Area and Volume Formulas

In this section we will derive the formulas used to get the area between two curves and the volume of a solid of revolution.

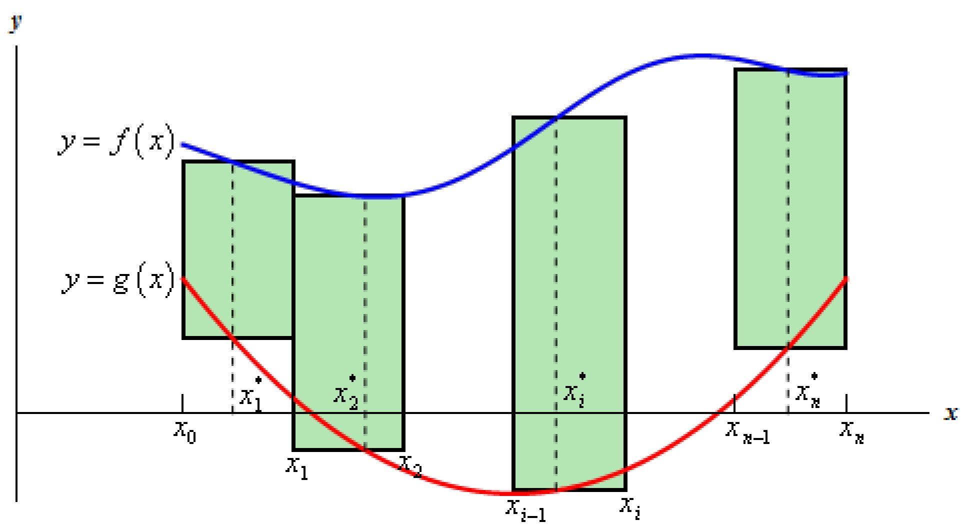
Area Between Two Curves

We will start with the formula for determining the area between $y = f(x)$ and $y = g(x)$ on the interval [a,b]. We will also assume that $f(x) \ge g(x)$ on [a,b].

We will now proceed much as we did when we looked that the Area Problem in the Integrals Chapter. We will first divide up the interval into n equal subintervals each with length,

$$\Delta x = \frac{b-a}{n}$$

Next, pick a point in each subinterval, x_i^*, and we can then use rectangles on each interval as follows.



The height of each of these rectangles is given by,

$$f(x_i^*) - g(x_i^*)$$

and the area of each rectangle is then,

$$\left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

So, the area between the two curves is then approximated by,

$$A \approx \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

The exact area is,

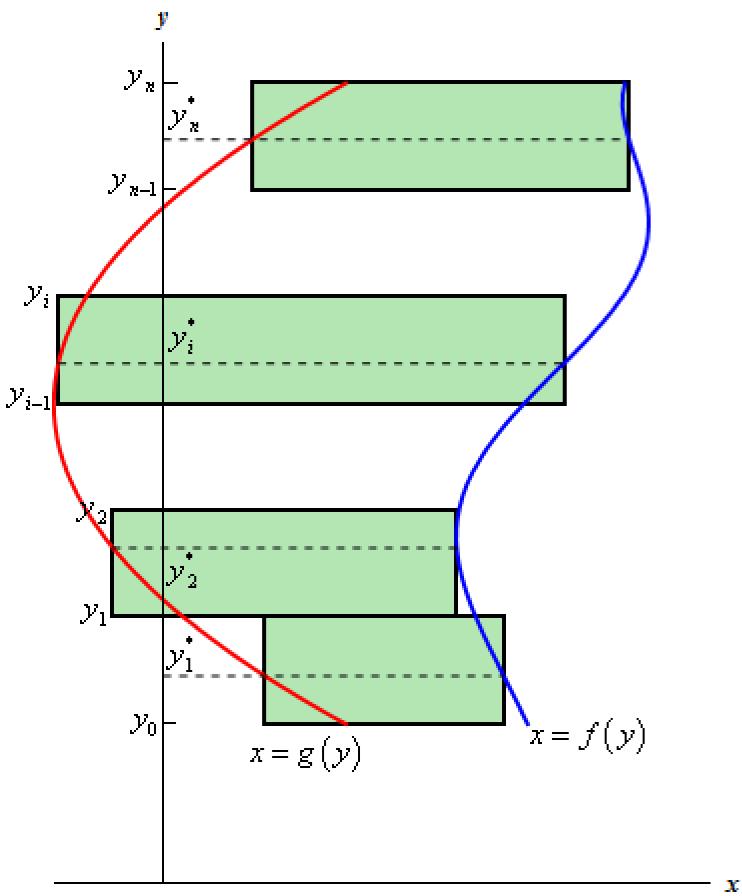
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

