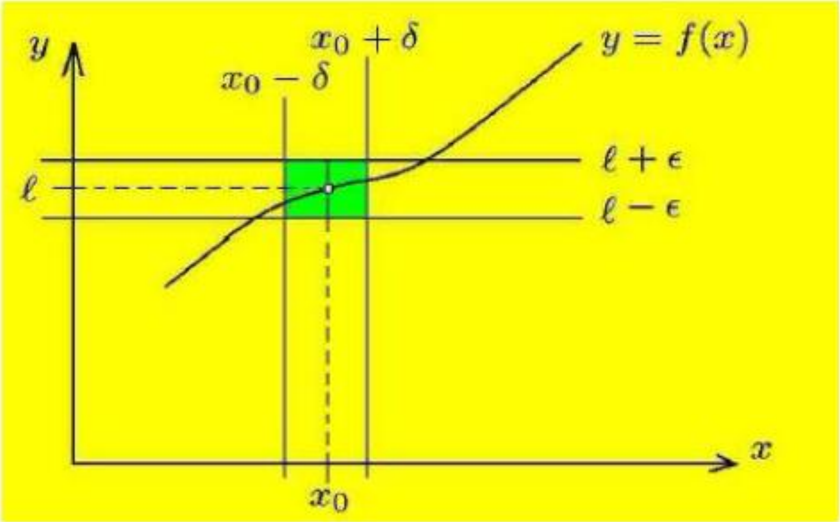
LIMITS AND CONTINUITY

Definition of Limit of a function.

Let $f(x)$ be defined and single-valued for all values of x in some deleted neighborhood of the point x_0. A number ℓ is called a limit of $f(x)$ as x approaches x_0, written $\lim\limits_{x\to x_0} f(x) = \ell$, if for every small positive number $\epsilon > 0$ there exists a number δ such that[6]

$$|f(x) - \ell| < \epsilon \qquad \textit{whenever} \qquad 0 < |x - x_0| < \delta$$

Then one can write $f(x) \to \ell$ ($f(x)$ approaches ℓ) as $x \to x_0$ (x approaches x_0). Note that $f(x)$ need not be defined at the point x_0 in order for a limit to exist.



The above definition must be modified if restrictions are placed upon how x approaches x_0. For example, the limits $\lim\limits_{x\to x_0^+} f(x) = \ell_1$ and $\lim\limits_{x\to x_0^-} f(x) = \ell_2$ are called the **right-hand and left-hand limits** associated with the function $f(x)$ as x approaches the point x_0. Sometimes the right-hand limit is expressed $\lim\limits_{x\to x_0^+} f(x) = f(x_0^+)$ and the left-hand limit is expressed $\lim\limits_{x\to x_0^-} f(x) = f(x_0^-)$. The $\epsilon - \delta$ definitions associated with these left and right-hand limits is exactly the same as given above with the understanding that for right-hand limits x is restricted to the set of values $x > x_0$ and for left-hand limits x is restricted to the set of values $x < x_0$.

Calculating Limits Using the Limit Laws

The Limit Laws

The next theorem tells how to calculate limits of functions that are arithmetic combinations of functions whose limits we already know.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c}(f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x\to c}(f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

EXAMPLE 1 Using the Limit Laws

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 8 in Section 2.1) and the properties of limits to find the following limits.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim_{x\to c}\dfrac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x\to -2}\sqrt{4x^2 - 3}$

Solution

(a)
$$\begin{aligned}\lim_{x\to c}(x^3 + 4x^2 - 3) &= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Product and Multiple Rules}\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)} && \text{Quotient Rule}\\ &= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5} && \text{Sum and Difference Rules}\\ &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x\to -2}\sqrt{4x^2 - 3} &= \sqrt{\lim_{x\to -2}(4x^2 - 3)} && \text{Power Rule with } r/s = {}^1\!/_2\\ &= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3} && \text{Difference Rule}\\ &= \sqrt{4(-2)^2 - 3} && \text{Product and Multiple Rules}\\ &= \sqrt{16 - 3}\\ &= \sqrt{13}\end{aligned}$$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

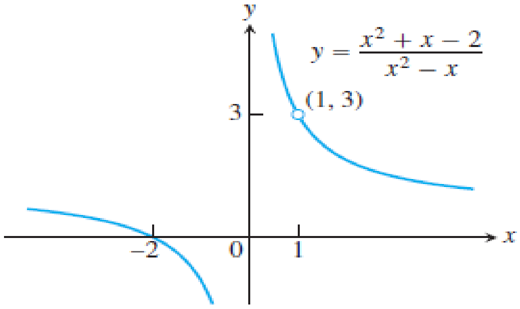
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 2 Limit of a Rational Function

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}=\frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}=\frac{0}{6}=0$$

EXAMPLE 3 Canceling a Common Factor

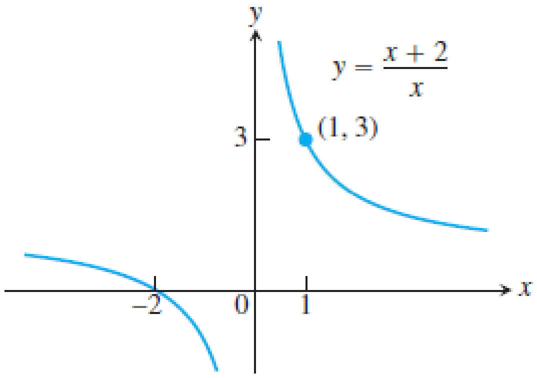


$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}.$$

$$\frac{x^2+x-2}{x^2-x}=\frac{(x-1)(x+2)}{x(x-1)}=\frac{x+2}{x},\qquad \text{if } x\neq 1.$$

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3.$$

EXAMPLE 4 Creating and Canceling a Common Factor



$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}.$$

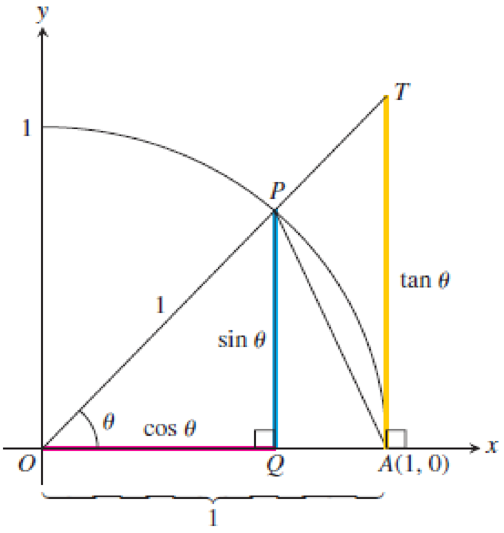
$$\frac{\sqrt{x^2+100}-10}{x^2}=\frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

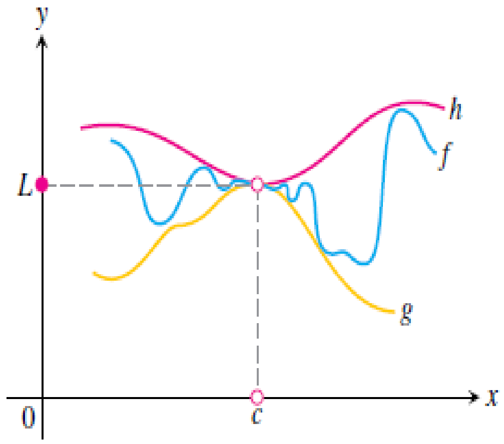
$$=\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}=\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}=\frac{1}{\sqrt{x^2+100}+10}.$$

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}\\ &= \frac{1}{\sqrt{0^2+100}+10}\\ &= \frac{1}{20}=0.05.\end{aligned}$$

Denominator not 0 at $x = 0$; substitute

The Sandwich Theorem





THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

THEOREM 7

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \qquad (\theta \text{ in radians}) \qquad (1)$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

Area $\Delta OAP <$ area sector $OAP <$ area ΔOAT.

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$ the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1. \qquad \lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

EXAMPLE 5 Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h}$$

$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \qquad \text{Let } \theta = h/2.$$

$$= -(1)(0) = 0.$$

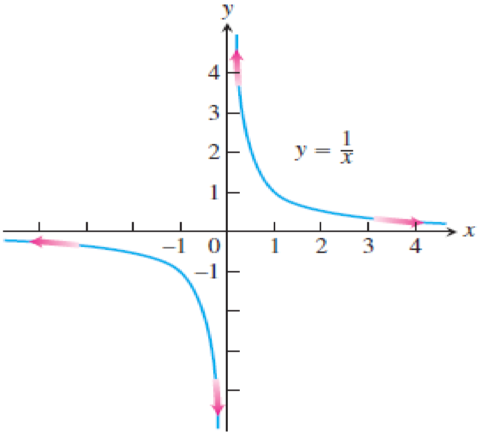
(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

Now, Eq. (1) applies with $\theta = 2x$.

■

Finite Limits as $x \to \pm\infty$



DEFINITIONS Limit as x approaches ∞ or $-\infty$

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty}(f(x)\cdot g(x)) = L\cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty}(k\cdot f(x)) = k\cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x\to\pm\infty}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 7 Using Theorem 8

$$\textbf{(a)} \quad \lim_{x\to\infty}\left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{Sum Rule}$$

$$= 5 + 0 = 5 \qquad \text{Known limits}$$

$$\textbf{(b)} \quad \lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$

$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} \qquad \text{Product rule}$$

$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0 \qquad \text{Known limits}$$

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator. What happens then depends on the degrees of the polynomials involved.

EXAMPLE 8 Numerator and Denominator of Same Degree

$$\lim_{x\to\infty}\frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty}\frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} \qquad \text{Divide numerator and denominator by } x^2.$$

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} \qquad \text{See Fig. 2.33.}$$

■

EXAMPLE 9 Degree of Numerator Less Than Degree of Denominator

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to-\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$$

Divide numerator and denominator by x^3.

$$=\frac{0+0}{2-0}=0$$

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

$$\lim_{x\to\infty}\frac{1}{x}=0 \qquad \lim_{x\to-\infty}\frac{1}{x}=0.$$

DEFINITION Horizontal Asymptote

A line $y=b$ is a **horizontal asymptote** of the graph of a function $y=f(x)$ if either

$$\lim_{x\to\infty}f(x)=b \qquad \text{or} \qquad \lim_{x\to-\infty}f(x)=b.$$

Oblique Asymptotes

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an **oblique (slanted) asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x\to\pm\infty$. Here's an example.

EXAMPLE Finding an Oblique Asymptote

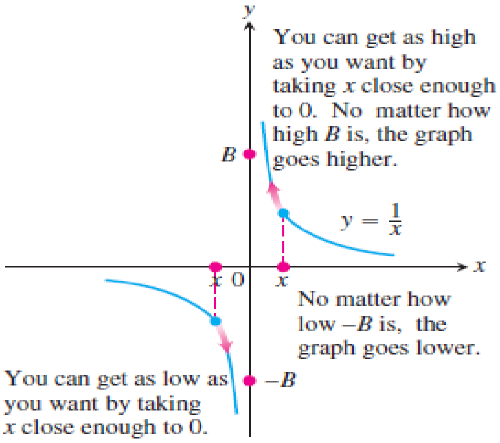
Find the oblique asymptote for the graph of

$$f(x)=\frac{2x^2-3}{7x+4}$$

Solution By long division, we find

$$f(x)=\frac{2x^2-3}{7x+4}=\left(\frac{2}{7}x-\frac{8}{49}\right)+\frac{-115}{49(7x+4)}$$

Infinite Limits and Vertical Asymptotes



You can get as high as you want by taking x close enough to 0. No matter how high B is, the graph goes higher.

No matter how low $-B$ is, the graph goes lower.

You can get as low as you want by taking x close enough to 0.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty .$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty .$$

The Precise Definition of a Limit

Now that we have gained some insight into the limit concept, working intuitively with the informal definition, we turn our attention to its precise definition. We replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a precise definition we will be able to prove conclusively the limit properties given in the preceding section, and we can establish other particular limits important to the study of calculus.

To show that the limit of $f(x)$ as $x \to x_0$ equals the number L, we need to show that the gap between $f(x)$ and L can be made “as small as we choose” if x is kept “close enough” to x_0. Let us see what this would require if we specified the size of the gap between $f(x)$ and L.

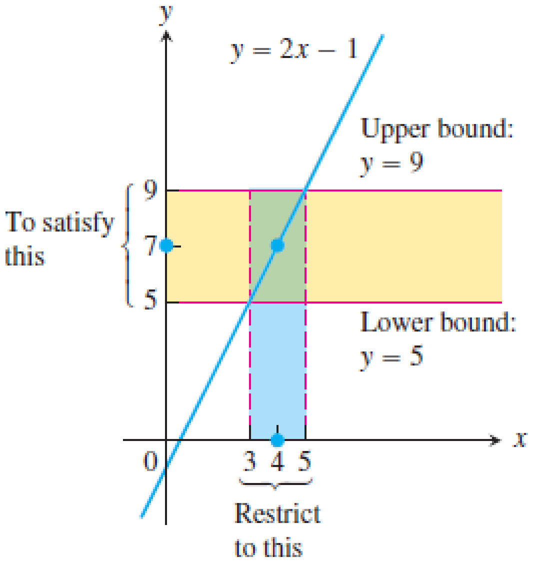
EXAMPLE 1 A Linear Function

Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is clear that y is close to 7 when x is close to 4, so $\lim_{x \to 4}(2x - 1) = 7$. However, how close to $x_0 = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:



$$
\begin{aligned}
|2x - 8| &< 2 \\
-2 < 2x - 8 &< 2 \\
6 < 2x &< 10 \\
3 < x &< 5 \\
-1 < x - 4 &< 1.
\end{aligned}
$$

One-Sided Limits and Limits at Infinity

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

EXAMPLE 2 Limits of the Function Graphed in Figure 2.24

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,

$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

$\lim_{x \to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

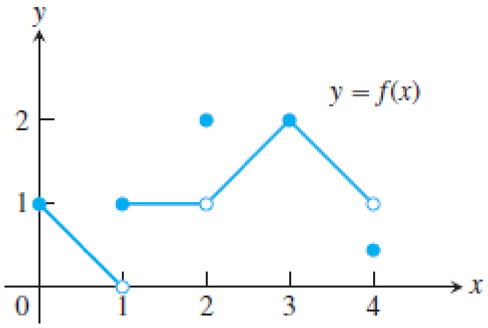
At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Continuity



DEFINITION **Continuous at a Point**

Interior point: A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

EXAMPLE 1

Find the points at which the function f in Figure 2.50 is continuous and the points at which f is discontinuous.

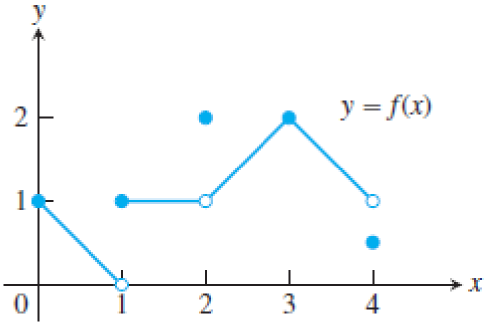
Solution The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1, x = 2$, and $x = 4$. At these points, there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\lim_{x \to 0^+} f(x) = f(0)$.

At $x = 3$, $\lim_{x \to 3} f(x) = f(3)$.

At $0 < c < 4, c \neq 1, 2$, $\lim_{x \to c} f(x) = f(c)$.



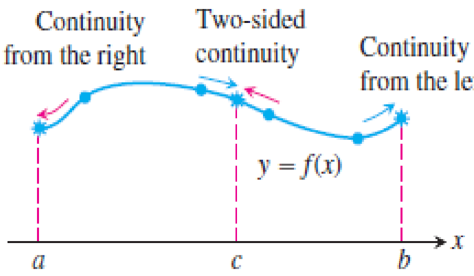
Points at which f is discontinuous:

At $x = 1$, $\lim\limits_{x \to 1} f(x)$ does not exist.

At $x = 2$, $\lim\limits_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.

At $x = 4$, $\lim\limits_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.

At $c < 0$, $c > 4$, these points are not in the domain of f.



Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Evaluate the following limit.

