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Chapter 14

Factorial Experiments (Two or More Factors)

14.1 Introduction

Consider a situation where it is of interest to study the effects of **two factors**, A and B, on some response. For example, in a chemical experiment, we would like to vary simultaneously the reaction pressure and reaction time and study the effect of each on the yield. In a biological experiment, it is of interest to study the effects of drying time and temperature on the amount of solids (percent by weight) left in samples of yeast. As in Chapter 13, the term **factor** is used in a general sense to denote any feature of the experiment such as temperature, time, or pressure that may be varied from trial to trial. We define the **levels** of a factor to be the actual values used in the experiment.

For each of these cases, it is important to determine not only if each of the two factors has an influence on the response, but also if there is a significant interaction between the two factors. As far as terminology is concerned, the experiment described here is a two-factor experiment and the experimental design may be either a completely randomized design, in which the various treatment combinations are assigned randomly to all the experimental units, or a randomized complete block design, in which factor combinations are assigned randomly within blocks. In the case of the yeast example, the various treatment combinations of temperature and drying time would be assigned randomly to the samples of yeast if we were using a completely randomized design.

Many of the concepts studied in Chapter 13 are extended in this chapter to two and three factors. The main thrust of this material is the use of the completely randomized design with a *factorial experiment*. A factorial experiment in two factors involves experimental trials (or a single trial) with all factor combinations. For example, in the temperature-drying-time example with, say, 3 levels of each and $n = 2$ runs at each of the 9 combinations, we have a *two-factor factorial experiment in a completely randomized design*. Neither factor is a blocking factor; we are interested in how each influences percent solids in the samples and whether or not they interact. The biologist would have available 18 physical samples of material which are experimental units. These would then be assigned randomly to the 18 combinations (9 treatment combinations, each duplicated).

Before we launch into analytical details, sums of squares, and so on, it may be of interest for the reader to observe the obvious connection between what we have described and the situation with the one-factor problem. Consider the yeast experiment. Explanation of degrees of freedom aids the reader or the analyst in visualizing the extension. We should initially view the 9 treatment combinations as if they represented one factor with 9 levels (8 degrees of freedom). Thus, an initial look at degrees of freedom gives

Treatment combinations	8
Error	9
Total	17

Main Effects and Interaction

The experiment could be analyzed as described in the above table. However, the F-test for combinations would probably not give the analyst the information he or she desires, namely, that which considers the role of temperature and drying time. Three drying times have 2 associated degrees of freedom; three temperatures have 2 degrees of freedom. The main factors, temperature and drying time, are called **main effects**. The main effects represent 4 of the 8 degrees of freedom for *factor combinations*. The additional 4 degrees of freedom are associated with *interaction* between the two factors. As a result, the analysis involves

Combinations		8
Temperature	2	
Drying time	2	
Interaction	4	
Error		9
Total		17

Recall from Chapter 13 that factors in an analysis of variance may be viewed as fixed or random, depending on the type of inference desired and how the levels were chosen. Here we must consider fixed effects, random effects, and even cases where effects are mixed. Most attention will be directed toward expected mean squares when we advance to these topics. In the following section, we focus on the concept of interaction.

14.2 Interaction in the Two-Factor Experiment

In the randomized block model discussed previously, it was assumed that one observation on each treatment is taken in each block. If the model assumption is correct, that is, if blocks and treatments are the only real effects and interaction does not exist, the expected value of the mean square error is the experimental error variance σ^2. Suppose, however, that there is interaction occurring between treatments and blocks as indicated by the model

$$y_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ij}$$

of Section 13.8. The expected value of the mean square error is then given as

$$E\left[\frac{SSE}{(b-1)(k-1)}\right] = \sigma^2 + \frac{1}{(b-1)(k-1)}\sum_{i=1}^{k}\sum_{j=1}^{b}(\alpha\beta)_{ij}^2.$$

The treatment and block effects do not appear in the expected mean square error, but the interaction effects do. Thus, if there is interaction in the model, the mean square error reflects variation due to experimental error plus an interaction contribution, and for this experimental plan, there is no way of separating them.