Now, recalling the definition of the definite integral this is nothing more than,

$$A = \int_a^b f(x) - g(x)\,dx$$

The formula above will work provided the two functions are in the form $y = f(x)$ and $y = g(x)$. However, not all functions are in that form. Sometimes we will be forced to work with functions in the form between $x = f(y)$ and $x = g(y)$ on the interval [c,d] (an interval of y values…).

When this happens the derivation is identical. First we will start by assuming that $f(y) \geq g(y)$ on [c,d]. We can then divide up the interval into equal subintervals and build rectangles on each of these intervals. Here is a sketch of this situation.



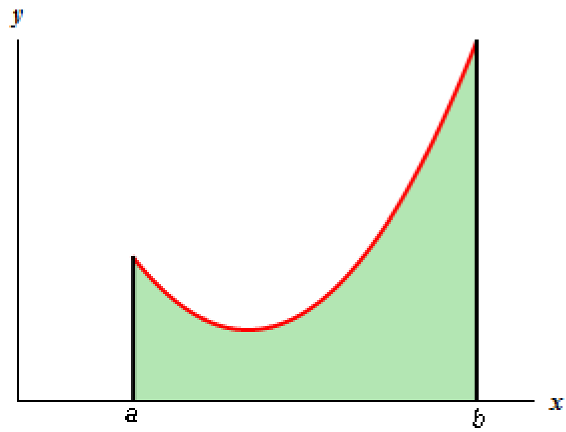
Following the work from above, we will arrive at the following for the area,

$$A = \int_c^d f(y) - g(y)\,dy$$

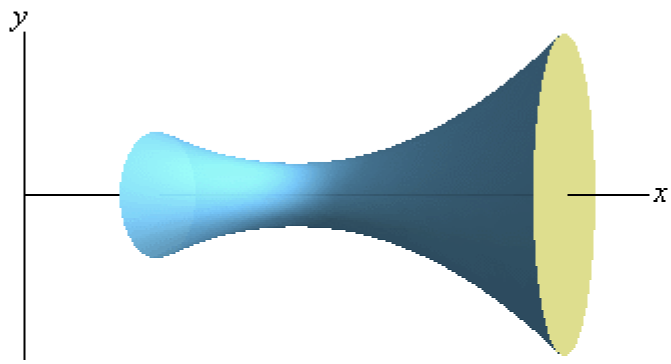
So, regardless of the form that the functions are in we use basically the same formula.

Volumes for Solid of Revolution

Before deriving the formula for this we should probably first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval [*a*,*b*].



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this derivation let's rotate the curve about the *x*-axis. Doing this gives the following three dimensional region.



We want to determine the volume of the interior of this object. To do this we will proceed much as we did for the area between two curves case. We will first divide up the interval into *n* subintervals of width,

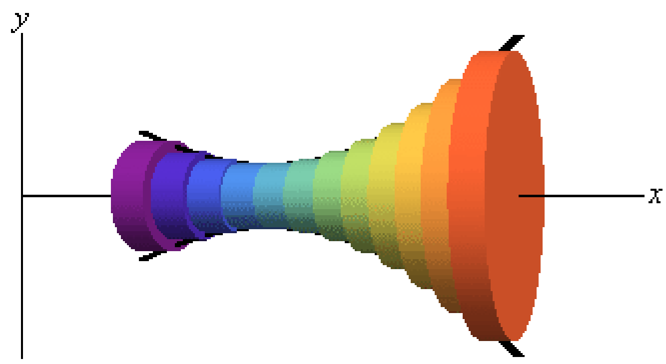
$$\Delta x = \frac{b-a}{n}$$

We will then choose a point from each subinterval, x_i^*.