Example 1 $\lim\limits_{x\to 2}\frac{x^2+4x-12}{x^2-2x}$ $\qquad$ $\lim\limits_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\frac{0}{0}$

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}=\lim_{x\to 2}\frac{x+6}{x}$$

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{x+6}{x}=\frac{8}{2}=4$$

Example 2 $\lim\limits_{h\to 0}\frac{2(-3+h)^2-18}{h}$

$$\lim_{h\to 0}\frac{2(-3+h)^2-18}{h}=\lim_{h\to 0}\frac{2(9-6h+h^2)-18}{h}$$

$$=\lim_{h\to 0}\frac{18-12h+2h^2-18}{h}=\lim_{h\to 0}\frac{h(-12+2h)}{h}=\lim_{h\to 0}\frac{-12h+2h^2}{h}$$

$$=\lim_{h\to 0}-12+2h=-12$$

Example 3

$$g(y)=\begin{cases}y^2+5 & \text{if } y<-2\\ 1-3y & \text{if } y\geq -2\end{cases}$$

Compute the following limits. **(a)** $\lim\limits_{y\to 6}g(y)$ **(b)** $\lim\limits_{y\to -2}g(y)$

(a) $\lim\limits_{y\to 6}g(y)$

$$\lim_{y\to 6}g(y)=\lim_{y\to 6}1-3y \quad =-17$$

(b) $\lim\limits_{y\to -2} g(y)$ $\quad \lim\limits_{y\to -2^-} g(y) = \lim\limits_{y\to -2^-} y^2+5 \quad = 9$

since $y \to 2^-$ implies $y < -2$

$\lim\limits_{y\to -2^+} g(y) = \lim\limits_{y\to -2^+} 1-3y \quad = 7$ since $y \to 2^+$ implies $y > -2$

$\lim\limits_{y\to -2^-} g(y) = 9 \neq 7 = \lim\limits_{y\to -2^+} g(y)$

and so since the two one sided limits aren't the same

Example 4

Evaluate the following limit.

$$\lim_{y\to -2} g(y) \qquad \text{where, } g(y) = \begin{cases} y^2+5 & \text{if } y < -2 \\ 3-3y & \text{if } y \geq -2 \end{cases}$$

Solution

The two one-sided limits this time are,

$$\lim_{y\to -2^-} g(y) = \lim_{y\to -2^-} y^2+5 \qquad \text{since } y \to 2^- \text{ implies } y < -2$$
$$= 9$$

$$\lim_{y\to -2^+} g(y) = \lim_{y\to -2^-} 3-3y \qquad \text{since } y \to 2^+ \text{ implies } y > -2$$
$$= 9$$

The one-sided limits are the same so we get,

$$\lim_{y\to -2} g(y) = 9$$

Example. 5

(a) $\lim\limits_{x\to\infty}\left(2x^4 - x^2 - 8x\right)$

(b) $\lim\limits_{t\to -\infty}\left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$

Solution

(a) $\lim\limits_{x\to\infty}\left(2x^4 - x^2 - 8x\right)$

So, let's see what we get if we do that. As x approaches infinity, then x to a power can only get larger and the coefficient on each term (the first and third) will only make the term even larger. So, if we look at what each term is doing in the limit we get the following,

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = \infty - \infty - \infty$$

This is one of those **indeterminate forms** that we first started seeing in a previous section. So, we need a way to get around this problem. What we'll do here is factor the largest power of x out of the whole polynomial as follows,

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = \lim_{x\to\infty}\left[x^4\left(2 - \frac{1}{x^2} - \frac{8}{x^3}\right)\right]$$

Now for each of the terms we have,

$$\lim_{x\to\infty} x^4 = \infty \qquad\qquad \lim_{x\to\infty}\left(2 - \frac{1}{x^2} - \frac{8}{x^3}\right) = 2$$

The first limit is clearly infinity and for the second limit we'll use the fact above on the last two terms. Therefore using Fact 2 from the previous section we see value of the limit will be,

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = \infty$$

(b) $\lim\limits_{t\to-\infty}\left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$

We'll work this part much quicker than the previous part. All we need to do is factor out the largest power of t to get the following,

$$\lim_{t\to-\infty}\left(\tfrac{1}{3}t^5 + 2t^3 - t^2 + 8\right) = \lim_{t\to-\infty}\left[t^5\left(\frac{1}{3} + \frac{2}{t^2} - \frac{1}{t^3} + \frac{8}{t^5}\right)\right]$$

Remember that all you need to do to get the factoring correct is divide the original polynomial by the power of t we're factoring out, t^5 in this case.

Now all we need to do is take the limit of the two terms. In the first don't forget that since we're going out towards $-\infty$ and we're raising t to the 5th power that the limit will be negative (negative number raised to an odd power is still negative). In the second term we'll again make heavy use of the fact above to see that is a finite number.

Therefore, using the a modification of the Facts from the previous section the value of the limit is,

$$\lim_{t\to-\infty}\left(\tfrac{1}{3}t^5 + 2t^3 - t^2 + 8\right) = -\infty$$

Example.6.

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}$$

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\frac{\infty}{-\infty}$$

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{8}{x^3}}{-5+\frac{7}{x^4}}$$

$$=\frac{2+0+0}{-5+0}\qquad=-\frac{2}{5}$$

Example 7.

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}\qquad\qquad \lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}$$

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{\sqrt{x^2\left(3+\frac{6}{x^2}\right)}}{x\left(\frac{5}{x}-2\right)}\qquad=\lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}\qquad\qquad \sqrt{x^2}=|x|$$

$$|x|=\begin{cases}x & \text{if } x\ge 0\\ -x & \text{if } x<0\end{cases}$$

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}=\lim_{x\to\infty}\frac{\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}=\frac{\sqrt{3+0}}{0-2}=-\frac{\sqrt{3}}{2}$$

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to-\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to-\infty}\frac{-x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}\quad=\lim_{x\to-\infty}\frac{-\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}$$

$$=\frac{\sqrt{3}}{2}$$

Example.8.

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3}\qquad\qquad\lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}$$

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3}=\lim_{z\to\infty}\frac{z^3\left(\frac{4}{z}+z^3\right)}{z^3\left(\frac{1}{z^3}-5\right)}=\lim_{z\to\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}$$

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3}=\frac{\infty}{-5}=-\infty$$

$$\lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}=\lim_{z\to-\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}=\frac{-\infty}{-5}=\infty$$

$\lim_{x\to\infty} \mathbf{e}^{x}$ $\lim_{x\to-\infty} \mathbf{e}^{x}$ $\lim_{x\to\infty} \mathbf{e}^{-x}$ $\lim_{x\to-\infty} \mathbf{e}^{-x}$

$\lim_{x\to\infty} \mathbf{e}^{x} = \infty$ $\lim_{x\to-\infty} \mathbf{e}^{x} = 0$ $\lim_{x\to\infty} \mathbf{e}^{-x} = 0$ $\lim_{x\to-\infty} \mathbf{e}^{-x} = \infty$

Example.9.

(a) $\lim_{x\to\infty} \mathbf{e}^{2-4x-8x^2}$ **(b)** $\lim_{t\to-\infty} \mathbf{e}^{t^4-5t^2+1}$ **(c)** $\lim_{z\to 0^+} \mathbf{e}^{\frac{1}{z}}$

Solution

(a) $\lim_{x\to\infty} \mathbf{e}^{2-4x-8x^2}$

In this part what we need to note (using Fact 2 above) is that in the limit the exponent of the exponential does the following,

$$\lim_{x\to\infty}\left(2-4x-8x^2\right) = -\infty$$

So, the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous set of examples. So, the answer here is,

$$\lim_{x\to\infty} \mathbf{e}^{2-4x-8x^2} = 0$$

(b) $\lim_{t\to-\infty} \mathbf{e}^{t^4-5t^2+1}$

Here let's first note that,

$$\lim_{t\to-\infty}\left(t^4-5t^2+1\right) = \infty$$

The exponent goes to infinity in the limit and so the exponential will also need to go to infinity in the limit. Or,

$$\lim_{t\to-\infty} \mathbf{e}^{t^4-5t^2+1} = \infty$$

(c) $\lim_{z\to 0^+} \mathbf{e}^{\frac{1}{z}}$

So, let's first note that using the idea from the previous section we have,

$$\lim_{z\to 0^+} \frac{1}{z} = \infty$$

Here's the answer to this part.

$$\lim_{z\to 0^+} \mathbf{e}^{\frac{1}{z}} = \infty$$

Example.10.

$$\textbf{(a)}\ \lim_{x\to\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)$$

$$\textbf{(b)}\ \lim_{x\to-\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)$$

Solution.

(a) $\lim_{x\to\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)$

Let's start by just taking the limit of each of the pieces and see what we get.

$$\lim_{x\to\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)=\infty-\infty+\infty+0-0$$

The first three are a problem however as they present us with another indeterminate form.
When dealing with polynomials we factored out the term with the largest exponent in it.
What do we mean by the "largest" exponent? So, since $10x$ is the largest of the three exponents there we'll "factor" an $\mathbf{e}^{10x}$ out of the whole thing.

$$\frac{-9\mathbf{e}^{-15x}}{\mathbf{e}^{10x}}=-9\mathbf{e}^{-15x-10x}=-9\mathbf{e}^{-25x}$$

Doing factoring on all terms then gives,

$$\lim_{x\to\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)=\lim_{x\to\infty}\left[\mathbf{e}^{10x}\left(1-4\mathbf{e}^{-4x}+3\mathbf{e}^{-9x}+2\mathbf{e}^{-12x}-9\mathbf{e}^{-25x}\right)\right]$$

$$\lim_{x\to\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)=\infty$$

To simplify the work here a little all we really needed to do was factor the $\mathbf{e}^{10x}$ out of the "problem" terms (the first three in this case) as follows,

$$\begin{aligned}\lim_{x\to\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}\right)&=\lim_{x\to\infty}\left[\mathbf{e}^{10x}\left(1-4\mathbf{e}^{-4x}+3\mathbf{e}^{-9x}\right)+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right]\\&=(\infty)(1)+0-0\\&=\infty\end{aligned}$$

(b) $\lim_{x\to-\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)$

Taking the limits gives,

$$\lim_{x\to-\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)=0-0+0+\infty-\infty$$

Example.11.

$$\textbf{(a)}\ \lim_{x\to\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}$$

$$\textbf{(b)}\ \lim_{x\to-\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}$$

$$\lim_{x\to\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}=\lim_{x\to\infty}\frac{\mathbf{e}^{4x}\left(6-\mathbf{e}^{-6x}\right)}{\mathbf{e}^{4x}\left(8-\mathbf{e}^{-2x}+3\mathbf{e}^{-5x}\right)}$$

$$=\lim_{x\to\infty}\frac{6-\mathbf{e}^{-6x}}{8-\mathbf{e}^{-2x}+3\mathbf{e}^{-5x}}$$

$$=\frac{6-0}{8-0+0}$$

$$=\frac{3}{4}$$

$$\lim_{x\to-\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}=\lim_{x\to-\infty}\frac{\mathbf{e}^{-x}\left(6\mathbf{e}^{5x}-\mathbf{e}^{-x}\right)}{\mathbf{e}^{-x}\left(8\mathbf{e}^{5x}-\mathbf{e}^{3x}+3\right)}$$

$$=\lim_{x\to-\infty}\frac{6\mathbf{e}^{5x}-\mathbf{e}^{-x}}{8\mathbf{e}^{5x}-\mathbf{e}^{3x}+3}$$

$$=\frac{0-\infty}{0-0+3}$$

$$=-\infty$$

Example.12.

(a) $\lim\limits_{x\to\infty} \ln\left(7x^3 - x^2 + 1\right)$

(b) $\lim\limits_{t\to-\infty} \ln\left(\dfrac{1}{t^2 - 5t}\right)$ [S

Solution

(a) $\lim\limits_{x\to\infty} \ln\left(7x^3 - x^2 + 1\right)$

So, let's first look to see what the argument of the log is doing,

$$\lim_{x\to\infty}\left(7x^3 - x^2 + 1\right) = \infty$$

The argument of the log is going to infinity and so the log must also be going to infinity in the limit. The answer to this part is then,

$$\lim_{x\to\infty} \ln\left(7x^3 - x^2 + 1\right) = \infty$$

(b) $\lim\limits_{t\to-\infty} \ln\left(\dfrac{1}{t^2 - 5t}\right)$

Using the techniques from earlier in this section we can see that,

$$\lim_{t\to-\infty} \frac{1}{t^2 - 5t} = 0$$

Therefore, not only does the argument go to zero, it goes to zero from the right. This is exactly what we need to do this limit.

So, the answer here is,

$$\lim_{t\to-\infty} \ln\left(\frac{1}{t^2 - 5t}\right) = -\infty$$

Computing limits:

Evaluate the limits, if it exists:

1. $\lim\limits_{x\to 2}\left(8-3x+12x^2\right)$

2. $\lim\limits_{t\to -3}\dfrac{6+4t}{t^2+1}$

3. $\lim\limits_{x\to -5}\dfrac{x^2-25}{x^2+2x-15}$

4. $\lim\limits_{z\to 8}\dfrac{2z^2-17z+8}{8-z}$

5. $\lim\limits_{y\to 7}\dfrac{y^2-4y-21}{3y^2-17y-28}$

6. $\lim\limits_{h\to 0}\dfrac{(6+h)^2-36}{h}$

7. $\lim\limits_{z\to 4}\dfrac{\sqrt{z}-2}{z-4}$

8. $\lim\limits_{x\to -3}\dfrac{\sqrt{2x+22}-4}{x+3}$

9. $\lim\limits_{x\to 0}\dfrac{x}{3-\sqrt{x+9}}$

Evaluate the following limits, if they exist.

(a) $\lim\limits_{z\to 7}h(z)$ (b) $\lim\limits_{z\to -4}h(z)$

Infinite Limits

1. For $f(x)=\dfrac{9}{(x-3)^5}$ evaluate,

(a) $\lim\limits_{x\to 3^-}f(x)$ **(b)** $\lim\limits_{x\to 3^+}f(x)$ **(c)** $\lim\limits_{x\to 3}f(x)$

2. For $h(t)=\dfrac{2t}{6+t}$ evaluate,

(a) $\lim\limits_{t\to -6^-}h(t)$ **(b)** $\lim\limits_{t\to -6^+}h(t)$ **(c)** $\lim\limits_{t\to -6}h(t)$

3. For $g(z)=\dfrac{z+3}{(z+1)^2}$ evaluate,

(a) $\lim\limits_{z\to-1^-} g(z)$ **(b)** $\lim\limits_{z\to-1^+} g(z)$ **(c)** $\lim\limits_{z\to-1} g(z)$

4. For $g(x)=\dfrac{x+7}{x^2-4}$ evaluate,

(a) $\lim\limits_{x\to2^-} g(x)$ **(b)** $\lim\limits_{x\to2^+} g(x)$ **(c)** $\lim\limits_{x\to2} g(x)$

find all the vertical asymptotes of the given function.

$$f(x)=\frac{7x}{(10-3x)^4} \qquad g(x)=\frac{-8}{(x+5)(x-9)}$$

Write down the equation(s) of any horizontal asymptotes for the function.

$$f(x)=\frac{8-4x^2}{9x^2+5x} \qquad f(x)=\frac{3x^7-4x^2+1}{5-10x^2}$$

$$f(x)=\frac{20x^4-7x^3}{2x+9x^2+5x^4} \qquad f(x)=\frac{x^6-x^4+x^2-1}{7x^6+4x^3+10}$$

$$f(x)=\frac{\sqrt{7+9x^2}}{1-2x} \qquad f(x)=\frac{x^3-2x+11}{3-6x^5}$$

$$f(x)=\frac{x+8}{\sqrt{2x^2+3}} \qquad f(x)=\frac{8+x-4x^2}{\sqrt{6+x^2+7x^4}}$$

Limits of Exponential at infinity.

evaluate **(a)** $\lim\limits_{x\to-\infty} f(x)$ and **(b)** $\lim\limits_{x\to\infty} f(x)$.