Interaction and the Interpretation of Main Effects

From an experimenter's point of view it should seem necessary to arrive at a significance test on the existence of interaction by separating true error variation from that due to interaction. The main effects, A and B, take on a different meaning in the presence of interaction. In the previous biological example, the effect that drying time has on the amount of solids left in the yeast might very well depend on the temperature to which the samples are exposed. In general, there could be experimental situations in which factor A has a positive effect on the response at one level of factor B, while at a different level of factor B the effect of A is negative. We use the term **positive effect** here to indicate that the yield or response increases as the levels of a given factor increase according to some defined order. In the same sense, a **negative effect** corresponds to a decrease in response for increasing levels of the factor.

Consider, for example, the following data on temperature (factor A at levels t_1, t_2, and t_3 in increasing order) and drying time d_1, d_2, and d_3 (also in increasing order). The response is percent solids. These data are completely hypothetical and given to illustrate a point.

	B			
A	d_1	d_2	d_3	**Total**
t_1	4.4	8.8	5.2	18.4
t_2	7.5	8.5	2.4	18.4
t_3	9.7	7.9	0.8	18.4
Total	21.6	25.2	8.4	55.2

Clearly the effect of temperature on percent solids is positive at the low drying time d_1 but negative for high drying time d_3. This **clear interaction** between temperature and drying time is obviously of interest to the biologist, but, based on the totals of the responses for temperatures t_1, t_2, and t_3, the temperature sum of squares, SSA, will yield a value of zero. We say then that the presence of interaction is **masking** the effect of temperature. Thus, if we consider the average effect of temperature, averaged over drying time, **there is no effect**. This then defines the main effect. But, of course, this is likely not what is pertinent to the biologist.

Before drawing any final conclusions resulting from tests of significance on the main effects and interaction effects, the **experimenter should first observe whether or not the test for interaction is significant**. If interaction is not significant, then the results of the tests on the main effects are meaningful. However, if interaction should be significant, then only those tests on the main effects that turn out to be significant are meaningful. Nonsignificant main effects in the presence of interaction might well be a result of masking and dictate the need to observe the influence of each factor at fixed levels of the other.

A Graphical Look at Interaction

The presence of interaction as well as its scientific impact can be interpreted nicely through the use of **interaction plots**. The plots clearly give a pictorial view of the tendency in the data to show the effect of changing one factor as one moves from one level to another of a second factor. Figure 14.1 illustrates the strong temperature by drying time interaction. The interaction is revealed in nonparallel lines.

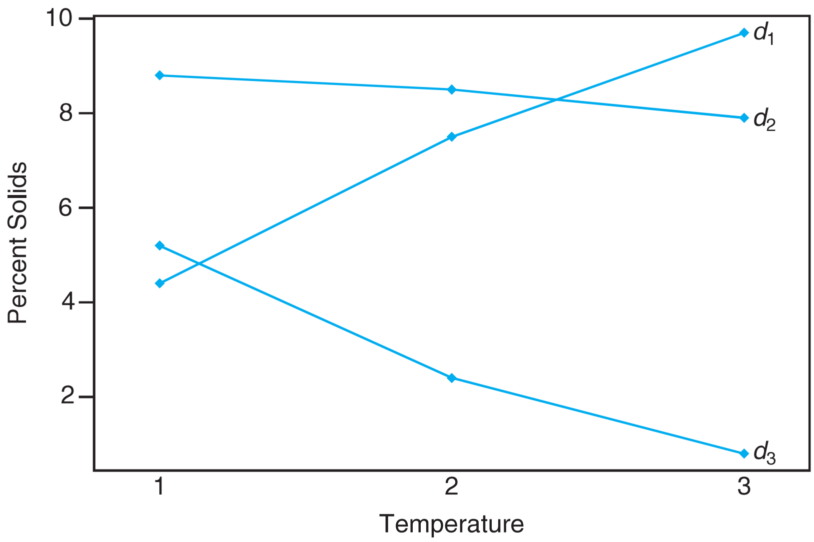

Figure 14.1: Interaction plot for temperature–drying time data.

The relatively strong *temperature effect* on percent solids at the lower drying time is reflected in the steep slope at d_1. At the middle drying time d_2 the temperature has very little effect, while at the high drying time d_3 the negative slope illustrates a negative effect of temperature. Interaction plots such as this set give the scientist a quick and meaningful interpretation of the interaction that is present. It should be apparent that **parallelism** in the plots signals an **absence of interaction**.

