrals Chapter. We will first divide up the interval into *n* equal subintervals each with length,

$$\Delta x = \frac{b-a}{n}$$

Next, pick a point in each subinterval, $x_i^*$, and we can then use rectangles on each interval as follows.




The height of each of these rectangles is given by,

$$f(x_i^*) - g(x_i^*)$$

and the area of each rectangle is then,

$$\left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

So, the area between the two curves is then approximated by,

$$A \approx \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

The exact area is,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$