1. $f(x)=\mathbf{e}^{8+2x-x^3}$

2. $f(x)=\mathbf{e}^{\frac{6x^2+x}{5+3x}}$

3. $f(x)=2\mathbf{e}^{6x}-\mathbf{e}^{-7x}-10\mathbf{e}^{4x}$

4. $f(x)=3\mathbf{e}^{-x}-8\mathbf{e}^{-5x}-\mathbf{e}^{10x}$

5. $f(x)=\dfrac{\mathbf{e}^{-3x}-2\mathbf{e}^{8x}}{9\mathbf{e}^{8x}-7\mathbf{e}^{-3x}}$

6. $f(x)=\dfrac{\mathbf{e}^{-7x}-2\mathbf{e}^{3x}-\mathbf{e}^{x}}{\mathbf{e}^{-x}+16\mathbf{e}^{10x}+2\mathbf{e}^{-4x}}$

Limits of Logarithm Functions.

evaluate the given limit.

7. $\lim\limits_{t\to-\infty}\ln\left(4-9t-t^3\right)$

8. $\lim\limits_{z\to-\infty}\ln\left(\dfrac{3z^4-8}{2+z^2}\right)$

9. $\lim\limits_{x\to\infty}\ln\left(\dfrac{11+8x}{x^3+7x}\right)$

10. $\lim\limits_{x\to-\infty}\tan^{-1}\left(7-x+3x^5\right)$

11. $\lim\limits_{t\to\infty}\tan^{-1}\left(\dfrac{4+7t}{2-t}\right)$

12. $\lim\limits_{w\to\infty}\tan^{-1}\left(\dfrac{3w^2-9w^4}{4w-w^3}\right)$

DIFFERENTIATION

The Derivative as a Function

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x)$$

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \frac{dy}{dx}\bigg|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{d}{dx}f(x)\bigg|_{x=a}.$$

DEFINITION **Derivative Function**
The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx}f(x)$$

EXAMPLE Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution Here we have $f(x) = \dfrac{x}{x-1}$

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad = \quad \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$$

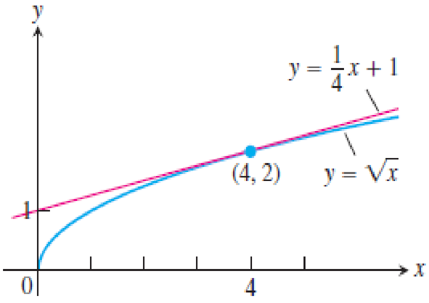
$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \qquad = \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.$$

EXAMPLE **(a)** Find the derivative of $y = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

(a) We use the equivalent form to calculate f':



$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \quad = \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)} \quad = \lim_{z\to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

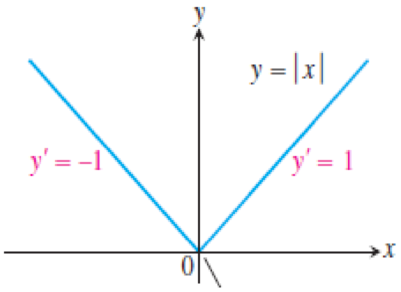
The tangent is the line through the point (4, 2) with slope 1/4 (

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$

EXAMPLE $y = |x|$

Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.



Solution To the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1.$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \qquad |x| = -x$$

There can be no derivative at the origin because the one-sided derivatives dif- fer there:

$$\text{Right-hand derivative of } |x| \text{ at zero} = \lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h}$$

$$= \lim_{h \to 0^+} 1 = 1 \qquad = \lim_{h \to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0.$$

$$\text{Left-hand derivative of } |x| \text{ at zero} = \lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h}$$

$$= \lim_{h \to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0. \qquad = \lim_{h \to 0^-} -1 = -1.$$

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + (f(c+h) - f(c)) = f(c) + \frac{f(c+h)-f(c)}{h} \cdot h.$$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \cdot \lim_{h\to 0} h$$

$$= f(c) + f'(c) \cdot 0 = f(c) + 0 = f(c).$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$ then f is continuous from that side at $x = c$.

The Intermediate Value Property of Derivatives

Not every function can be some function's derivative, as we see from the following theorem.

THEOREM 2

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Differentiation Rules

This section introduces a few rules that allow us to differentiate a great variety of functions. By proving these rules here, we can differentiate functions without having to apply the definition of the derivative each time.

Powers, Multiples, Sums, and Differences

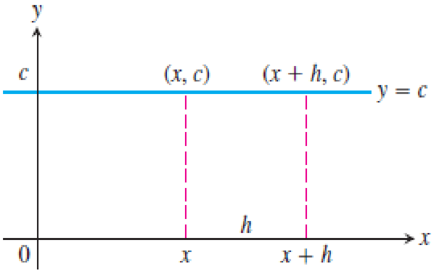
The first rule of differentiation is that the derivative of every constant function is zero.

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

EXAMPLE If f has the constant value $f(x) = 8$, then



$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

Similarly,

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0$$

$$\frac{d}{dx}\left(\sqrt{3}\right) = 0.$$

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.8). At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

The second rule tells how to differentiate x^n if n is a positive integer.

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

EXAMPLE 2 Interpreting Rule 2

f	x	x^2	x^3	x^4	...
f'	1	$2x$	$3x^2$	$4x^3$	...

To apply the Power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

EXAMPLE **(a)** The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 4 Derivative of a Sum

$$\begin{aligned} y &= x^4 + 12x \\ \frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \\ &= 4x^3 + 12 \end{aligned}$$

EXAMPLE 5 Derivative of a Polynomial

$$\begin{aligned} y &= x^3 + \frac{4}{3}x^2 - 5x + 1 \\ \frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\ &= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 \\ &= 3x^2 + \frac{8}{3}x - 5 \end{aligned}$$

EXAMPLE 6 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x:

$$4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$x = 0, 1, -1.$$

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

EXAMPLE 9 Differentiating a Product in Two Ways

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[\left(x^2 + 1\right)\left(x^3 + 3\right)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 5x^4 + 3x^2 + 6x. \qquad = 3x^4 + 3x^2 + 2x^4 + 6x$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

In function notation, $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 10 Find the derivative of

$$y = \frac{t^2 - 1}{t^2 + 1}.$$

Solution

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\begin{aligned} \frac{dy}{dt} &= \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} \\ &= \frac{4t}{(t^2+1)^2}. \end{aligned}$$

■

Negative Integer Powers of *x*

The Power Rule for negative integers is the same as the rule for positive integers.

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 11

(a) $$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

(b) $$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

EXAMPLE 12 Tangent to a Curve

Find an equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point (1, 3)

Solution The slope of the curve is

$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2}.$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1.$$

The line through (1, 3) with slope $m = -1$ is

$$y - 3 = (-1)(x - 1) \quad y = -x + 4.$$

EXAMPLE 13 Choosing Which Rule to Use

Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4},$$

expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. Notationally,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$$

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n}$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n.

EXAMPLE 14 Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$
Second derivative: $y'' = 6x - 6$
Third derivative: $y''' = 6$
Fourth derivative: $y^{(4)} = 0.$

The Derivative as a Rate of Change

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x + h) - f(x))/h$ as the average rate of change in f over the interval from x to $x + h$, we can interpret its limit as $h \to 0$ as the rate at which f is changing at the point x.

DEFINITION **Instantaneous Rate of Change**

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time.

The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

DEFINITION **Velocity**

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION **Speed**

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

DEFINITIONS Acceleration,

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Derivatives of Trigonometric Functions

Derivative of the Sine Function

If $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Sine angle sum identity} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 5(a) and} \\
&= \cos x. && \text{Theorem 7, Section 2.4}
\end{aligned}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x.$$

(b) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x \qquad \text{Product Rule}$$

$$= x^2 \cos x + 2x \sin x.$$

(c) $y = \frac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}.$$

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine,

$\cos(x + h) = \cos x \cos h - \sin x \sin h$,

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE

(a) $y = 5x + \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5x) + \frac{d}{dx}\left(\cos x\right) && \text{Sum Rule}\\ &= 5 - \sin x.\end{aligned}$$

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}\left(\cos x\right) + \cos x \frac{d}{dx}\left(\sin x\right) && \text{Product Rule}\\ &= \sin x(-\sin x) + \cos x(\cos x)\\ &= \cos^2 x - \sin^2 x.\end{aligned}$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(1 - \sin x\right)\frac{d}{dx}\left(\cos x\right) - \cos x \frac{d}{dx}\left(1 - \sin x\right)}{(1 - \sin x)^2} && \text{Quotient Rule}\\ &= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}\\ &= \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}.\end{aligned}$$

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE Find $d(\tan x)/dx$.

Solution

$$\frac{d}{dx}\left(\tan x\right) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}\left(\sin x\right) - \sin x \frac{d}{dx}\left(\cos x\right)}{\cos^2 x} \quad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

■

EXAMPLE Find y'' if $y = \sec x$.

Solution

$$y = \sec x$$

$$y' = \sec x \tan x$$

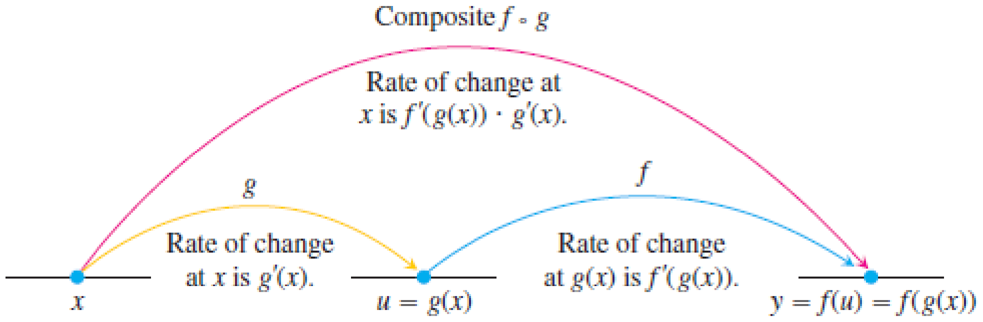
$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}\left(\tan x\right) + \tan x \frac{d}{dx}\left(\sec x\right) \quad \text{Product Rule}$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$

The Chain Rule and Parametric Equations



THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{du}\cdot\frac{du}{dt}\\ &= -\sin(u)\cdot 2t && \frac{dx}{du} \text{ evaluated at } u\\ &= -\sin(t^2+1)\cdot 2t\\ &= -2t\sin(t^2+1).\end{aligned}$$

"Outside-Inside" Rule

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

$$\frac{dy}{dx} = f'(g(x))\cdot g'(x).$$

EXAMPLE Differentiate $\sin(x^2+x)$ with respect to x.

Solution

$$\frac{d}{dx}\sin\underbrace{(x^2+x)}_{\text{inside}} = \cos\underbrace{(x^2+x)}_{\substack{\text{inside}\\ \text{left alone}}}\cdot\underbrace{(2x+1)}_{\substack{\text{derivative of}\\ \text{the inside}}}$$

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example.

EXAMPLE Find the derivative of $g(t) = \tan(5-\sin 2t)$.

Solution Notice here that the tangent is a function of $5-\sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned}g'(t) &= \frac{d}{dt}\big(\tan(5-\sin 2t)\big)\\ &= \sec^2(5-\sin 2t)\cdot\frac{d}{dt}(5-\sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5-\sin 2t\\ &= \sec^2(5-\sin 2t)\cdot\left(0-\cos 2t\cdot\frac{d}{dt}(2t)\right) && \text{Derivative of } 5-\sin u \text{ with } u = 2t\\ &= \sec^2(5-\sin 2t)\cdot(-\cos 2t)\cdot 2\\ &= -2(\cos 2t)\sec^2(5-\sin 2t).\end{aligned}$$

■

The Chain Rule with Powers of a Function

If f is a differentiable function of u and if u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

EXAMPLE

(a) $$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}\left(5x^3 - x^4\right)$$

Power Chain Rule with $u = 5x^3 - x^4, n = 7$

$$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$$
$$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$$

(b) $$\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$

$$= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2)$$

Power Chain Rule with $u = 3x - 2, n = -1$

$$= -1(3x - 2)^{-2}(3)$$
$$= -\frac{3}{(3x - 2)^2}$$

EXAMPLE

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Solution

(a) $$\frac{dy}{dx} = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$

Power Chain Rule with $u = \sin x, n = 5$

$$= 5\sin^4 x \cos x$$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

(b) $\frac{dy}{dx} = \frac{d}{dx}(1 - 2x)^{-3}$

$$= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \qquad \text{Power Chain Rule with } u = (1 - 2x), n = -3$$

$$= -3(1 - 2x)^{-4} \cdot (-2)$$

$$= \frac{6}{(1 - 2x)^4}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers.

Parametric Equations

DEFINITION **Parametric Curve**

If x and y are given as functions

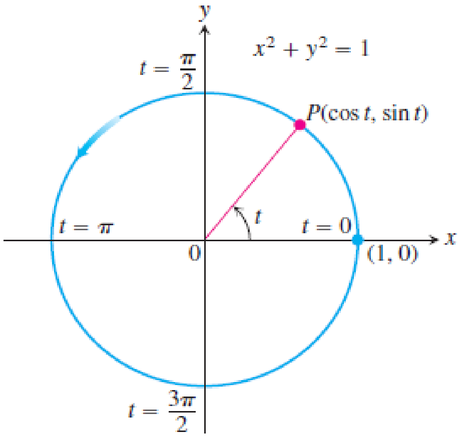
$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve, and its domain I is the **parameter interval**. If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point** of the curve. The point $(f(b), g(b))$ is the **terminal point**. When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval together constitute a **parametrization** of the curve.

EXAMPLE Moving Counterclockwise on a Circle

Graph the parametric curves



(a) $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi.$

(b) $x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi.$

Solution

(a) Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the parametric curve lies along the unit circle $x^2 + y^2 = 1$. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at

$(1, 0)$ and traces the entire circle once counterclockwise

(b) For $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$, we have $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$. The parametrization describes a motion that begins at the point $(a, 0)$ and traverses the circle $x^2 + y^2 = a^2$ once counterclockwise, returning to $(a, 0)$ at $t = 2\pi$. ■

Slopes of Parametrized Curves

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

EXAMPLE If $x = 2t + 3$ and $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

Solution $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t = \frac{x - 3}{2}.$$

When $t = 6$, $dy/dx = 6$. Notice that we are also able to find the derivative dy/dx as a function of x. ■

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{3}$$

EXAMPLE Finding d^2y/dx^2 for a Parametrized Curve

Find d^2y/dx^2 as a function of t if $x = t - t^2$, $y = t - t^3$.

Solution

1. Express $y' = dy/dx$ in terms of t.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

2. Differentiate y' with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2}$$

3. Divide dy'/dt by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3}$$

Implicit Differentiation

Implicitly Defined Functions

We begin with an example.

EXAMPLE Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$. We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?

The answer is yes. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$\begin{aligned} y^2 &= x \\ 2y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{2y}. \end{aligned}$$

The Chain Rule gives $\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left[f(x)\right]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2\left(-\sqrt{x}\right)} = -\frac{1}{2\sqrt{x}}.$$

EXAMPLE Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x. Rather it is the combined

$$y_1 = \sqrt{25 - x^2} \text{ and } y_2 = -\sqrt{25 - x^2}$$

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

EXAMPLE Find dy/dx if $y^2 = x^2 + \sin xy$

Solution

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy)$$

Differentiate both sides with respect to x . . .

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$

. . . treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$

Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$

Collect terms with dy/dx . . .

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

. . . and factor out dy/dx.

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

Solve for dy/dx by dividing.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Lenses, Tangents, and Normal Lines

EXAMPLE

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and

normal to the curve there

Solution The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}\left(9xy\right) = \frac{d}{dx}\left(0\right)$$

Differentiate both sides with respect to x.

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

Treat xy as a product and y as a function of x.

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$

Solve for dy/dx.

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope 4/5:

$$y = 4 + \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives. Here is an example.

EXAMPLE Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}\left(2x^3 - 3y^2\right) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y}, \qquad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$

Rational Powers of Differentiable Functions

We know that the rule $\frac{d}{dx}x^n = nx^{n-1}$

holds when n is an integer. Using implicit differentiation we can show that it holds when n is any rational number.