Need for Multiple Observations

Interaction and experimental error are separated in the two-factor experiment only if multiple observations are taken at the various treatment combinations. For maximum efficiency, there should be the same number n of observations at each combination. These should be true replications, not just repeated measurements. For example, in the yeast illustration, if we take $n = 2$ observations at each combination of temperature and drying time, there should be two separate samples and not merely repeated measurements on the same sample. This allows variability due to experimental units to appear in "error," so the variation is not merely measurement error.

14.3 Two-Factor Analysis of Variance

To present general formulas for the analysis of variance of a two-factor experiment using repeated observations in a completely randomized design, we shall consider the case of n replications of the treatment combinations determined by a levels of factor A and b levels of factor B. The observations may be classified by means of a rectangular array where the rows represent the levels of factor A and the columns represent the levels of factor B. Each treatment combination defines a cell in our array. Thus, we have ab cells, each cell containing n observations. Denoting the kth observation taken at the ith level of factor A and the jth level of factor B by y_{ijk}, Table 14.1 shows the abn observations.

Table 14.1: Two-Factor Experiment with n Replications

	B					
A	**1**	**2**	**⋯**	***b***	**Total**	**Mean**
1	y_{111}	y_{121}	$\cdots$	y_{1b1}	$Y_{1..}$	$\bar{y}_{1..}$
	y_{112}	y_{122}	$\cdots$	y_{1b2}		
	$\vdots$	$\vdots$		$\vdots$		
	y_{11n}	y_{12n}	$\cdots$	y_{1bn}		
2	y_{211}	y_{221}	$\cdots$	y_{2b1}	$Y_{2..}$	$\bar{y}_{2..}$
	y_{212}	y_{222}	$\cdots$	y_{2b2}		
	$\vdots$	$\vdots$		$\vdots$		
	y_{21n}	y_{22n}	$\cdots$	y_{2bn}		
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
a	y_{a11}	y_{a21}	$\cdots$	y_{ab1}	$Y_{a..}$	$\bar{y}_{a..}$
	y_{a12}	y_{a22}	$\cdots$	y_{ab2}		
	$\vdots$	$\vdots$		$\vdots$		
	y_{a1n}	y_{a2n}	$\cdots$	y_{abn}		
Total	$Y_{.1.}$	$Y_{.2.}$	$\cdots$	$Y_{.b.}$	$Y_{...}$	
Mean	$\bar{y}_{.1.}$	$\bar{y}_{.2.}$	$\cdots$	$\bar{y}_{.b.}$		$\bar{y}_{...}$

The observations in the (ij)th cell constitute a random sample of size n from a population that is assumed to be normally distributed with mean μ_{ij} and variance σ^2. All ab populations are assumed to have the same variance σ^2. Let us define the following useful symbols, some of which are used in Table 14.1:

$Y_{ij.}$ = sum of the observations in the (ij)th cell,
$Y_{i..}$ = sum of the observations for the ith level of factor A,
$Y_{.j.}$ = sum of the observations for the jth level of factor B,
$Y_{...}$ = sum of all abn observations,
$\bar{y}_{ij.}$ = mean of the observations in the (ij)th cell,
$\bar{y}_{i..}$ = mean of the observations for the ith level of factor A,
$\bar{y}_{.j.}$ = mean of the observations for the jth level of factor B,
$\bar{y}_{...}$ = mean of all abn observations.

Unlike in the one-factor situation covered at length in Chapter 13, here we are assuming that the **populations**, where n independent identically distributed observations are taken, are **combinations** of factors. Also we will assume throughout that an equal number (n) of observations are taken at each factor combination. In cases in which the sample sizes per combination are unequal, the computations are more complicated but the concepts are transferable.