Now, in the area between two curves case we approximated the area using rectangles on each subinterval. For volumes we will use disks on each subinterval to approximate the area. The area of the face of each disk is given by $A(x_i^*)$ and the volume of each disk is

$$V_i = A(x_i^*)\Delta x$$

Here is a sketch of this,



The volume of the region can then be approximated by,

$$V \approx \sum_{i=1}^{n} A\left(x_i^*\right)\Delta x$$

The exact volume is then,

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n} A\left(x_i^*\right)\Delta x \\ &= \int_a^b A\left(x\right)dx \end{aligned}$$

So, in this case the volume will be the integral of the cross-sectional area at any *x*, $A\left(x\right)$. Note as well that, in this case, the cross-sectional area is a circle and we could go farther and get a formula for that as well. However, the formula above is more general and will work for any way of getting a cross section so we will leave it like it is.

In the sections where we actually use this formula we will also see that there are ways of generating the cross section that will actually give a cross-sectional area that is a function of *y* instead of *x*. In these cases the formula will be,

$$V = \int_c^d A\left(y\right)dy, \qquad c \le y \le d$$

In this case we looked at rotating a curve about the *x*-axis, however, we could have just as easily rotated the curve about the *y*-axis. In fact we could rotate the curve about any vertical or horizontal axis and in all of these, case we can use one or both of the following formulas.

$$V = \int_a^b A\left(x\right)dx \qquad\qquad V = \int_c^d A\left(y\right)dy$$

Types of Infinity

Most students have run across infinity at some point in time prior to a calculus class. However, when they have dealt with it, it was just a symbol used to represent a really, really large positive or really, really large negative number and that was the extent of it. Once they get into a calculus class students are asked to do some basic algebra with infinity and this is where they get into trouble. Infinity is NOT a number and for the most part doesn't behave like a number. However, despite that we'll think of infinity in this section as a really, really, really large number that is so large there isn't another number larger than it. This is not correct of course, but may help with the discussion in this section. Note as well that everything that we'll be discussing in this section applies only to real numbers. If you move into complex numbers for instance things can and do change.

So, let's start thinking about addition with infinity. When you add two non-zero numbers you get a new number. For example, $4+7=11$. With infinity this is not true. With infinity you have the following.

$$\infty + a = \infty \qquad \text{where } a \neq -\infty$$
$$\infty + \infty = \infty$$

In other words, a really, really large positive number (∞) plus any positive number, regardless of the size, is still a really, really large positive number. Likewise, you can add a negative number (*i.e.* $a<0$) to a really, really large positive number and stay really, really large and positive. So, addition involving infinity can be dealt with in an intuitive way if you're careful. Note as well that the *a* must NOT be negative infinity. If it is, there are some serious issues that we need to deal with as we'll see in a bit.

Subtraction with negative infinity can also be dealt with in an intuitive way in most cases as well. A really, really large negative number minus any positive number, regardless of its size, is still a really, really large negative number. Subtracting a negative number (*i.e.* $a<0$) from a really, really large negative number will still be a really, really large negative number. Or,

$$-\infty - a = -\infty \qquad \text{where } a \neq -\infty$$
$$-\infty - \infty = -\infty$$

Again, *a* must not be negative infinity to avoid some potentially serious difficulties.

Multiplication can be dealt with fairly intuitively as well. A really, really large number (positive, or negative) times any number, regardless of size, is still a really, really large number we'll just need to be careful with signs. In the case of multiplication we have

$$(a)(\infty) = \infty \quad \text{if } a > 0 \qquad\qquad (a)(\infty) = -\infty \quad \text{if } a < 0$$
$$(\infty)(\infty) = \infty \qquad\qquad (-\infty)(-\infty) = \infty \qquad\qquad (-\infty)(\infty) = -\infty$$

What you know about products of positive and negative numbers is still true here.

Some forms of division can be dealt with intuitively as well. A really, really large number divided by a number that isn't too large is still a really, really large number.

$$\frac{\infty}{a} = \infty \qquad \text{if } a > 0, a \neq \infty \qquad\qquad \frac{\infty}{a} = -\infty \qquad \text{if } a < 0, a \neq -\infty$$
$$\frac{-\infty}{a} = -\infty \qquad \text{if } a > 0, a \neq \infty \qquad\qquad \frac{-\infty}{a} = \infty \qquad \text{if } a < 0, a \neq -\infty$$

Division of a number by infinity is somewhat intuitive, but there are a couple of subtleties that you need to be aware of. When we talk about division by infinity we are really talking about a limiting process in which the denominator is going towards infinity. So, a number that isn't too large divided an increasingly large number is an increasingly small number. In other words in the limit we have,

$$\frac{a}{\infty} = 0 \qquad\qquad \frac{a}{-\infty} = 0$$

So, we've dealt with almost every basic algebraic operation involving infinity. There are two cases that that we haven't dealt with yet. These are

$$\infty - \infty = ? \qquad\qquad \frac{\pm\infty}{\pm\infty} = ?$$

The problem with these two cases is that intuition doesn't really help here. A really, really large number minus a really, really large number can be anything ($-\infty$, a constant, or ∞). Likewise, a really, really large number divided by a really, really large number can also be anything ($\pm\infty$ – this depends on sign issues, 0, or a non-zero constant).