THEOREM 4 Power Rule for Rational Powers
If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

EXAMPLE Using the Rational Power Rule

(a) $$\frac{d}{dx}\overbrace{\left(1 - x^2\right)^{1/4}}^{\text{function defined on } [-1,\,1]} = \frac{1}{4}\left(1 - x^2\right)^{-3/4}(-2x) \qquad \text{Power Chain Rule with } u = 1 - x^2$$

$$= \underbrace{\frac{-x}{2\left(1 - x^2\right)^{3/4}}}_{\text{derivative defined only on } (-1,\,1)}$$

(b) $$\frac{d}{dx}(\cos x)^{-1/5} = -\frac{1}{5}(\cos x)^{-6/5}\frac{d}{dx}(\cos x)$$

$$= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$

$$= \frac{1}{5}(\sin x)(\cos x)^{-6/5}$$

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to measure directly the rate of increase of the volume than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt.

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

EXAMPLE

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

Solution We draw a picture of a partially filled vertical cylindrical tank, calling its radius r and the height of the fluid h Call the volume of the fluid V.

As time passes, the radius remains constant, but V and h change. We think of V and h as differentiable functions of time and use t to represent time. We are told that

$$\frac{dV}{dt} = -3000.$$

We pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.

We are asked to find

$$\frac{dh}{dt}.$$

How fast will the fluid level drop?

To find dh/dt, we first write an equation that relates h to V. The equation depends on the units chosen for V, r, and h. With V in liters and r and h in meters, the appropriate equation for the cylinder's volume is

$$V = 1000\pi r^2 h$$

because a cubic meter contains 1000 L.

Since V and h are differentiable functions of t, we can differentiate both sides of the equation $V = 1000\pi r^2 h$ with respect to t to get an equation that relates dh/dt to dV/dt:

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}.$$

r is a constant.

We substitute the known value $dV/dt = -3000$ and solve for dh/dt:

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}.$$

The fluid level will drop at the rate of $3/(\pi r^2)$ m/min.

The equation $dh/dt = -3/\pi r^2$ shows how the rate at which the fluid level drops depends on the tank's radius. If r is small, dh/dt will be large; if r is large, dh/dt will be small.

If $r = 1$ m: $\frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95 \text{ m/min} = -95 \text{ cm/min}.$

If $r = 10$ m: $\frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095 \text{ m/min} = -0.95 \text{ cm/min}.$ ■

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants.* Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

EXAMPLE

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.43). The variables in the picture are

$\theta =$ the angle in radians the range finder makes with the ground.

$y =$ the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$.

4. *Write an equation that relates the variables y and θ.*

$$\frac{y}{500} = \tan\theta \qquad \text{or} \qquad y = 500\tan\theta$$

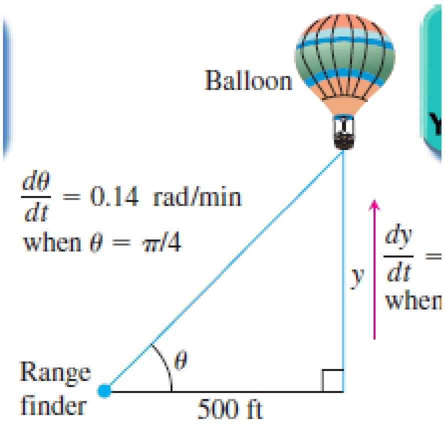
5. *Differentiate with respect to t using the Chain Rule.* The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

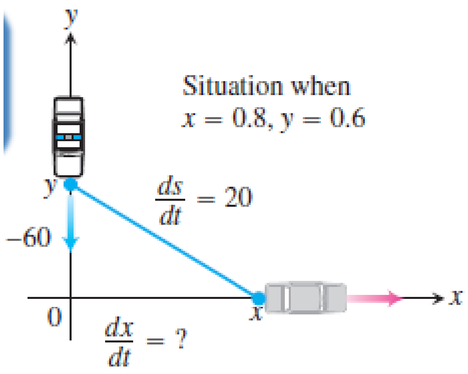
$$\frac{dy}{dt} = 500\,(\sec^2\theta)\,\frac{d\theta}{dt}$$

6. *Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.*

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ■





A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 3.44). We let t represent time and set

$$\begin{aligned} x &= \text{position of car at time } t \\ y &= \text{position of cruiser at time } t \\ s &= \text{distance between car and cruiser at time } t. \end{aligned}$$

We assume that x, y, and s are differentiable functions of t.

We want to find dx/dt when

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

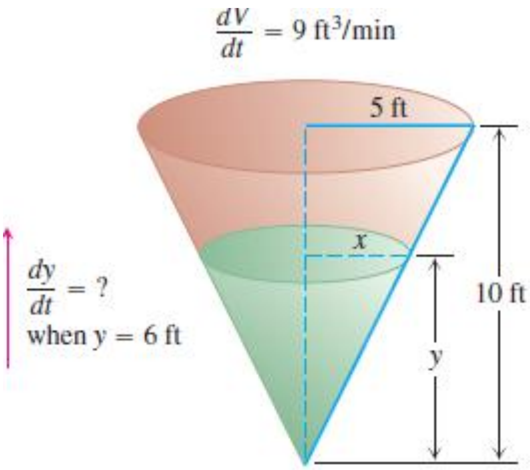
(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned} 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \\ &= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right). \end{aligned}$$

Finally, use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$\begin{aligned} 20 &= \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right) \\ \frac{dx}{dt} &= \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70 \end{aligned}$$

At the moment in question, the car's speed is 70 mph.



Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution Figure 3.45 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)
x = radius (ft) of the surface of the water at time t
y = depth (ft) of water in tank at time t.

We assume that V, x, and y are differentiable functions of t. The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6 \text{ ft} \qquad \text{and} \qquad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

$$\frac{x}{y} = \frac{5}{10} \qquad \text{or} \qquad x = \frac{y}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4}y^2 \frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt.

$$9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min.

power rule.

Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

(b) $g(t) = 2t^{6} + 7t^{-6}$

Solution

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

$$f'(x) = 15(100)x^{99} - 3(12)x^{11} + 5(1)x^{0} - 0$$
$$= 1500x^{99} - 36x^{11} + 5$$

(b) $g(t) = 2t^{6} + 7t^{-6}$

$$g'(t) = 2(6)t^{5} + 7(-6)t^{-7}$$
$$= 12t^{5} - 42t^{-7}$$

Example 2

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

Solution

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

We can simplify this rational expression however as follows.

$$h(t) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'(t) = 6t^2 + 10t^{-3}$$

Product and Quotient Rule.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $f(x) = \left(6x^3 - x\right)\left(10 - 20x\right)$

Solution

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

$$y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}\left(2 - 2x\right)$$

(b) $f(x) = (6x^3 - x)(10 - 20x)$

This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'(x) &= (18x^2 - 1)(10 - 20x) + (6x^3 - x)(-20) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Solution

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'(z) &= \frac{3(2 - z) - (3z + 9)(-1)}{(2 - z)^2} \\ &= \frac{15}{(2 - z)^2} \end{aligned}$$

EXAMPLE

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$\begin{aligned} h'(x) &= \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}(x^2 - 2) - 4x^{\frac{1}{2}}(2x)}{(x^2 - 2)^2} \\ &= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{(x^2 - 2)^2} \\ &= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{(x^2 - 2)^2} \end{aligned}$$

EXAMPLE:

Suppose that the amount of air in a balloon at any time t is given by

$$V(t)=\frac{6\sqrt[3]{t}}{4t+1}$$

Determine if the balloon is being filled with air or being drained of air at $t=8$.

Solution

$$V'(t)=\frac{2t^{-\frac{2}{3}}(4t+1)-6t^{\frac{1}{3}}(4)}{(4t+1)^2}\quad=\frac{-16t^{\frac{1}{3}}+2t^{-\frac{2}{3}}}{(4t+1)^2}$$

$$=\frac{-16t^{\frac{1}{3}}+\frac{2}{t^{\frac{2}{3}}}}{(4t+1)^2}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate. The rate of change of the volume at t=8 is then,

$$V'(8)=\frac{-16(2)+\frac{2}{4}}{(33)^2}\qquad (8)^{\frac{1}{3}}=2\qquad (8)^{\frac{2}{3}}=\left((8)^{\frac{1}{3}}\right)^2=(2)^2=4$$

$$=-\frac{63}{2178}=-\frac{7}{242}$$

So, the rate of change of the volume at $t=8$ is negative and so the volume must be decreasing. Therefore air is being drained out of the balloon at $t=8$.

Derivatives of Trig Functions.

Example

(a) $\lim\limits_{\theta\to 0}\frac{\sin\theta}{6\theta}$ **(b)** $\lim\limits_{x\to 0}\frac{\sin(6x)}{x}$ **(c)** $\lim\limits_{x\to 0}\frac{x}{\sin(7x)}$

(d) $\lim\limits_{t\to 0}\frac{\sin(3t)}{\sin(8t)}$ **(e)** $\lim\limits_{x\to 4}\frac{\sin(x-4)}{x-4}$ **(f)** $\lim\limits_{z\to 0}\frac{\cos(2z)-1}{z}$

Solution

(a) $\lim\limits_{\theta\to 0}\frac{\sin\theta}{6\theta}$

$$\lim_{\theta\to 0}\frac{\sin\theta}{6\theta}=\frac{1}{6}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{1}{6}(1)=\frac{1}{6}$$

(b) $\lim\limits_{x\to 0}\frac{\sin(6x)}{x}$

Doing the change of variables on this limit gives,

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(6x)}{x}&=6\lim_{x\to 0}\frac{\sin(6x)}{6x} \qquad \text{let } \theta=6x\\&=6\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\\&=6(1)\\&=6\end{aligned}$$

(c) $\lim\limits_{x\to 0}\dfrac{x}{\sin(7x)}$

$$\lim_{x\to 0}\frac{x}{\sin(7x)}=\frac{1}{\lim\limits_{x\to 0}\dfrac{7\sin(7x)}{7x}}=\frac{1}{7\lim\limits_{x\to 0}\dfrac{\sin(7x)}{7x}}$$

$$=\frac{1}{(7)(1)}=\frac{1}{7}$$

(d) $\lim\limits_{t\to 0}\dfrac{\sin(3t)}{\sin(8t)}$

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}=\lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}$$

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}=\lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}\frac{t}{t}=\lim_{t\to 0}\frac{\sin(3t)}{t}\frac{t}{\sin(8t)}$$

$$=\left(\lim_{t\to 0}\frac{\sin(3t)}{t}\right)\left(\lim_{t\to 0}\frac{t}{\sin(8t)}\right)$$

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}=\left(\lim_{t\to 0}\frac{3\sin(3t)}{3t}\right)\left(\lim_{t\to 0}\frac{8t}{8\sin(8t)}\right)$$

$$=\left(3\lim_{t\to 0}\frac{\sin(3t)}{3t}\right)\left(\frac{1}{8}\lim_{t\to 0}\frac{8t}{\sin(8t)}\right)=(3)\left(\frac{1}{8}\right)=\frac{3}{8}$$

(e) $\lim\limits_{x\to 4}\dfrac{\sin(x-4)}{x-4}$

So, let $\theta = x-4$ and then notice that as $x\to 4$ we have $\theta\to 0$. Therefore, after doing the change of variable the limit becomes,

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

(f) $\lim\limits_{z\to 0}\dfrac{\cos(2z)-1}{z}$

$$\lim_{z\to 0}\frac{\cos(2z)-1}{z}=\lim_{z\to 0}\frac{2(\cos(2z)-1)}{2z}$$

$$=2\lim_{z\to 0}\frac{\cos(2z)-1}{2z}\qquad =2(0)\qquad 0$$

Example

(a) $g(x)=3\sec(x)-10\cot(x)$

(b) $h(w)=3w^{-4}-w^{2}\tan(w)$

(c) $y=5\sin(x)\cos(x)+4\csc(x)$

(d) $P(t)=\dfrac{\sin(t)}{3-2\cos(t)}$

Solution

(a) $g(x) = 3\sec(x) - 10\cot(x)$

$$g'(x) = 3\sec(x)\tan(x) - 10\left(-\csc^2(x)\right)$$
$$= 3\sec(x)\tan(x) + 10\csc^2(x)$$

(b) $h(w) = 3w^{-4} - w^2\tan(w)$

$$h'(w) = -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right)$$
$$= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

$$y' = 5\cos(x)\cos(x) + 5\sin(x)\left(-\sin(x)\right) - 4\csc(x)\cot(x)$$
$$= 5\cos^2(x) - 5\sin^2(x) - 4\csc(x)\cot(x)$$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

$$P'(t) = \frac{\cos(t)\left(3 - 2\cos(t)\right) - \sin(t)\left(2\sin(t)\right)}{\left(3 - 2\cos(t)\right)^2}$$
$$= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{\left(3 - 2\cos(t)\right)^2}$$

$$P'(t)=\frac{3\cos(t)-2\left(\cos^2(t)+\sin^2(t)\right)}{\left(3-2\cos(t)\right)^2}$$

$$=\frac{3\cos(t)-2}{\left(3-2\cos(t)\right)^2}$$

Example

(a) $R(w)=4^w-5\log_9 w$

(b) $f(x)=3\mathbf{e}^x+10x^3\ln x$

(c) $y=\dfrac{5\mathbf{e}^x}{3\mathbf{e}^x+1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w)=4^w\ln 4-\frac{5}{w\ln 9}$$

(b) Not much to this one. Just remember to use the product rule on the second term.

$$f'(x)=3\mathbf{e}^x+30x^2\ln x+10x^3\left(\frac{1}{x}\right)$$

$$=3\mathbf{e}^x+30x^2\ln x+10x^2$$

(c) We'll need to use the quotient rule on this one.

$$y'=\frac{5\mathbf{e}^x\left(3\mathbf{e}^x+1\right)-\left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x+1\right)^2}=\frac{15\mathbf{e}^{2x}+5\mathbf{e}^x-15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x+1\right)^2}$$

$$=\frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x+1\right)^2}$$

Chain Rule.

Example

(a) $f(x) = \sin(3x^2 + x)$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$

(d) $g(x) = \ln(x^{-4} + x^4)$

(e) $y = \sec(1 - 5x)$

(f) $P(t) = \cos^4(t) + \cos(t^4)$

Solution

(a) $f(x) = \sin(3x^2 + x)$

It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(3x^2 + x)}_{\substack{\text{leave inside}\\ \text{function alone}}} \underbrace{(6x + 1)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

$$f'(x) = (6x + 1)\cos(3x^2 + x)$$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\begin{aligned} f'(t) &= 50(2t^3 + \cos(t))^{49}(6t^2 - \sin(t)) \\ &= 50(6t^2 - \sin(t))(2t^3 + \cos(t))^{49} \end{aligned}$$

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$

$$h'(w) = \mathbf{e}^{w^4 - 3w^2 + 9}\left(4w^3 - 6w\right)$$
$$= \left(4w^3 - 6w\right)\mathbf{e}^{w^4 - 3w^2 + 9}$$

(d) $g(x) = \ln\left(x^{-4} + x^4\right)$

$$g'(x) = \frac{1}{x^{-4} + x^4}\left(-4x^{-5} + 4x^3\right) = \frac{-4x^{-5} + 4x^3}{x^{-4} + x^4}$$

(e) $y = \sec(1 - 5x)$

$$y' = \sec(1 - 5x)\tan(1 - 5x)(-5)$$
$$= -5\sec(1 - 5x)\tan(1 - 5x)$$

(f) $P(t) = \cos^4(t) + \cos\left(t^4\right)$

$$P'(t) = 4\cos^3(t)(-\sin(t)) - \sin\left(t^4\right)\left(4t^3\right)$$
$$= -4\sin(t)\cos^3(t) - 4t^3\sin\left(t^4\right)$$

(a) $T(x) = \tan^{-1}(2x)\sqrt[3]{1 - 3x^2}$

(b) $y = \dfrac{\left(x^3 + 4\right)^5}{\left(1 - 2x^2\right)^3}$

Solution

(a) $T(x) = \tan^{-1}(2x)\sqrt[3]{1-3x^2}$

This requires the product rule and each derivative in the product rule will require a chain rule application as well.

$$T'(x) = \frac{1}{1+(2x)^2}(2)\left(1-3x^2\right)^{\frac{1}{3}} + \tan^{-1}(2x)\left(\frac{1}{3}\right)\left(1-3x^2\right)^{-\frac{2}{3}}(-6x)$$

$$= \frac{2\left(1-3x^2\right)^{\frac{1}{3}}}{1+(2x)^2} - 2x\left(1-3x^2\right)^{-\frac{2}{3}}\tan^{-1}(2x)$$

(b) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

In this case we will be using the chain rule in concert with the quotient rule.

$$y' = \frac{5\left(x^3+4\right)^4\left(3x^2\right)\left(1-2x^2\right)^3 - \left(x^3+4\right)^5(3)\left(1-2x^2\right)^2(-4x)}{\left(\left(1-2x^2\right)^3\right)^2}$$

These tend to be a little messy. Notice that when we go to simplify that we'll be able to a fair amount of factoring in the numerator and this will often greatly simplify the derivative.

$$y' = \frac{\left(x^3+4\right)^4\left(1-2x^2\right)^2\left(5\left(3x^2\right)\left(1-2x^2\right) - \left(x^3+4\right)(3)(-4x)\right)}{\left(1-2x^2\right)^6}$$

$$= \frac{3x\left(x^3+4\right)^4\left(5x-6x^3+16\right)}{\left(1-2x^2\right)^4}$$

Implicit Differentiation.