Model and Hypotheses for the Two-Factor Problem

Each observation in Table 14.1 may be written in the form

$$y_{ijk} = \mu_{ij} + \epsilon_{ijk},$$

where ϵ_{ijk} measures the deviations of the observed y_{ijk} values in the (ij)th cell from the population mean μ_{ij}. If we let $(\alpha\beta)_{ij}$ denote the interaction effect of the ith level of factor A and the jth level of factor B, α_i the effect of the ith level of factor A, β_j the effect of the jth level of factor B, and μ the overall mean, we can write

$$\mu_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij},$$

and then

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk},$$

on which we impose the restrictions

$$\sum_{i=1}^{a} \alpha_i = 0, \qquad \sum_{j=1}^{b} \beta_j = 0, \qquad \sum_{i=1}^{a} (\alpha\beta)_{ij} = 0, \qquad \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0.$$

The three hypotheses to be tested are as follows:

1. H_0': $\alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$,
H_1': At least one of the α_i is not equal to zero.

2. H_0'': $\beta_1 = \beta_2 = \cdots = \beta_b = 0$,
H_1'': At least one of the β_j is not equal to zero.

3. $H_0''': (\alpha\beta)_{11} = (\alpha\beta)_{12} = \cdots = (\alpha\beta)_{ab} = 0,$
H_1''': At least one of the $(\alpha\beta)_{ij}$ is not equal to zero.

We warned the reader about the problem of masking of main effects when interaction is a heavy contributor in the model. It is recommended that the interaction test result be considered first. The interpretation of the main effect test follows, and the nature of the scientific conclusion depends on whether interaction is found. If interaction is ruled out, then hypotheses 1 and 2 above can be tested and the interpretation is quite simple. However, if interaction is found to be present the interpretation can be more complicated, as we have seen from the discussion of the drying time and temperature in the previous section. In what follows, the structure of the tests of hypotheses 1, 2, and 3 will be discussed. Interpretation of results will be incorporated in the discussion of the analysis in Example 14.1.

The tests of the hypotheses above will be based on a comparison of independent estimates of σ^2 provided by splitting the total sum of squares of our data into four components by means of the following identity.

Partitioning of Variability in the Two-Factor Case

Theorem 14.1: **Sum-of-Squares Identity**

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2 = bn\sum_{i=1}^{a}(\bar{y}_{i..} - \bar{y}_{...})^2 + an\sum_{j=1}^{b}(\bar{y}_{.j.} - \bar{y}_{...})^2$$
$$+ n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij.})^2$$

Symbolically, we write the sum-of-squares identity as

$$SST = SSA + SSB + SS(AB) + SSE,$$

where SSA and SSB are called the sums of squares for the main effects A and B, respectively, $SS(AB)$ is called the interaction sum of squares for A and B, and SSE is the error sum of squares. The degrees of freedom are partitioned according to the identity

$$abn - 1 = (a-1) + (b-1) + (a-1)(b-1) + ab(n-1).$$

Formation of Mean Squares

If we divide each of the sums of squares on the right side of the sum-of-squares identity by its corresponding number of degrees of freedom, we obtain the four statistics

$$S_1^2 = \frac{SSA}{a-1}, \qquad S_2^2 = \frac{SSB}{b-1}, \qquad S_3^2 = \frac{SS(AB)}{(a-1)(b-1)}, \qquad S^2 = \frac{SSE}{ab(n-1)}.$$

All of these variance estimates are independent estimates of σ^2 under the condition that there are no effects α_i, β_j, and, of course, $(\alpha\beta)_{ij}$. If we interpret the sums of squares as functions of the independent random variables $y_{111}, y_{112}, \ldots, y_{abn}$, it is not difficult to verify that

$$E(S_1^2) = E\left[\frac{SSA}{a-1}\right] = \sigma^2 + \frac{nb}{a-1}\sum_{i=1}^{a}\alpha_i^2,$$

$$E(S_2^2) = E\left[\frac{SSB}{b-1}\right] = \sigma^2 + \frac{na}{b-1}\sum_{j=1}^{b}\beta_j^2,$$

$$E(S_3^2) = E\left[\frac{SS(AB)}{(a-1)(b-1)}\right] = \sigma^2 + \frac{n}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij}^2,$$

$$E(S^2) = E\left[\frac{SSE}{ab(n-1)}\right] = \sigma^2,$$

from which we immediately observe that all four estimates of σ^2 are unbiased when H_0', H_0'', and H_0''' are true.