What we've got to remember here is that there are really, really large numbers and then there are really, really, really large numbers. In other words, some infinities are larger than other infinities. With addition, multiplication and the first sets of division we worked this wasn't an issue. The general size of the infinity just doesn't affect the answer in those cases. However, with the subtraction and division cases listed above, it does matter as we will see.

Here is one way to think of this idea that some infinities are larger than others. This is a fairly dry and technical way to think of this and your calculus problems will probably never use this stuff, but this it is a nice way of looking at this. Also, please note that I'm not trying to give a precise proof of anything here. I'm just trying to give you a little insight into the problems with infinity and how some infinities can be thought of as larger than others. For a much better (and definitely more precise) discussion see,

http://www.math.vanderbilt.edu/~schectex/courses/infinity.pdf

Let's start by looking at how many integers there are. Clearly, I hope, there are an infinite number of them, but let's try to get a better grasp on the "size" of this infinity. So, pick any two integers completely at random. Start at the smaller of the two and list, in increasing order, all the integers that come after that. Eventually we will reach the larger of the two integers that you picked.

Depending on the relative size of the two integers it might take a very, very long time to list all the integers between them and there isn't really a purpose to doing it. But, it could be done if we wanted to and that's the important part.

Because we could list all these integers between two randomly chosen integers we say that the integers are *countably infinite*. Again, there is no real reason to actually do this, it is simply something that can be done if we should chose to do so.

In general a set of numbers is called countably infinite if we can find a way to list them all out. In a more precise mathematical setting this is generally done with a special kind of function called a bijection that associates each number in the set with exactly one of the positive integers. To see some more details of this see the pdf given above.

It can also be shown that the set of all fractions are also countably infinite, although this is a little harder to show and is not really the purpose of this discussion. To see a proof of this see the pdf given above. It has a very nice proof of this fact.

Let's contrast this by trying to figure out how many numbers there are in the interval (0,1). By numbers, I mean all possible fractions that lie between zero and one as well as all possible decimals (that aren't fractions) that lie between zero and one. The following is similar to the proof given in the pdf above, but was nice enough and easy enough (I hope) that I wanted to include it here.

To start let's assume that all the numbers in the interval (0,1) are countably infinite. This means that there should be a way to list all of them out. We could have something like the following,

$$\begin{aligned} x_1 &= 0.692096\cdots \\ x_2 &= 0.171034\cdots \\ x_3 &= 0.993671\cdots \\ x_4 &= 0.045908\cdots \\ \vdots & \qquad \vdots \end{aligned}$$

Now, select the i^{th} decimal out of x_i as shown below

$$\begin{aligned} x_1 &= 0.\underline{6}92096\cdots \\ x_2 &= 0.1\underline{7}1034\cdots \\ x_3 &= 0.99\underline{3}671\cdots \\ x_4 &= 0.045\underline{9}08\cdots \\ \vdots & \qquad \vdots \end{aligned}$$

and form a new number with these digits. So, for our example we would have the number

$$x = 0.6739\cdots$$

In this new decimal replace all the 3's with a 1 and replace every other numbers with a 3. In the case of our example this would yield the new number

$$\bar{x} = 0.3313\cdots$$

Notice that this number is in the interval (0,1) and also notice that given how we choose the digits of the number this number will not be equal to the first number in our list, x_1, because the first digit of each is guaranteed to not be the same. Likewise, this new number will not get the same number as the second in our list, x_2, because the second digit of each is guaranteed to not be the same. Continuing in this manner we can see that this new number we constructed, $\bar{x}$, is guaranteed to not be in our listing. But this contradicts the initial assumption that we could list out all the numbers in the interval (0,1). Hence, it must not be possible to list out all the numbers in the interval (0,1).

Sets of numbers, such as all the numbers in (0,1), that we can't write down in a list are called *uncountably* infinite.