Example

Find y' for $xy = 1$.

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

Solution 2 :

$$xy = x\,y(x) = 1$$

$$\frac{d}{dx}\left(x\,y(x)\right) = \frac{d}{dx}(1)$$

$$(1)\,y(x) + x\frac{d}{dx}\left(y(x)\right) = 0$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}\left(y(x)\right) = \frac{dy}{dx} = y'$$

Using this we get the following,

$$y + xy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = -\frac{1/x}{x} = -\frac{1}{x^2}$$

Example

(a) $\left(5x^3 - 7x + 1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

(b) $\sin\left(3 - 6x\right)$, $\sin\left(y(x)\right)$

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

Solution

(a) $\left(5x^3-7x+1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

With the first function here we're being asked to do the following,

$$\frac{d}{dx}\left[\left(5x^3-7x+1\right)^5\right]=5\left(5x^3-7x+1\right)^4\left(15x^2-7\right)$$

$$\frac{d}{dx}\left[f(x)\right]^5=5\left[f(x)\right]^4 f'(x)$$

(b) $\sin(3-6x)$

$$\frac{d}{dx}\left[\sin(3-6x)\right]=-6\cos(3-6x)$$

$$\frac{d}{dx}\left[\sin(y(x))\right]=y'(x)\cos(y(x))$$

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

$$\frac{d}{dx}\left(\mathbf{e}^{x^2-9x}\right)=(2x-9)\mathbf{e}^{x^2-9x}$$

$$\frac{d}{dx}\left(\mathbf{e}^{y(x)}\right)=y'(x)\mathbf{e}^{y(x)}$$

Example

Find y' for the following function. $x^2 + y^2 = 9$

$$\frac{d}{dx}\left(x^2 + \left[y(x)\right]^2\right) = \frac{d}{dx}(9)$$

All we need to do for the second term is use the chain rule. After taking the derivative we have,

$$2x + 2\left[y(x)\right]^1 y'(x) = 0$$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Example

Find the equation of the tangent line to $x^2 + y^2 = 9$

at the point $\left(2, \sqrt{5}\right)$.

The tangent line then is given by,

$$y = f(a) + f'(a)(x - a)$$

Solution

$$m = y'\big|_{x=2,\, y=\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

The tangent line is then. $y = \sqrt{5} - \frac{2}{\sqrt{5}}(x-2)$

Example Find y' for each of the following.

(a) $x^3y^5 + 3x = 8y^3 + 1$

(b) $x^2\tan(y) + y^{10}\sec(x) = 2x$

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln(xy^3)$

Solution

(a) $x^3y^5 + 3x = 8y^3 + 1$

Here is the differentiation of each side for this function.

$$3x^2y^5 + 5x^3y^4y' + 3 = 24y^2y'$$

Then factor y' out of all the terms containing it and divide both

sides by the "coefficient" of the y'. Here is the solving work for this one,

$$3x^2y^5 + 3 = 24y^2y' - 5x^3y^4y'$$

$$3x^2y^5 + 3 = (24y^2 - 5x^3y^4)y'$$

$$y' = \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4}$$

(b) $x^2\tan(y)+y^{10}\sec(x)=2x$

We've got two product rules to deal with this time. Here is the derivative of this function

$$2x\tan(y)+x^2\sec^2(y)\,y'+10y^9y'\sec(x)+y^{10}\sec(x)\tan(x)=2$$

Now, solve for the derivative.

$$\left(x^2\sec^2(y)+10y^9\sec(x)\right)y'=2-y^{10}\sec(x)\tan(x)-2x\tan(y)$$

$$y'=\frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2\sec^2(y)+10y^9\sec(x)}$$

(c) $\mathbf{e}^{2x+3y}=x^2-\ln\left(xy^3\right)$

Here is the derivative of this equation.

$$\mathbf{e}^{2x+3y}\left(2+3y'\right)=2x-\frac{y^3+3xy^2y'}{xy^3}$$

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y}=2x-\frac{y^3}{xy^3}-\frac{3xy^2y'}{xy^3}$$

$$2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y}=2x-\frac{1}{x}-\frac{3y'}{y}$$

$$\left(3\mathbf{e}^{2x+3y}+3y^{-1}\right)y'=2x-x^{-1}-2\mathbf{e}^{2x+3y}$$

$$y'=\frac{2x-x^{-1}-2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y}+3y^{-1}}$$

Higher Order Derivatives.

Example

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

(b) $y = \cos x$

(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$

Solution

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

There really isn't a lot to do here other than do the derivatives.

$$R'(t) = 6t + 4t^{-\frac{1}{2}} + \mathbf{e}^t$$

$$R''(t) = 6 - 2t^{-\frac{3}{2}} + \mathbf{e}^t$$

$$R'''(t) = 3t^{-\frac{5}{2}} + \mathbf{e}^t$$

$$R^{(4)}(t) = -\frac{15}{2}t^{-\frac{7}{2}} + \mathbf{e}^t$$

(b) $y = \cos x$

Again, let's just do some derivatives.

$$y = \cos x$$
$$y' = -\sin x$$
$$y'' = -\cos x$$
$$y''' = \sin x$$
$$y^{(4)} = \cos x$$

(c) $f(y)=\sin(3y)+\mathbf{e}^{-2y}+\ln(7y)$

$$f'(y)=3\cos(3y)-2\mathbf{e}^{-2y}+\frac{1}{y}=3\cos(3y)-2\mathbf{e}^{-2y}+y^{-1}$$

$$f''(y)=-9\sin(3y)+4\mathbf{e}^{-2y}-y^{-2}$$

$$f'''(y)=-27\cos(3y)-8\mathbf{e}^{-2y}+2y^{-3}$$

$$f^{(4)}(y)=81\sin(3y)+16\mathbf{e}^{-2y}-6y^{-4}$$

Example

$$y=\frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. However, that would be a fairly messy process. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y=\ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y=\ln\left(x^5\right)-\ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y=\ln\left(x^5\right)-\ln(1-10x)-\ln\left(\sqrt{x^2+2}\right)$$

What we need to do at this point is differentiate both sides with respect to x. Note that this is really implicit differentiation.

$$\frac{y'}{y}=\frac{5x^4}{x^5}-\frac{-10}{1-10x}-\frac{\frac{1}{2}\left(x^2+2\right)^{-\frac{1}{2}}(2x)}{\left(x^2+2\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y}=\frac{5}{x}+\frac{10}{1-10x}-\frac{x}{x^2+2}$$

To finish the problem all that we need to do is multiply both sides by y and the plug in for y since we do know what that is.

$$y' = y\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$
$$= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

Problems.

1. $f(x) = 6x^3 - 9x + 4$
2. $y = 2t^4 - 10t^2 + 13t$
3. $g(z) = 4z^7 - 3z^{-7} + 9z$
4. $h(y) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$
5. $y = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$
6. $f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$
7. $f(t) = \frac{4}{t} - \frac{1}{6t^3} + \frac{8}{t^5}$
8. $R(z) = \frac{6}{\sqrt{z^3}} + \frac{1}{8z^4} - \frac{1}{3z^{10}}$
9. $z = x(3x^2 - 9)$
10. $g(y) = (y-4)(2y + y^2)$
11. $h(x) = \frac{4x^3 - 7x + 8}{x}$
12. $f(y) = \frac{y^5 - 5y^3 + 2y}{y^3}$

Product and Quotient Rule

For problems 1 – 6 use the Product Rule or the Quotient Rule to find the derivative of the given function.

1. $f(t) = (4t^2 - t)(t^3 - 8t^2 + 12)$

2. $y = \left(1 + \sqrt{x^3}\right)\left(x^{-3} - 2\sqrt[3]{x}\right)$

3. $h(z) = (1 + 2z + 3z^2)(5z + 8z^2 - z^3)$

4. $g(x) = \dfrac{6x^2}{2 - x}$

5. $R(w) = \dfrac{3w + w^4}{2w^2 + 1}$

6. $f(x) = \dfrac{\sqrt{x} + 2x}{7x - 4x^2}$

7. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$ determine the value of $(f\,g)'(2)$.

8. If $f(x) = x^3 g(x)$, $g(-7) = 2$, $g'(-7) = -9$ determine the value of $f'(-7)$.

9. Find the equation of the tangent line to $f(x) = \left(1 + 12\sqrt{x}\right)(4 - x^2)$ at $x = 9$.

10. Determine where $f(x) = \dfrac{x - x^2}{1 + 8x^2}$ is increasing and decreasing.

11. Determine where $V(t) = (4 - t^2)(1 + 5t^2)$ is increasing and decreasing.

Derivatives of Trig Functions

1. $\lim\limits_{z\to 0}\dfrac{\sin(10z)}{z}$

2. $\lim\limits_{\alpha\to 0}\dfrac{\sin(12\alpha)}{\sin(5\alpha)}$

3. $\lim\limits_{x\to 0}\dfrac{\cos(4x)-1}{x}$

4. $f(x)=2\cos(x)-6\sec(x)+3$

5. $g(z)=10\tan(z)-2\cot(z)$

6. $f(w)=\tan(w)\sec(w)$

Derivatives of Exponential and Logarithm Functions

1. $f(x)=2\mathbf{e}^x-8^x$

2. $g(t)=4\log_3(t)-\ln(t)$

3. $R(w)=3^w\log(w)$

4. $y=z^5-\mathbf{e}^z\ln(z)$

5. Find the tangent line to $f(x)=7^x+4\mathbf{e}^x$ at $x=0$.

6. Find the tangent line to $f(x)=\ln(x)\log_2(x)$ at $x=2$.

Chain Rule

1. $f(x)=\left(6x^2+7x\right)^4$

7. $f(t)=5+\mathbf{e}^{4t+t^7}$

2. $g(t)=\left(4t^2-3t+2\right)^{-2}$

8. $g(x)=\mathbf{e}^{1-\cos(x)}$

3. $y=\sqrt[3]{1-8z}$

9. $H(z)=2^{1-6z}$

4. $R(w)=\csc(7w)$

10. $u(t)=\tan^{-1}(3t-1)$

5. $G(x)=2\sin(3x+\tan(x))$

11. $F(y)=\ln\left(1-5y^2+y^3\right)$

6. $h(u)=\tan(4+10u)$

12. $V(x)=\ln(\sin(x)-\cot(x))$

Implicit Differentiation

For problems 1 – 3 do each of the following.

(a) Find y' by solving the equation for y and differentiating directly.
(b) Find y' by implicit differentiation.
(c) Check that the derivatives in **(a)** and **(b)** are the same.

1. $\dfrac{x}{y^3}=1$

2. $x^2+y^3=4$

3. $x^2+y^2=2$

For problems 4 – 9 find y' by implicit differentiation.

4. $2y^3 + 4x^2 - y = x^6$

5. $7y^2 + \sin(3x) = 12 - y^4$

6. $\mathbf{e}^x - \sin(y) = x$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

8. $\cos\left(x^2 + 2y\right) + x\,\mathbf{e}^{y^2} = 1$

9. $\tan\left(x^2y^4\right) = 3x + y^2$

APPLICATIONS OF DERIVATIVES

Extreme Values of Functions

This section shows how to locate and identify extreme values of a continuous function from its derivative. Once we can do this, we can solve a variety of *optimization problems* in which we find the optimal (best) way to do something in a given situation.

DEFINITIONS **Absolute Maximum, Absolute Minimum**

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

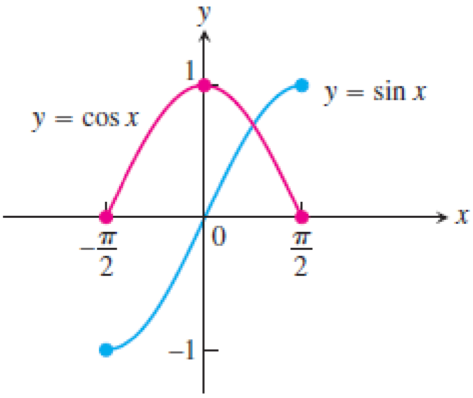
$$f(x) \le f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in } D.$$

Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*). Absolute extrema are also called **global** extrema, to distinguish them from *local extrema* defined below.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1



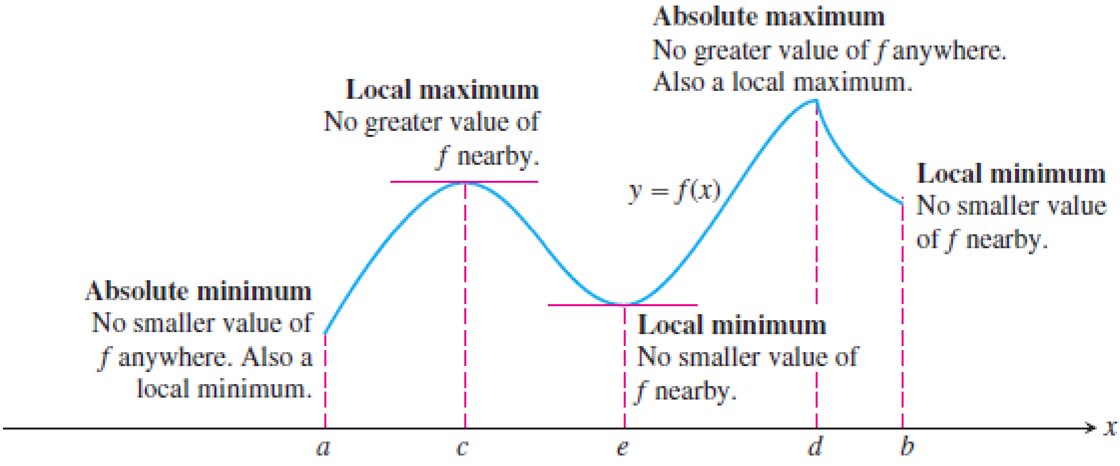
EXAMPLE

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

Local (Relative) Extreme Values



smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

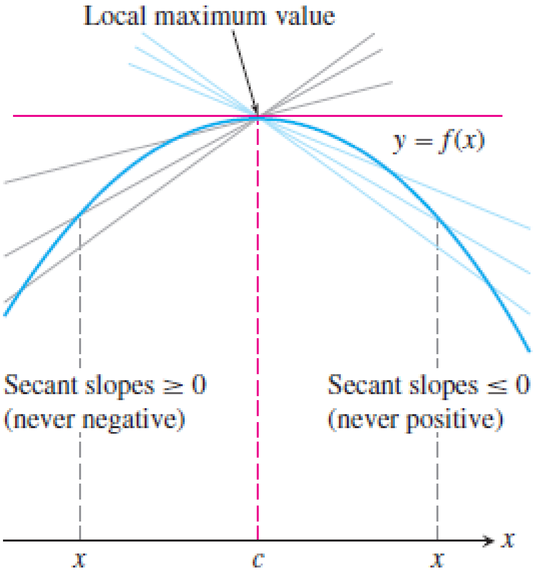
To begin, suppose that f has a local maximum value at $x = c$ so that

$f(x) - f(c) \leq 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c} \le 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \le f(c) \qquad (1)$$



Similarly,

$$f'(c) = \lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} \ge 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \le f(c) \qquad (2)$$

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \ge f(c)$, which reverses the inequalities in Equations (1) and (2). ■

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

DEFINITION **Critical Point**

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Thus the only domain points where a function can assume extreme values are critical points and endpoints.