To test the hypothesis H_0', that the effects of factors A are all equal to zero, we compute the following ratio:

F-Test for Factor A

$$f_1 = \frac{s_1^2}{s^2},$$

which is a value of the random variable F_1 having the F-distribution with $a-1$ and $ab(n-1)$ degrees of freedom when H_0' is true. The null hypothesis is rejected at the α-level of significance when $f_1 > f_\alpha[a-1, ab(n-1)]$.

Similarly, to test the hypothesis H_0'' that the effects of factor B are all equal to zero, we compute the following ratio:

F-Test for Factor B

$$f_2 = \frac{s_2^2}{s^2},$$

which is a value of the random variable F_2 having the F-distribution with $b-1$ and $ab(n-1)$ degrees of freedom when H_0'' is true. This hypothesis is rejected at the α-level of significance when $f_2 > f_\alpha[b-1, ab(n-1)]$.

Finally, to test the hypothesis H_0''', that the interaction effects are all equal to zero, we compute the following ratio:

F-Test for Interaction

$$f_3 = \frac{s_3^2}{s^2},$$

which is a value of the random variable F_3 having the F-distribution with $(a-1)(b-1)$ and $ab(n-1)$ degrees of freedom when H_0''' is true. We conclude that, at the α-level of significance, interaction is present when $f_3 > f_\alpha[(a-1)(b-1), ab(n-1)]$.

As indicated in Section 14.2, it is advisable to interpret the test for interaction before attempting to draw inferences on the main effects. If interaction is not significant, there is certainly evidence that the tests on main effects are interpretable. Rejection of hypothesis 1 on page 566 implies that the response means at the levels of factor A are significantly different, while rejection of hypothesis 2 implies a similar condition for the means at levels of factor B. However, a significant interaction could very well imply that the data should be analyzed in a somewhat different manner—**perhaps observing the effect of factor A at fixed levels of factor B**, and so forth.

The computations in an analysis-of-variance problem, for a two-factor experiment with n replications, are usually summarized as in Table 14.2.

Table 14.2: Analysis of Variance for the Two-Factor Experiment with n Replications

Source of Variation	**Sum of Squares**	**Degrees of Freedom**	**Mean Square**	**Computed f**
Main effect:				
A	SSA	$a-1$	$s_1^2 = \frac{SSA}{a-1}$	$f_1 = \frac{s_1^2}{s^2}$
B	SSB	$b-1$	$s_2^2 = \frac{SSB}{b-1}$	$f_2 = \frac{s_2^2}{s^2}$
Two-factor interactions:				
AB	$SS(AB)$	$(a-1)(b-1)$	$s_3^2 = \frac{SS(AB)}{(a-1)(b-1)}$	$f_3 = \frac{s_3^2}{s^2}$
Error	SSE	$ab(n-1)$	$s^2 = \frac{SSE}{ab(n-1)}$	
Total	SST	$abn-1$		

Example 14.1: In an experiment conducted to determine which of 3 different missile systems is preferable, the propellant burning rate for 24 static firings was measured. Four different propellant types were used. The experiment yielded duplicate observations of burning rates at each combination of the treatments.

The data, after coding, are given in Table 14.3. Test the following hypotheses: (a) $H_0^{'}$: there is no difference in the mean propellant burning rates when different missile systems are used, (b) $H_0^{''}$: there is no difference in the mean propellant burning rates of the 4 propellant types, (c) $H_0^{'''}$: there is no interaction between the different missile systems and the different propellant types.

Table 14.3: Propellant Burning Rates

Missile System	**Propellant Type** b_1	b_2	b_3	b_4
a_1	34.0	30.1	29.8	29.0
	32.7	32.8	26.7	28.9
a_2	32.0	30.2	28.7	27.6
	33.2	29.8	28.1	27.8
a_3	28.4	27.3	29.7	28.8
	29.3	28.9	27.3	29.1

Solution: 1. (a) $H_0^{'}$: $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
(b) $H_0^{''}$: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$.

(c) $H_0''': (\alpha\beta)_{11} = (\alpha\beta)_{12} = \cdots = (\alpha\beta)_{34} = 0.$

2. (a) H_1': At least one of the α_i is not equal to zero.
   (b) H_1'': At least one of the β_j is not equal to zero.
   (c) H_1''': At least one of the $(\alpha\beta)_{ij}$ is not equal to zero.