The reason for going over this is the following. An infinity that is uncountably infinite is significantly larger than an infinity that is only countably infinite. So, if we take the difference of two infinities we have a couple of possibilities.

$$\infty\left(\text{uncountable}\right) - \infty\left(\text{countable}\right) = \infty$$
$$\infty\left(\text{countable}\right) - \infty\left(\text{uncountable}\right) = -\infty$$
$$\infty\left(\text{countable}\right) - \infty\left(\text{countable}\right) = \text{ a constant}$$

Notice that we didn't put down a difference of two uncountable infinities. There is still have some ambiguity about just what the answer would be in this case, but that is a whole different topic.

We could also do something similar for quotients of infinities.

$$\frac{\infty\left(\text{countable}\right)}{\infty\left(\text{uncountable}\right)} = 0$$
$$\frac{\infty\left(\text{uncountable}\right)}{\infty\left(\text{countable}\right)} = \infty$$
$$\frac{\infty\left(\text{countable}\right)}{\infty\left(\text{countable}\right)} = \text{ a constant}$$

Again, we avoided a quotient of two uncountable infinities since there will still be ambiguities about its value.

So, that' it and hopefully you've learned something from this discussion. Infinity simply isn't a number and because there are different kinds of infinity it generally doesn't behave as a number does. Be careful when dealing with infinity.

Summation Notation

In this section we need to do a brief review of summation notation or sigma notation. We'll start out with two integers, *n* and *m*, with $n < m$ and a list of numbers denoted as follows,

$$a_n,\ a_{n+1},\ a_{n+2},\ \ldots,\ a_{m-2},\ a_{m-1},\ a_m$$

We want to add them up, in other words we want,

$$a_n + a_{n+1} + a_{n+2} + \ldots + a_{m-2} + a_{m-1} + a_m$$

For large lists this can be a fairly cumbersome notation so we introduce summation notation to denote these kinds of sums. The case above is denoted as follows.

$$\sum_{i=n}^{m} a_i = a_n + a_{n+1} + a_{n+2} + \ldots + a_{m-2} + a_{m-1} + a_m$$

The *i* is called the index of summation. This notation tells us to add all the a_i's up for all integers starting at *n* and ending at *m*.

For instance,

$$\sum_{i=0}^{4} \frac{i}{i+1} = \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} = \frac{163}{60} = 2.7166\overline{6}$$

$$\sum_{i=4}^{6} 2^i x^{2i+1} = 2^4 x^9 + 2^5 x^{11} + 2^6 x^{13} = 16x^9 + 32x^{11} + 64x^{13}$$

$$\sum_{i=1}^{4} f\left(x_i^*\right) = f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + f\left(x_4^*\right)$$

Properties

Here are a couple of formulas for summation notation.

1. $\displaystyle\sum_{i=i_0}^{n} c a_i = c \sum_{i=i_0}^{n} a_i$ where *c* is any number. So, we can factor constants out of a summation.

2. $\displaystyle\sum_{i=i_0}^{n} \left(a_i \pm b_i\right) = \sum_{i=i_0}^{n} a_i \pm \sum_{i=i_0}^{n} b_i$ So we can break up a summation across a sum or difference.

Note that we started the series at i_0 to denote the fact that they can start at any value of i that we need them to. Also note that while we can break up sums and differences as we did in **2** above we can't do the same thing for products and quotients. In other words,

$$\sum_{i=i_0}^{n} \left(a_i b_i\right) \neq \left(\sum_{i=i_0}^{n} a_i\right)\left(\sum_{i=i_0}^{n} b_i\right) \qquad\qquad \sum_{i=i_0}^{n} \frac{a_i}{b_i} \neq \frac{\displaystyle\sum_{i=i_0}^{n} a_i}{\displaystyle\sum_{i=i_0}^{n} b_i}$$

Formulas

Here are a couple of nice formulas that we will find useful in a couple of sections. Note that these formulas are only true if starting at $i = 1$. You can, of course, derive other formulas from these for different starting points if you need to.

1. $\sum_{i=1}^{n} c = cn$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Here is a quick example on how to use these properties to quickly evaluate a sum that would not be easy to do by hand.