Be careful not to misinterpret Theorem 2 because its converse is false. A differentiable function may have a critical point at $x = c$ without having a local extreme value there. For instance, the function $f(x) = x^3$ has a critical point at the origin and zero value there, but is positive to the right of the origin and negative to the left. So it cannot have a local extreme value at the origin. Instead, it has a *point of inflection* there. This idea is de-

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

EXAMPLE

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

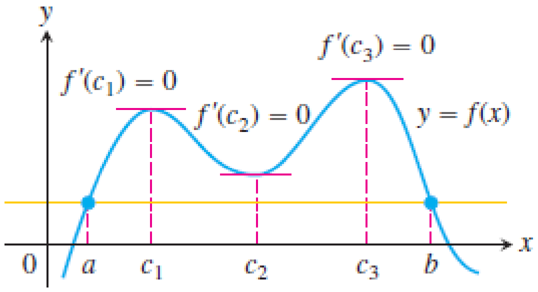
Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

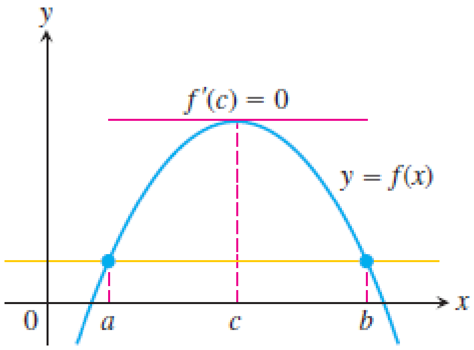
$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

The Mean Value Theorem

Rolle's Theorem



(b)



(a)

THEOREM 3 Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

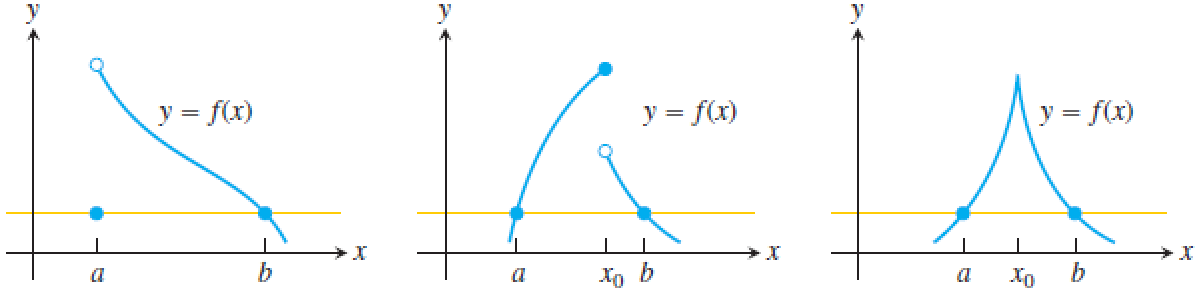
$$f'(c) = 0.$$

1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c between a and b, then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's theorem.

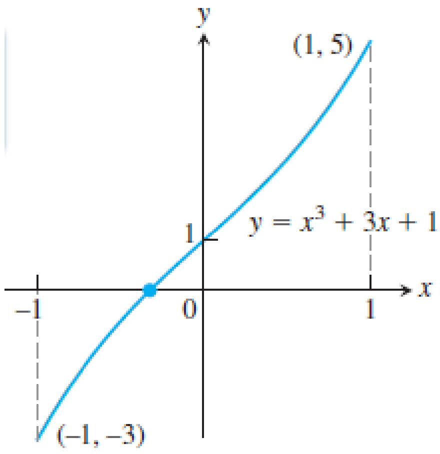
If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b). ■



(a) Discontinuous at an endpoint of $[a, b]$ (b) Discontinuous at an interior point of $[a, b]$ (c) Continuous on $[a, b]$ but not differentiable at an interior point

EXAMPLE Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution Let $y = f(x) = x^3 + 3x + 1$.



$$f'(x) = 3x^2 + 3$$

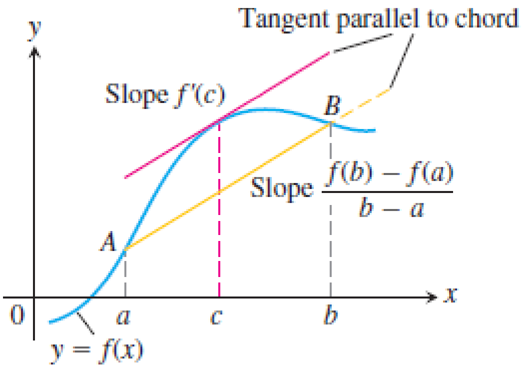
is never zero (because it is always positive). Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. Therefore, f has no more than one zero. It does in fact have one zero, because the Intermediate Value Theorem tells us that the graph of $y = f(x)$ crosses the x-axis somewhere between $x = -1$ (where $y = -3$) and $x = 0$

The Mean Value Theorem

THEOREM 4 The Mean Value Theorem
Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$



(point-slope equation). The vertical difference between the graphs of f and g at x is

$$h(x) = f(x) - g(x)$$

$$= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3)} \ldots$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \ldots \text{with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

EXAMPLE Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0, 2).

Solution Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$ for some constant C. The value of C can be determined from the condition that $f(0) = 2$ (the graph of f passes through (0, 2)):

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$.

Finding Velocity and Position from Acceleration

Here is how to find the velocity and displacement functions of a body falling freely from rest with acceleration 9.8 $\mathrm{m/sec^2}$.

We know that $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \qquad \text{and} \qquad C = 0.$$

The velocity function must be $v(t) = 9.8t$. How about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $f(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C$$

for some constant C. If the initial height is $s(0) = h$, measured positive downward from the rest position, then

$$4.9(0)^2 + C = h, \qquad \text{and} \qquad C = h.$$

The position function must be $s(t) = 4.9t^2 + h$.

Monotonic Functions and The First Derivative Test

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section defines precisely what it means for a function to be increasing or decreasing over an interval, and gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function for the presence of local extreme values.

Increasing Functions and Decreasing Functions

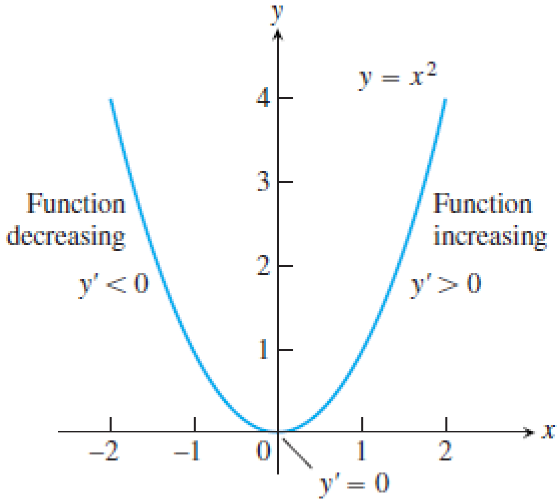
What kinds of functions have positive derivatives or negative derivatives? The answer, provided by the Mean Value Theorem's third corollary, is this: The only functions with positive derivatives are increasing functions; the only functions with negative derivatives are decreasing functions.

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.



COROLLARY 3 First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

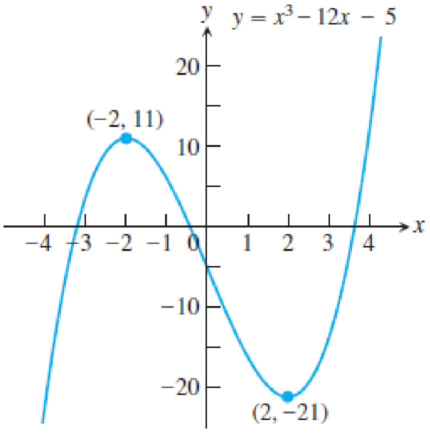
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

EXAMPLE

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$



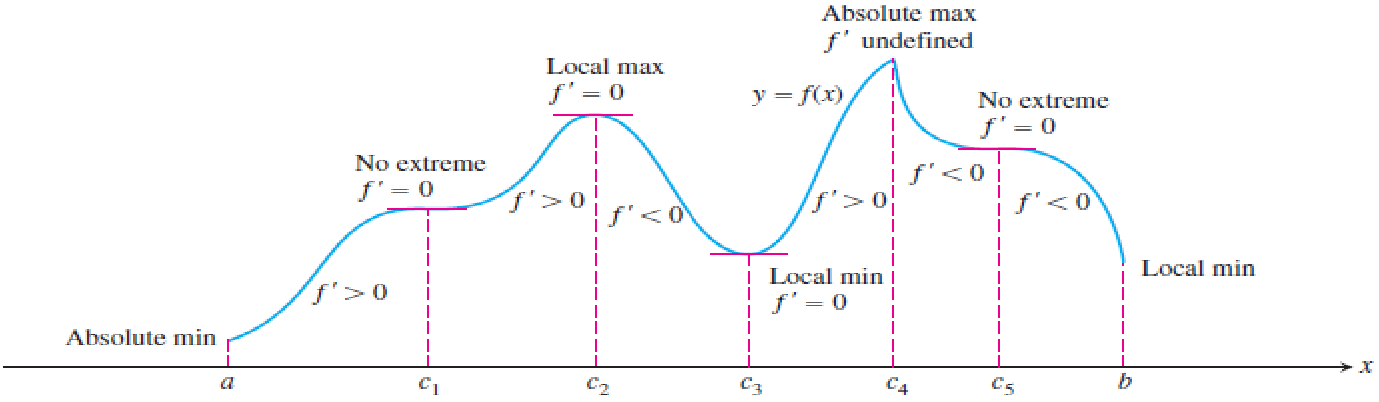
is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f into intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f at a convenient point in each subinterval. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summa-

Intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' Evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

First Derivative Test for Local Extrema

at the points where f has a minimum value, $f' < 0$ immediately to the left

and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.



These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

EXAMPLE Find the critical points of $f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}\left(x - 1\right) = \frac{4(x - 1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points. We can display the information in a table like the following:

Intervals	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Concavity and Curve Sketching

Concavity

the curve $y = x^3$ rises as x increases, but the portions de- fined on the intervals

$(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the ori- gin

from the left along the curve, the curve turns to our right and falls below its tangents.

The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

DEFINITION **Concave Up, Concave Down**

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

EXAMPLE Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is

negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive

Points of Inflection

DEFINITION **Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the Intermediate Value Property) or undefined. If y is a twice-differentiable function, $y'' = 0$ at a point of inflection and y' has a local maximum or minimum.

A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero when $t = 1$ and $t = 11/3$.

Intervals	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest), at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Intervals	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

The accelerating force is directed toward the left during the time interval $[0, 7/3]$, is momentarily zero at $t = 7/3$, and is directed toward the right thereafter. ■

Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and character of local extrema.

THEOREM 5 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

EXAMPLE Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or th origin (Section 1.4).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$, y-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2\cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative Test. We will see in Step 6 that both are absolute extrema as well.

4. *Increasing and decreasing*. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $\left(-\infty, -\sqrt{3}\right)$, positive on $\left(-\sqrt{3}, 0\right)$, negative on $\left(0, \sqrt{3}\right)$, and positive again on $\left(\sqrt{3}, \infty\right)$. Thus each point is a point of inflection. The curve is concave down on the interval $\left(-\infty, -\sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$, concave down on $\left(0, \sqrt{3}\right)$, and concave up again on $\left(\sqrt{3}, \infty\right)$.

6. *Asymptotes*. Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

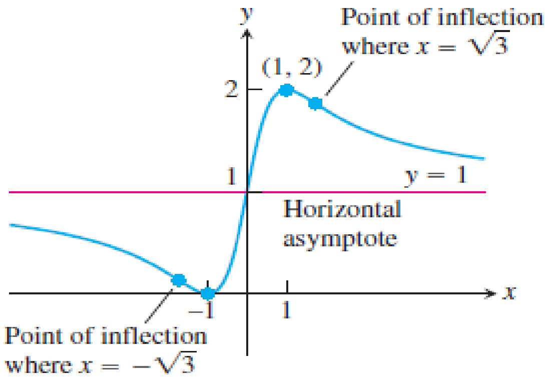
$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$

$$= \frac{1+(2/x)+(1/x^2)}{(1/x^2)+1}. \qquad \text{Dividing by } x^2$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \leq y \leq 2$).

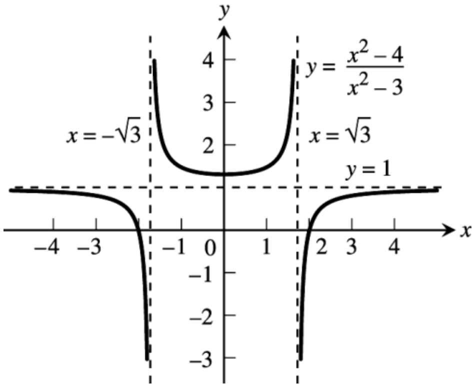
7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■



Graph the function: $y = \frac{x^2-4}{x^2-3}$

* ***X-Intercept*** $\xrightarrow{yields}$ $y = 0 \xrightarrow{yields} x = \pm 2$

* ***y-intercept*** $\xrightarrow{yields} x = 0 \xrightarrow{yields} y = \frac{4}{3}$



* ***Vertical asymipitot*** $\xrightarrow{yields}$**x=**$\pm\sqrt{3}$

* ***Horizontal asym.*** $\lim_{x\to\infty} \frac{x^2-4}{x^2-3} = 1$

* $\frac{dy}{dx} = \frac{(x^2-3)(2x)-(x^2+1).2x}{(x^2-3)^2} = \frac{2x}{(x^2-3)^2}$

*$\frac{d^2y}{dx^2} = \frac{(x^2-3)^2.2-(2x).2(x^2-3).2x}{(x^2-3)^4} = \frac{-6(x^2+1)}{(x^2-3)^3}$

* $\frac{dy}{dx} ------*^{-\sqrt{3}}-----*^{0}++++++*^{\sqrt{3}}++++++$

* $\frac{d^2y}{dx^2}$ $-----*^{-\sqrt{3}}+++++++++++++*^{\sqrt{3}}------$

Graph the function : $y = \frac{x^2}{x^2-9}$

**Graph symmetric about y-axis*

**x-intercept y=0→* $x = 0 \to \quad point(0,0)$

**y-intercept x=0→* $y = 0 \to \quad point(0,0)$

**vertical asymptote ,*$x^2 - 9 = 0 \to x = \pm 3$

**Horizontal asymptote,* $\lim_{x\to\to\infty} \frac{x^2}{x^2+9} = 1$

*$\frac{dy}{dx} = \frac{2x(x^2-)-x^2.2x}{(x^2-9)^2} = \frac{-18x}{(x^2-9)^2}$

$\frac{dy}{dx}$ *++++++++-3++++++0---------3-----------*

*$\frac{d^2y}{dx^2} = \frac{-18(x^2-9)^2+18x\{2(x^2-9).2x\}}{(x^2-9)^2} = \frac{54x^2-162}{(x^2-9)^3} = \frac{54(x^2-3)}{(x^2-9)^3}$

$\frac{d^2y}{dx^2} +++++3-----3+++$

3. _Graph the equation:_ $y = -\frac{x^2-2}{x^2-1}$

$* \;\; x - intercept \xrightarrow{yields} y = 0 \xrightarrow{yields} -x^2 + 2 = 0 \xrightarrow{yields} x = \pm\sqrt{2},$

The point$(\sqrt{2}, 0), (-\sqrt{2}. 0)$

$* \;\; y - intercept \xrightarrow{yields} x = 0 \xrightarrow{yields} -\frac{0-2}{0-1} = -2$

the point$(0 \;\;, -2)$

$* \, vertical\ asymptote \rightarrow x^2 - 1 = 0 \xrightarrow{yields} x = \pm 1$

$* \, horizontal\ asymptote \rightarrow \lim_{x\to\infty} -\frac{x^2-2}{x^2-1} = -1$

$$\frac{dy}{dx} = \frac{(x^2-1)*(-2x)-(-x^2+2)*(2x)}{(x^2-1)^2} = \frac{-2x^3+2x+2x^3-4x}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

$$\left(\frac{dy}{dx}\right) \quad \pm\pm\pm\pm\pm\frac{\infty}{-1}\pm\pm\pm\pm\pm\frac{0}{0}======\frac{\infty}{1}======$$

$$\frac{d^2y}{dx^2} = \frac{(x^2-1)^2*(-2)-(-2x)\left[2(x^2-1)*(2x)\right]}{(x^2-1)^4}$$

$$= \frac{(x^2-1)^2*(-2)+8x^2(x^2-1)}{(x^2-1)^4}$$

$$= \frac{-2x^2+2+8x^2}{(x^2-1)^3} = \frac{6x^2+2}{(x^2-1)^3}$$

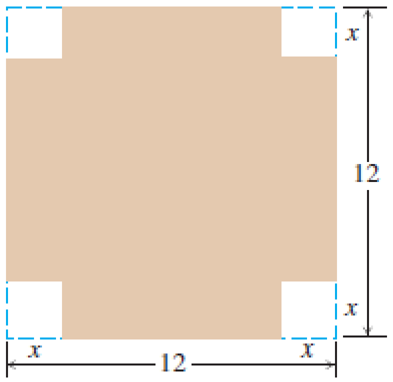
$$\left(\frac{d^2y}{dx^2}\right) \quad +++++\frac{\infty}{-1}------\frac{\infty}{1}+++++++$$

Applied Optimization Problems

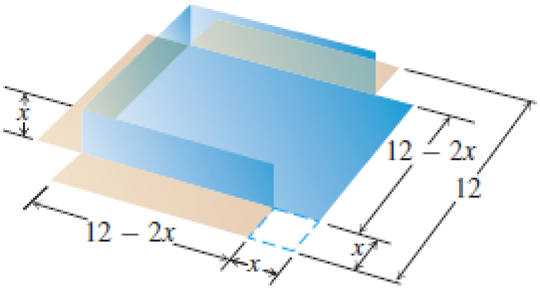
To optimize something means to maximize or minimize some aspect of it. What are the dimensions of a rectangle with fixed perimeter having maximum area? What is the least expensive shape for a cylindrical can? What is the size of the most profitable production run? The differential calculus is a powerful tool for solving problems that call for maximizing or minimizing a function. In this section we solve a variety of optimization problems from business, mathematics, physics, and economics.