The sum-of-squares formula is used as described in Theorem 14.1. The analysis of variance is shown in Table 14.4.

Table 14.4: Analysis of Variance for the Data of Table 14.3

Source of Variation	**Sum of Squares**	**Degrees of Freedom**	**Mean Square**	**Computed f**
Missile system	14.52	2	7.26	5.84
Propellant type	40.08	3	13.36	10.75
Interaction	22.16	6	3.69	2.97
Error	14.91	12	1.24	
Total	91.68	23		

The reader is directed to a *SAS* GLM Procedure (General Linear Models) for analysis of the burning rate data in Figure 14.2. Note how the "model" (11 degrees of freedom) is initially tested and the system, type, and system by type interaction are tested separately. The F-test on the model ($P = 0.0030$) is testing the accumulation of the two main effects and the interaction.

(a) Reject H_0' and conclude that different missile systems result in different mean propellant burning rates. The P-value is approximately 0.0169.

(b) Reject H_0'' and conclude that the mean propellant burning rates are not the same for the four propellant types. The P-value is approximately 0.0010.

(c) Interaction is barely insignificant at the 0.05 level, but the P-value of approximately 0.0513 would indicate that interaction must be taken seriously.

At this point we should draw some type of interpretation of the interaction. It should be emphasized that statistical significance of a main effect merely implies that *marginal means are significantly different.* However, consider the two-way table of averages in Table 14.5.

Table 14.5: Interpretation of Interaction

	b_1	b_2	b_3	b_4	**Average**
a_1	33.35	31.45	28.25	28.95	30.50
a_2	32.60	30.00	28.40	27.70	29.68
a_3	28.85	28.10	28.50	28.95	28.60
Average	31.60	29.85	28.38	28.53	

It is apparent that more important information exists in the body of the table—trends that are inconsistent with the trend depicted by marginal averages. Table 14.5 certainly suggests that the effect of propellant type depends on the system

```
                              The GLM Procedure
Dependent Variable: rate
                                    Sum of
 Source               DF          Squares    Mean Square    F Value    Pr > F
 Model                11      76.76833333     6.97893939       5.62    0.0030
 Error                12      14.91000000     1.24250000
 Corrected Total      23      91.67833333

 R-Square     Coeff Var      Root MSE     rate Mean
 0.837366      3.766854      1.114675      29.59167

 Source               DF      Type III SS    Mean Square   F Value     Pr > F
 system                2      14.52333333     7.26166667      5.84     0.0169
 type                  3      40.08166667    13.36055556     10.75     0.0010
 system*type           6      22.16333333     3.69388889      2.97     0.0512
```

Figure 14.2: *SAS* printout of the analysis of the propellant rate data of Table 14.3.

being used. For example, for system 3 the propellant-type effect does not appear to be important, although it does have a large effect if either system 1 or system 2 is used. This explains the "significant" interaction between these two factors. More will be revealed subsequently concerning this interaction.

Example 14.2: Referring to Example 14.1, choose two orthogonal contrasts to partition the sum of squares for the missile systems into single-degree-of-freedom components to be used in comparing systems 1 and 2 versus 3, and system 1 versus system 2.

***Solution*:** The contrast for comparing systems 1 and 2 with 3 is

$$w_1 = \mu_{1.} + \mu_{2.} - 2\mu_{3.}.$$

A second contrast, orthogonal to w_1, for comparing system 1 with system 2, is given by $w_2 = \mu_{1.} - \mu_{2.}$. The single-degree-of-freedom sums of squares are

$$SSw_1 = \frac{[244.0 + 237.4 - (2)(228.8)]^2}{(8)[(1)^2 + (1)^2 + (-2)^2]} = 11.80$$

and

$$SSw_2 = \frac{(244.0 - 237.4)^2}{(8)[(1)^2 + (-1)^2]} = 2.72.$$

Notice that $SSw_1 + SSw_2 = SSA$, as expected. The computed f-values corresponding to w_1 and w_2 are, respectively,

$$f_1 = \frac{11.80}{1.24} = 9.5 \qquad \text{and} \qquad f_2 = \frac{2.72}{1.24} = 2.2.$$

Compared to the critical value $f_{0.05}(1, 12) = 4.75$, we find f_1 to be significant. In fact, the P-value is less than 0.01. Thus, the first contrast indicates that the hypothesis

$$H_0\text{: } \frac{1}{2}(\mu_{1.} + \mu_{2.}) = \mu_{3.}$$

is rejected. Since $f_2 < 4.75$, the mean burning rates of the first and second systems are not significantly different.