Example 1 Using the formulas and properties from above determine the value of the following summation.

$$\sum_{i=1}^{100} (3-2i)^2$$

Solution

The first thing that we need to do is square out the stuff being summed and then break up the summation using the properties as follows,

$$\begin{aligned}\sum_{i=1}^{100} (3-2i)^2 &= \sum_{i=1}^{100} 9 - 12i + 4i^2 \\ &= \sum_{i=1}^{100} 9 - \sum_{i=1}^{100} 12i + \sum_{i=1}^{100} 4i^2 \\ &= \sum_{i=1}^{100} 9 - 12\sum_{i=1}^{100} i + 4\sum_{i=1}^{100} i^2\end{aligned}$$

Now, using the formulas, this is easy to compute,

$$\begin{aligned}\sum_{i=1}^{100} (3-2i)^2 &= 9(100) - 12\left(\frac{100(101)}{2}\right) + 4\left(\frac{100(101)(201)}{6}\right) \\ &= 1293700\end{aligned}$$

Doing this by hand would definitely taken some time and there’s a good chance that we might have made a minor mistake somewhere along the line.

Constants of Integration

In this section we need to address a couple of topics about the constant of integration. Throughout most calculus classes we play pretty fast and loose with it and because of that many students don't really understand it or how it can be important.

First, let's address how we play fast and loose with it. Recall that technically when we integrate a sum or difference we are actually doing multiple integrals. For instance,

$$\int 15x^4 - 9x^{-2}\,dx = \int 15x^4\,dx - \int 9x^{-2}\,dx$$

Upon evaluating each of these integrals we should get a constant of integration for each integral since we really are doing two integrals.

$$\begin{aligned}\int 15x^4 - 9x^{-2}\,dx &= \int 15x^4\,dx - \int 9x^{-2}\,dx \\ &= 3x^5 + c + 9x^{-1} + k \\ &= 3x^5 + 9x^{-1} + c + k\end{aligned}$$

Since there is no reason to think that the constants of integration will be the same from each integral we use different constants for each integral.

Now, both c and k are unknown constants and so the sum of two unknown constants is just an unknown constant and we acknowledge that by simply writing the sum as a c.

So, the integral is then,

$$\int 15x^4 - 9x^{-2}\,dx = 3x^5 + 9x^{-1} + c$$

We also tend to play fast and loose with constants of integration in some substitution rule problems. Consider the following problem,

$$\int \cos(1+2x) + \sin(1+2x)\,dx = \frac{1}{2}\int \cos u + \sin u\,du \qquad\qquad u = 1 + 2x$$

Technically when we integrate we should get,

$$\int \cos(1+2x) + \sin(1+2x)\,dx = \frac{1}{2}\left(\sin u - \cos u + c\right)$$

Since the whole integral is multiplied by $\frac{1}{2}$, the whole answer, including the constant of integration, should be multiplied by $\frac{1}{2}$. Upon multiplying the $\frac{1}{2}$ through the answer we get,

$$\int \cos(1+2x) + \sin(1+2x)\,dx = \frac{1}{2}\sin u - \frac{1}{2}\cos u + \frac{c}{2}$$

However, since the constant of integration is an unknown constant dividing it by 2 isn't going to change that fact so we tend to just write the fraction as a *c*.

$$\int \cos\left(1+2x\right)+\sin\left(1+2x\right)dx = \frac{1}{2}\sin u - \frac{1}{2}\cos u + c$$

In general, we don't really need to worry about how we've played fast and loose with the constant of integration in either of the two examples above.

The real problem however is that because we play fast and loose with these constants of integration most students don't really have a good grasp oF them and don't understand that there are times where the constants of integration are important and that we need to be careful with them.

To see how a lack of understanding about the constant of integration can cause problems consider the following integral.

$$\int \frac{1}{2x}\,dx$$

This is a really simple integral. However, there are two ways (both simple) to integrate it and that is where the problem arises.

The first integration method is to just break up the fraction and do the integral.

$$\int \frac{1}{2x}\,dx = \int \frac{1}{2}\frac{1}{x}\,dx = \frac{1}{2}\ln\left|x\right| + c$$

The second way is to use the following substitution.

$$u = 2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln\left|u\right| + c = \frac{1}{2}\ln\left|2x\right| + c$$

Can you see the problem? We integrated the same function and got very different answers. This doesn't make any sense. Integrating the same function should give us the same answer. We only used different methods to do the integral and both are perfectly legitimate integration methods. So, how can using different methods produce different answer?

The first thing that we should notice is that because we used a different method for each there is no reason to think that the constant of integration will in fact be the same number and so we really should use different letters for each.