EXAMPLE

An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



(a)



(b)

Solution We start with a picture (Figure 4.32). In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Figure 4.33) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

Critical-point value: $V(2) = 128$
Endpoint values: $V(0) = 0, \qquad V(6) = 0.$

The maximum volume is 128 in.^{3}. The cutout squares should be 2 in. on a side. ■

Related Rates.

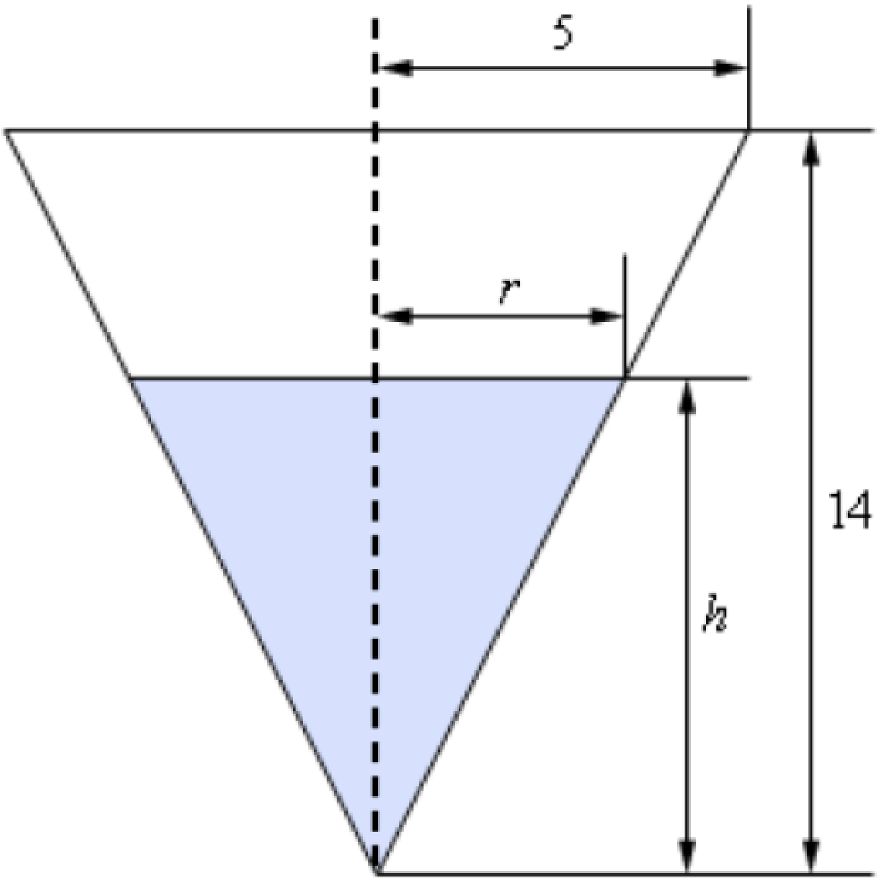
Example.

A tank of water in the shape of a cone is leaking water at a constant rate of $2\,\mathrm{ft}^3/\mathrm{hour}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

Solution.



The volume of water in the tank at any time *t* is given by,

$$V = \frac{1}{3}\pi r^2 h$$

and we've been given that $V' = -2$.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

For this part we need to determine h' when $h = 6$ and now we have a problem.

$$V' = \frac{2}{3}\pi r r' h + \frac{1}{3}\pi r^2 h'$$

When we have two similar triangles then ratios of any two sides will be equal. For our set this means that we have,

$$\frac{r}{h} = \frac{5}{14} \qquad \Rightarrow \qquad r = \frac{5}{14}h$$

If we take this and plug it into our volume formula we have,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 h = \frac{25}{588}\pi h^3$$

$$V' = \frac{25}{196}\pi h^2 h'$$

At this point all we need to do is plug in what we know and solve for h'.

$$-2 = \frac{25}{196}\pi\left(6^2\right)h' \qquad \Rightarrow \qquad h' = \frac{-98}{225\pi} = -0.1386$$

So, it looks like the height is decreasing at a rate of 0.1386 ft/hr.

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

In this case we are asking for r'

$$\frac{h}{r} = \frac{14}{5} \qquad \Rightarrow \qquad h = \frac{14}{5}r$$

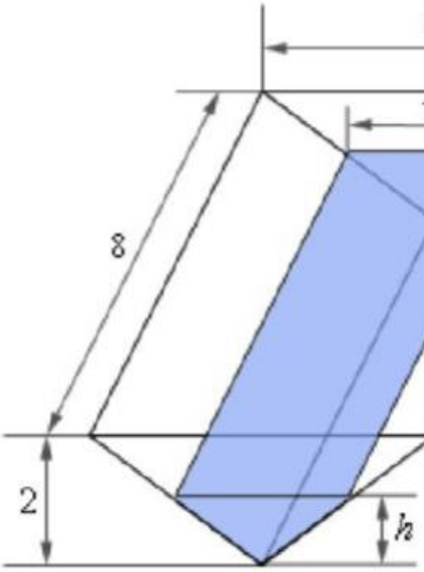
$$r = \frac{5}{14}h \qquad \Rightarrow \qquad r' = \frac{5}{14}h'$$

At this point all we need to do here is use the result from the first part to get,

$$r' = \frac{5}{14}\left(\frac{-98}{225\pi}\right) = -\frac{7}{45\pi} = -0.04951$$

Example .

A trough of water is 8 meters deep and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of $6\ \text{m}^3/\text{sec}$. At what rate is the height of the water changing when the water has a height of 120 cm?



For our case the volume of the water in the tank is,

$$\begin{aligned} V &= (\text{Area of End})(\text{depth}) \\ &= \left(\tfrac{1}{2}\text{base} \times \text{height}\right)(\text{depth}) \\ &= \tfrac{1}{2}hw(8) \\ &= 4hw \end{aligned}$$

with similar triangles ratios of sides must be equal. In our case we'll use,

$$\frac{w}{5} = \frac{h}{2} \qquad \Rightarrow \qquad w = \frac{5}{2}h$$

Plugging this into the volume gives a formula for the volume (and only for this tank) that only involved the height of the water.

$$V = 4hw = 4h\left(\frac{5}{2}h\right) = 10h^2$$

We can now differentiate this to get,

$$V' = 20hh'$$

Finally, all we need to do is plug in and solve for h' .

$$6 = 20(1.2)h' \qquad \Rightarrow \qquad h' = 0.25 \text{ m/sec}$$

So, the height of the water is rising at a rate of 0.25 m/sec.

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Indeterminate Forms and L'Hôpital's Rule

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

Indeterminate Form 0/0

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

from which we calculate derivatives and which always produces the equivalent of 0/0 when we substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x)}{g(x)}. \qquad \blacksquare$$

EXAMPLE Using L'Hôpital's Rule

(a) $\displaystyle \lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1} \bigg|_{x=0} = 2$

(b) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \frac{\dfrac{1}{2\sqrt{1 + x}}}{1} \Bigg|_{x=0} = \frac{1}{2}$

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

EXAMPLE Applying the Stronger Form of L'Hôpital's Rule

(a) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ $\quad \frac{0}{0}$

$$= \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \frac{0}{0}\text{; differentiate again.}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

(b) $\displaystyle \lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\quad \frac{0}{0}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

THEOREM 8 Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Using L'Hôpital's Rule

To find

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE Incorrectly Applying the Stronger Form of L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1+2x} = \frac{0}{1} = 0 \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is wrong. L'Hôpital's Rule can only be applied to limits which give indeterminate forms, and $0/1$ is not an indeterminate form. ■

EXAMPLE Using L'Hôpital's Rule with One-Sided Limits

(a) $\displaystyle \lim_{x\to 0^+} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0.$$

(b) $\displaystyle \lim_{x\to 0^-} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0.$$

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an ambiguous expression like ∞/∞, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form ∞/∞.

In more advanced books it is proved that l'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

EXAMPLE Working with the Indeterminate Form ∞/∞

Find

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$

(b) $\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x}$

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x} \qquad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x} = \lim_{x\to\infty} \frac{1-4x}{6x+5} = \lim_{x\to\infty} \frac{-4}{6} = -\frac{2}{3}.$

EXAMPLE Working with the Indeterminate Form $\infty \cdot 0$

Find

$$\lim_{x\to\infty}\left(x \sin\frac{1}{x}\right)$$

Solution

$$\lim_{x\to\infty}\left(x \sin\frac{1}{x}\right) \qquad \infty\cdot 0$$

$$= \lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) \qquad \text{Let } h = 1/x.$$

$$= 1$$

EXAMPLE Working with the Indeterminate Form $\infty - \infty$

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x}-\frac{1}{x}\to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x}-\frac{1}{x}\to -\infty-(-\infty) = -\infty+\infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x} \qquad \text{Common denominator is } x\sin x$$

Then apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} && \frac{0}{0}\\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} && \text{Still } \frac{0}{0}\\
&= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x}=\frac{0}{2}=0.
\end{aligned}$$

■

INTEGRATION

The Definite Integral

DEFINITION **The Definite Integral as a Limit of Riemann Sums**

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

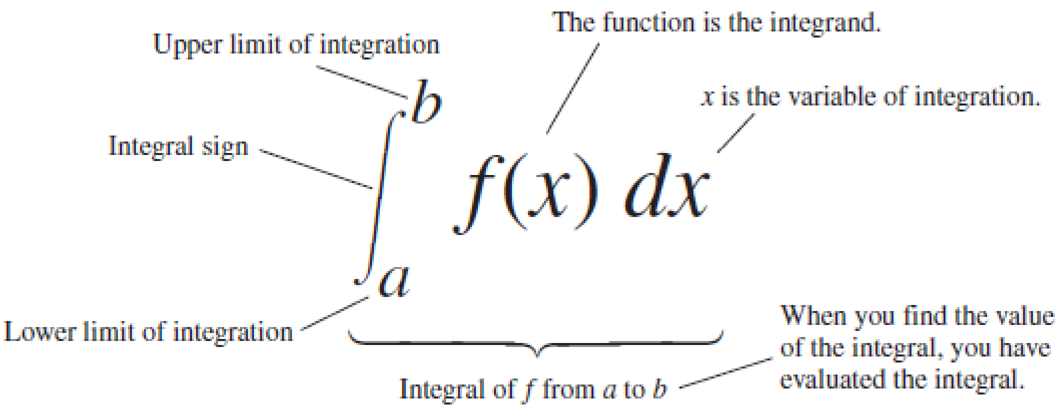
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon .$$

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,dx$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \quad \text{or} \quad \int_a^b f(u)\,du \quad \text{instead of} \quad \int_a^b f(x)\,dx.$$

THEOREM 1 The Existence of Definite Integrals

A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

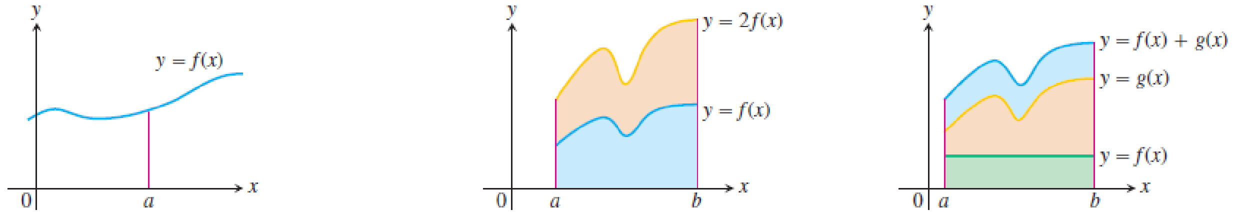
1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ Any Number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

(The area over a point is 0.)

(b) *Constant Multiple:*

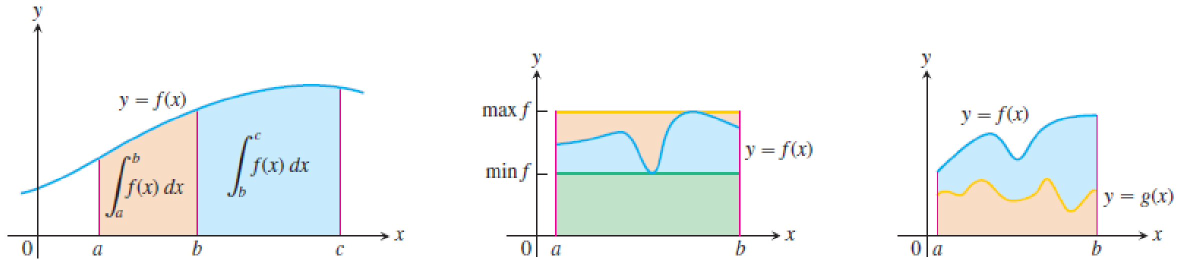
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

(Shown for $k = 2$.)

(c) *Sum:*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)



(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(e) *Max-Min Inequality:*

$$\min f\cdot(b-a) \le \int_a^b f(x)\,dx$$
$$\le \max f\cdot(b-a)$$

(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11

DEFINITION **Area Under a Curve as a Definite Integral**

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

The Fundamental Theorem of Calculus

THEOREM 4 **The Fundamental Theorem of Calculus Part 1**

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (2)$$

EXAMPLE : Use the Fundamental Theorem to find

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\, dt$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\, dt$

(c) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_x^5 3t \sin t\, dt$

(d) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_1^{x^2} \cos t\, dt$

Solution

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\, dt = \cos x$ Eq. 2 with $f(t) = \cos t$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\, dt = \frac{1}{1+x^2}$ Eq. 2 with $f(t) = \dfrac{1}{1+t^2}$

(c) Rule 1 for integrals in Table 5.3 of Section 5.3 sets this up for the Fundamental Theorem.