Impact of Significant Interaction in Example 14.1

If the hypothesis of no interaction in Example 14.1 is true, we could make the *general* comparisons of Example 14.2 regarding our missile systems rather than separate comparisons for each propellant. Similarly, we might make general comparisons among the propellants rather than separate comparisons for each missile system. For example, we could compare propellants 1 and 2 with 3 and 4 and also propellant 1 versus propellant 2. The resulting f-ratios, each with 1 and 12 degrees of freedom, turn out to be 24.81 and 7.39, respectively, and both are quite significant at the 0.05 level.

From propellant averages there appears to be evidence that propellant 1 gives the highest mean burning rate. A prudent experimenter might be somewhat cautious in drawing overall conclusions in a problem such as this one, where the f-ratio for interaction is barely below the 0.05 critical value. For example, the overall evidence, 31.60 versus 29.85 on the average for the two propellants, certainly indicates that propellant 1 is superior, in terms of a higher burning rate, to propellant 2. However, if we restrict ourselves to system 3, where we have an average of 28.85 for propellant 1 as opposed to 28.10 for propellant 2, there appears to be little or no difference between these two propellants. In fact, there appears to be a stabilization of burning rates for the different propellants if we operate with system 3. There is certainly overall evidence which indicates that system 1 gives a higher burning rate than system 3, but if we restrict ourselves to propellant 4, this conclusion does not appear to hold.

The analyst can conduct a simple t-test using average burning rates for system 3 in order to display conclusive evidence that interaction is *producing considerable difficulty in allowing broad conclusions on main effects*. Consider a comparison of propellant 1 against propellant 2 only using system 3. Borrowing an estimate of σ^2 from the overall analysis, that is, using $s^2 = 1.24$ with 12 degrees of freedom, we have

$$|t| = \frac{0.75}{\sqrt{2s^2/n}} = \frac{0.75}{\sqrt{1.24}} = 0.67,$$

which is not even close to being significant. This illustration suggests that one must be cautious about strict interpretation of main effects in the presence of interaction.

Graphical Analysis for the Two-Factor Problem of Example 14.1

Many of the same types of graphical displays that were suggested in the one-factor problems certainly apply in the two-factor case. Two-dimensional plots of cell means or treatment combination means can provide insight into the presence of interactions between the two factors. In addition, a plot of residuals against fitted values may well provide an indication of whether or not the homogeneous variance assumption holds. Often, of course, a violation of the homogeneous variance assumption involves an increase in the error variance as *the response numbers get larger*. As a result, this plot may point out the violation.

Figure 14.3 shows the plot of cell means in the case of the missile system propellant illustration in Example 14.1. Notice how graphically (in this case) the lack of parallelism shows through. Note the flatness of the part of the figure showing the propellant effect for system 3. This illustrates interaction among the factors. Figure 14.4 shows the plot of residuals against fitted values for the same data. There is no apparent sign of difficulty with the homogeneous variance assumption.