More appropriate answers would be,

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\ln\left|x\right| + c \qquad\qquad \int \frac{1}{2x}\,dx = \frac{1}{2}\ln\left|2x\right| + k$$

Now, let's take another look at the second answer. Using a property of logarithms we can write the answer to the second integral as follows,

$$\begin{aligned}\int \frac{1}{2x}\,dx &= \frac{1}{2}\ln\left|2x\right| + k \\ &= \frac{1}{2}\left(\ln 2 + \ln\left|x\right|\right) + k \\ &= \frac{1}{2}\ln\left|x\right| + \frac{1}{2}\ln 2 + k\end{aligned}$$

Upon doing this we can see that the answers really aren't that different after all. In fact they only differ by a constant and we can even find a relationship between *c* and *k*. It looks like,

$$c = \frac{1}{2}\ln 2 + k$$

So, without a proper understanding of the constant of integration, in particular using different integration techniques on the same integral will likely produce a different constant of integration, we might never figure out why we got "different" answers for the integral.

Note as well that getting answers that differ by a constant doesn't violate any principles of calculus. In fact, we've actually seen a fact that suggested that this might happen. We saw a fact in the Mean Value Theorem section that said that if $f'(x) = g'(x)$ then $f(x) = g(x) + c$. In other words, if two functions have the same derivative then they can differ by no more than a constant.

This is exactly what we've got here. The two functions,

$$f(x) = \frac{1}{2}\ln\left|x\right| \qquad\qquad g(x) = \frac{1}{2}\ln\left|2x\right|$$

have exactly the same derivative,

$$\frac{1}{2x}$$

and as we've shown they really only differ by a constant.

There is another integral that also exhibits this behavior. Consider,

$$\int \sin(x)\cos(x)\,dx$$

There are actually three different methods for doing this integral.

Method 1 :
This method uses a trig formula,

$$\sin(2x) = 2\sin(x)\cos(x)$$

Using this formula (and a quick substitution) the integral becomes,

$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\int \sin(2x)\,dx = -\frac{1}{4}\cos(2x) + c_1$$

Method 2 :
This method uses the substitution,

$$u = \cos(x) \qquad\qquad du = -\sin(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + c_2 = -\frac{1}{2}\cos^2(x) + c_2$$

Method 3 :
Here is another substitution that could be done here as well.

$$u = \sin(x) \qquad\qquad du = \cos(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{1}{2}u^2 + c_3 = \frac{1}{2}\sin^2(x) + c_3$$

So, we've got three different answers each with a different constant of integration. However, according to the fact above these three answers should only differ by a constant since they all have the same derivative.

In fact they do only differ by a constant. We'll need the following trig formulas to prove this.

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad\qquad \cos^2(x) + \sin^2(x) = 1$$

Start with the answer from the first method and use the double angle formula above.

$$-\frac{1}{4}\left(\cos^2(x) - \sin^2(x)\right) + c_1$$

Now, from the second identity above we have,

$$\sin^2(x) = 1 - \cos^2(x)$$

so, plug this in,

$$\begin{aligned} -\frac{1}{4}\left(\cos^2(x) - \left(1 - \cos^2(x)\right)\right) + c_1 &= -\frac{1}{4}\left(2\cos^2(x) - 1\right) + c_1 \\ &= -\frac{1}{2}\cos^2(x) + \frac{1}{4} + c_1 \end{aligned}$$

This is then answer we got from the second method with a slightly different constant. In other words,

$$c_2 = \frac{1}{4} + c_1$$

We can do a similar manipulation to get the answer from the third method. Again, starting with the answer from the first method use the double angle formula and then substitute in for the cosine instead of the sine using,

$$\cos^2(x) = 1 - \sin^2(x)$$

Doing this gives,

$$-\frac{1}{4}\left(\left(1-\sin^2(x)\right)-\sin^2(x)\right)+c_1 = -\frac{1}{4}\left(1-2\sin^2(x)\right)+c_1$$
$$= \frac{1}{2}\sin^2(x) - \frac{1}{4} + c_1$$

which is the answer from the third method with a different constant and again we can relate the two constants by,

$$c_3 = -\frac{1}{4} + c_1$$

So, what have we learned here? Hopefully we've seen that constants of integration are important and we can't forget about them. We often don't work with them in a Calculus I course, yet without a good understanding of them we would be hard pressed to understand how different integration methods and apparently produce different answers.