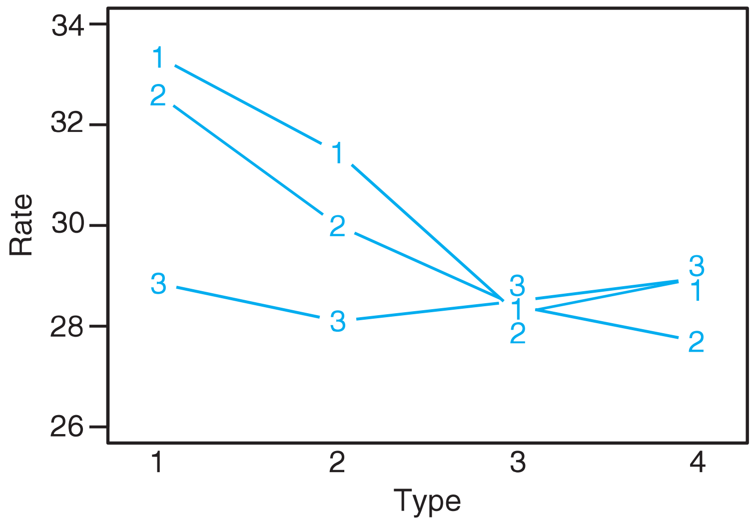

Figure 14.3: Plot of cell means for data of Example 14.1. Numbers represent missile systems.

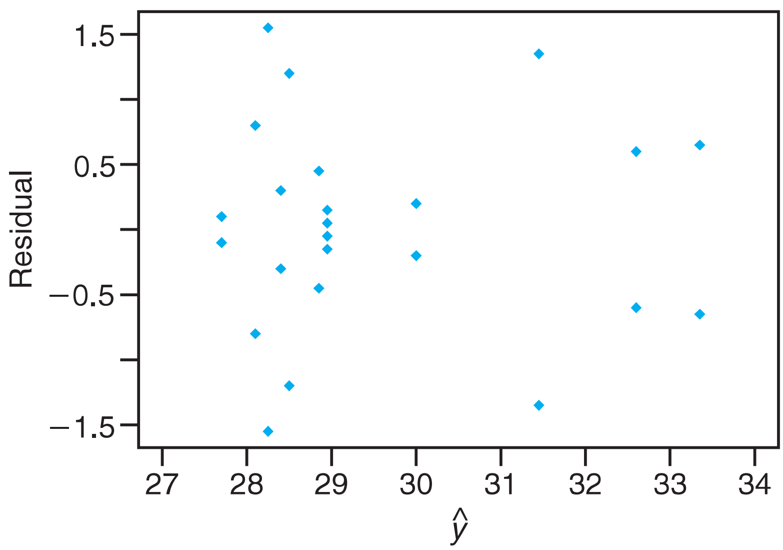

Figure 14.4: Residual plot of data of Example 14.1.

Example 14.3: An electrical engineer is investigating a plasma etching process used in semiconductor manufacturing. It is of interest to study the effects of two factors, the C_2F_6 gas flow rate (A) and the power applied to the cathode (B). The response is the etch rate. Each factor is run at 3 levels, and 2 experimental runs on etch rate are made for each of the 9 combinations. The setup is that of a completely randomized design. The data are given in Table 14.6. The etch rate is in A°/min.

Table 14.6: Data for Example 14.3

	Power Supplied		
C_2F_6 Flow Rate	**1**	**2**	**3**
1	288	488	670
	360	465	720
2	385	482	692
	411	521	724
3	488	595	761
	462	612	801

The levels of the factors are in ascending order, with level 1 being low level and level 3 being the highest.

(a) Show an analysis of variance table and draw conclusions, beginning with the test on interaction.

(b) Do tests on main effects and draw conclusions.

***Solution*:** A *SAS* output is given in Figure 14.5. From the output we learn the following.

```
                         The GLM Procedure
Dependent Variable: etchrate
                               Sum of
 Source             DF        Squares    Mean Square    F Value    Pr > F
 Model               8    379508.7778     47438.5972      61.00    <.0001
 Error               9      6999.5000       777.7222
 Corrected Total    17    386508.2778

 R-Square      Coeff Var      Root MSE     etchrate Mean
 0.981890       5.057714      27.88767          551.3889

 Source             DF    Type III SS    Mean Square    F Value    Pr > F
 c2f6                2     46343.1111     23171.5556      29.79    0.0001
 power               2    330003.4444    165001.7222     212.16    <.0001
 c2f6*power          4      3162.2222       790.5556       1.02    0.4485
```

Figure 14.5: *SAS* printout for Example 14.3.

(a) The P-value for the test of interaction is 0.4485. We can conclude that there is no significant interaction.

(b) There is a significant difference in mean etch rate for the 3 levels of C_2F_6 flow rate. Duncan's test shows that the mean etch rate for level 3 is significantly