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\, dt &= \frac{d}{dx}\left(-\int_5^x 3t \sin t\, dt\right) \qquad \text{Rule 1}\\ &= -\frac{d}{dx}\int_5^x 3t\sin t\, dt\\ &= -3x\sin x\end{aligned}$$

(d) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t\, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{d}{du}\int_1^u \cos t\, dt\right)\cdot\frac{du}{dx} = \cos u \cdot \frac{du}{dx} = \cos(x^2)\cdot 2x$$

$$= 2x\cos x^2$$

EXAMPLE Find a function $y = f(x)$ on the domain $(-\pi/2, \pi/2)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

Solution The Fundamental Theorem makes it easy to construct a function with derivative $\tan x$ that equals 0 at $x = 3$:

$$y = \int_3^x \tan t \, dt.$$

Since $y(3) = \int_3^3 \tan t \, dt = 0$, we have only to add 5 to this function to construct one with derivative $\tan x$ whose value at $x = 3$ is 5:

$$f(x) = \int_3^x \tan t \, dt + 5.$$ ■

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2
If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

The theorem says that to calculate the definite integral of f over $[a, b]$ all we need to do is:

1. Find an antiderivative F of f, and
2. Calculate the number $\int_a^b f(x) \, dx = F(b) - F(a)$.

The usual notation for $F(b) - F(a)$ is

(a) $$\int_0^\pi \cos x \, dx = \sin x \Big]_0^\pi = \sin \pi - \sin 0 = 0 - 0 = 0$$

(b) $$\int_{-\pi/4}^0 \sec x \tan x \, dx = \sec x \Big]_{-\pi/4}^0 = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

(c) $$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\ &= [8 + 1] - [5] = 4.\end{aligned}$$

Total Area

EXAMPLE Calculate the area bounded by the x-axis and the parabola $y = 6 - x - x^2$.

Solution We find where the curve crosses the x-axis by setting

$$y = 0 = 6 - x - x^2 = (3 + x)(2 - x),$$

which gives $x = -3$ or $x = 2$.

The curve is sketched in Figure 5.21, and is nonnegative on $[-3, 2]$.

The area is

$$\int_{-3}^{2} (6 - x - x^2)\, dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right) = 20\tfrac{5}{6}.$$

EXAMPLE the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$.

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

Solution The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\, dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x\, dx = -\cos x\Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2.$$

$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2.$$

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$\text{Area} = |2| + |-2| = 4.$$

■

EXAMPLE

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.23). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals,

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

Indefinite Integrals and the Substitution Rule

A definite integral is a number defined by taking the limit of Riemann sums associated with partitions of a finite closed interval whose norms go to zero. The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed easily if we can find an antiderivative of the function. Antiderivatives generally turn out to be more difficult to find than derivatives. However, it is well worth the effort to learn techniques for computing them.

The Power Rule in Integral Form

If u is a differentiable function of x and n is a rational number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n \, du,$$

If u is any differentiable function, then

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1, n \text{ rational}). \tag{1}$$

EXAMPLE

$$\begin{aligned}
\int \sqrt{1+y^2}\cdot 2y\,dy &= \int \sqrt{u}\cdot\left(\frac{du}{dy}\right)dy && \text{Let } u = 1+y^2,\ du/dy = 2y \\
&= \int u^{1/2}\,du \\
&= \frac{u^{(1/2)+1}}{(1/2)+1} + C && \text{Integrate, using Eq. (1) with } n = 1/2. \\
&= \frac{2}{3}u^{3/2} + C && \text{Simpler form} \\
&= \frac{2}{3}(1+y^2)^{3/2} + C && \text{Replace } u \text{ by } 1+y^2.
\end{aligned}$$

■

EXAMPLE

$$\begin{aligned}
\int \sqrt{4t-1}\,dt &= \int \frac{1}{4}\cdot\sqrt{4t-1}\cdot 4\,dt \\
&= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right)dt && \text{Let } u = 4t-1,\ du/dt = 4. \\
&= \frac{1}{4}\int u^{1/2}\,du && \text{With the } 1/4 \text{ out front, the integral is now in standard form.} \\
&= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C && \text{Integrate, using Eq. (1) with } n = 1/2. \\
&= \frac{1}{6}u^{3/2} + C && \text{Simpler form} \\
&= \frac{1}{6}(4t-1)^{3/2} + C && \text{Replace } u \text{ by } 4t-1.
\end{aligned}$$

■

Substitution: Running the Chain Rule Backwards

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof The rule is true because, by the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) \qquad \text{Chain Rule}$$

$$= f(g(x)) \cdot g'(x). \qquad \text{Because } F' = f$$

If we make the substitution $u = g(x)$ then

$$\int f(g(x))g'(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx$$

$$= F(g(x)) + C \qquad \text{Fundamental Theorem} \qquad = F(u) + C \qquad u = g(x)$$

$$= \int F'(u)\,du \qquad \text{Fundamental Theorem} \qquad = \int f(u)\,du \qquad F' = f$$

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x)\,dx, \qquad \int f(u)\,du.$$

when f and g' are continuous functions:

1. Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral
2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

EXAMPLE Using Substitution

$$\int \cos(7\theta + 5)\,d\theta = \int \cos u \cdot \frac{1}{7}\,du \qquad \text{Let } u = 7\theta + 5,\ du = 7\,d\theta,\ (1/7)\,du = d\theta.$$

$$= \frac{1}{7}\int \cos u\,du \qquad \text{With the } (1/7) \text{ out front, the integral is now in standard form.}$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate with respect to } u, \text{ Table 4.2.}$$

$$= \frac{1}{7}\sin(7\theta + 5) + C \qquad \text{Replace } u \text{ by } 7\theta + 5.$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 5)$. ■

EXAMPLE Using Substitution

$$\int x^2 \sin(x^3)\, dx = \int \sin(x^3) \cdot x^2\, dx$$

$$= \int \sin u \cdot \frac{1}{3}\, du \qquad \text{Let } u = x^3,\; du = 3x^2\, dx,\; (1/3)\, du = x^2\, dx.$$

$$= \frac{1}{3}\int \sin u\, du \quad = \frac{1}{3}(-\cos u) + C \qquad \text{Integrate with respect to } u.$$

$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

EXAMPLE Using Identities and Substitution

$$\int \frac{1}{\cos^2 2x}\, dx = \int \sec^2 2x\, dx \qquad \frac{1}{\cos 2x} = \sec 2x$$

$$= \int \sec^2 u \cdot \frac{1}{2}\, du \qquad u = 2x,\; du = 2\, dx,\; dx = (1/2)\, du$$

$$= \frac{1}{2}\int \sec^2 u\, du \quad = \frac{1}{2}\tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \frac{1}{2}\tan 2x + C \qquad u = 2x$$

EXAMPLE Using Different Substitutions $\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}}$.

Solution We can use the substitution method of integration as an exploratory tool: Substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case.

Solution 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\; du = 2z\, dz.$$

$$= \int u^{-1/3}\,du$$ In the form $\int u^n\,du$

$$= \frac{u^{2/3}}{2/3} + C$$ Integrate with respect to u.

$$= \frac{3}{2}u^{2/3} + C \qquad = \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $z^2 + 1$.

Solution 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$

Let $u = \sqrt[3]{z^2+1}$,
$u^3 = z^2 + 1$,
$3u^2\,du = 2z\,dz$.

$$= 3\int u\,du \qquad = 3\cdot\frac{u^2}{2} + C$$ Integrate with respect to u.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $(z^2+1)^{1/3}$.

The Integrals of $\sin^2 x$ and $\cos^2 x$

Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can using the substitution rule. Here is an example giving the integral formulas for $\sin^2 x$ and $\cos^2 x$ which arise frequently in applications.

EXAMPLE

(a)
$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b)
$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$

As in part (a), but with a sign change

EXAMPLE Area Beneath the Curve $y = \sin^2 x$

(a) the definite integral of $g(x)$ over $[0, 2\pi]$.

(b) the area between the graph of the function and the x-axis over $[0, 2\pi]$.

Solution

$$\int_0^{2\pi} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi} = \left[\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right] - \left[\frac{0}{2} - \frac{\sin 0}{4}\right]$$
$$= [\pi - 0] - [0 - 0] = \pi.$$

(b) The function $\sin^2 x$ is nonnegative, so the area is equal to the definite integral, or π.

Substitution and Area Between Curves

Substitution Formula

In the following formula, the limits of integration change when the variable of integration is changed by substitution.

THEOREM 6 Substitution in Definite Integrals
If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

EXAMPLE Substitution by Two Methods

$$\text{Evaluate } \int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx.$$

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 6.

$$\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx$$

$$= \int_0^2 \sqrt{u}\, du$$

Let $u = x^3 + 1$, $du = 3x^2\, dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3 + 1}\, dx = \int \sqrt{u}\, du$$

Let $u = x^3 + 1$, $du = 3x^2\, dx$.

$$= \frac{2}{3}u^{3/2} + C$$ Integrate with respect to u.

$$= \frac{2}{3}(x^3 + 1)^{3/2} + C$$ Replace u by $x^3 + 1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\,dx = \frac{2}{3}(x^3 + 1)^{3/2}\Big]_{-1}^{1}$$ Use the integral just found, with limits of integration for x.

$$= \frac{2}{3}\left[((1)^3 + 1)^{3/2} - ((-1)^3 + 1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

EXAMPLE Using the Substitution Formula

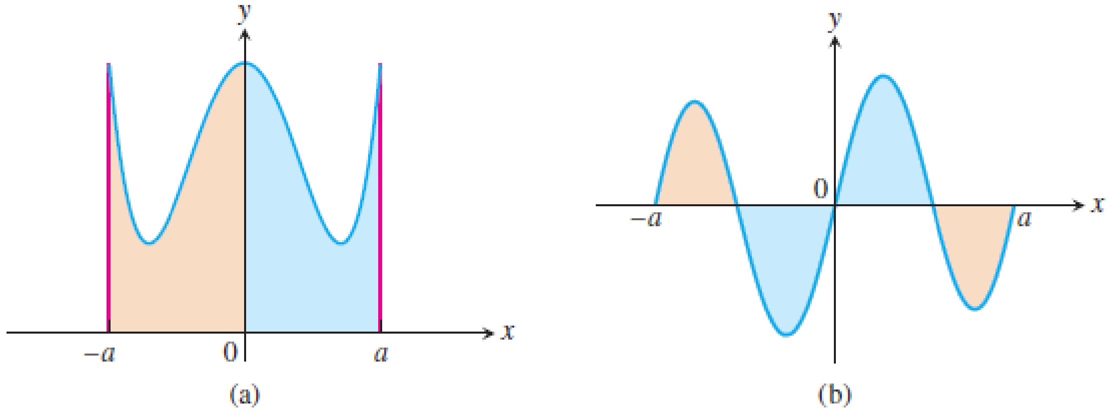
$$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta\, d\theta = \int_{1}^{0} u\cdot(-du)$$

$$= -\int_{1}^{0} u\,du$$

Let $u = \cot\theta$, $du = -\csc^2\theta\, d\theta$,
$-du = \csc^2\theta\, d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$= -\left[\frac{u^2}{2}\right]_{1}^{0} = -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

Definite Integrals of Symmetric Functions



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ (b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

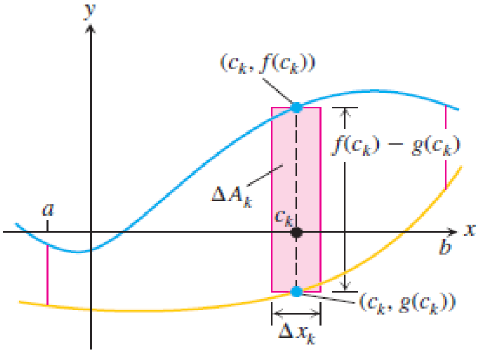
EXAMPLE Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2 \quad = 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

Areas Between Curves



DEFINITION **Area Between Curves**

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

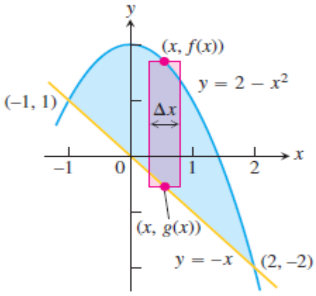
$$A = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves The limits of integration are found

by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.



$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x + 1)(x - 2) &= 0 && \text{Factor.} \\ x = -1, \qquad x &= 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^2 [(2 - x^2) - (-x)]\, dx$$

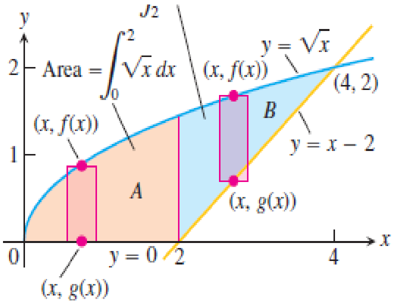
$$= \int_{-1}^2 (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

EXAMPLE

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:



$$\begin{aligned}
\sqrt{x} &= x - 2 \\
x &= (x-2)^2 = x^2 - 4x + 4 \\
x^2 - 5x + 4 &= 0 \\
(x-1)(x-4) &= 0 \\
x &= 1, \qquad x = 4.
\end{aligned}$$

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

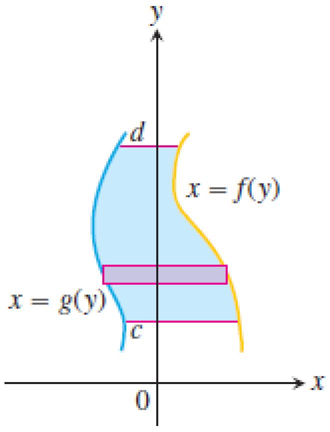
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

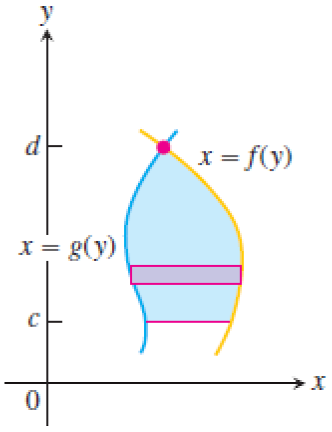
$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

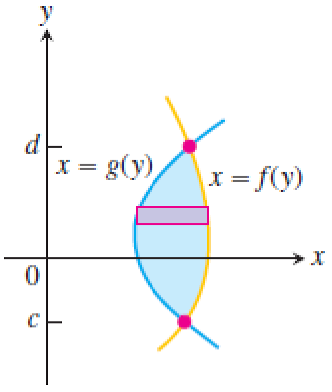
$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

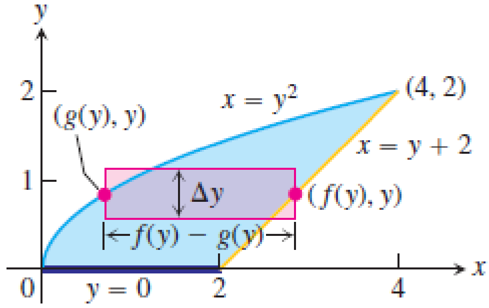






$$A = \int_c^d [f(y) - g(y)]\,dy.$$

EXAMPLE



$$y + 2 = y^2$$

$$y^2 - y - 2 = 0$$

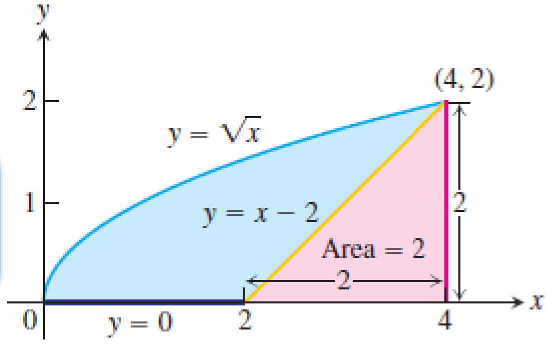
$$(y + 1)(y - 2) = 0$$

$$y = -1, \qquad y = 2$$

$$A = \int_a^b [f(y) - g(y)]\, dy = \int_0^2 [y + 2 - y^2]\, dy$$

$$= \int_0^2 [2 + y - y^2]\, dy \quad = 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \quad = \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$

EXAMPLE find the area



$$\text{Area} = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2)$$

$$= \frac{2}{3} x^{3/2} \Big]_0^4 - 2$$

$$= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.$$

APPLICATIONS OF DEFINITE INTEGRALS

Volumes by Slicing and Rotation About an Axis

Volume = area × height = $A \cdot h$.

DEFINITION **Volume**

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

Solids of Revolution: The Disk Method

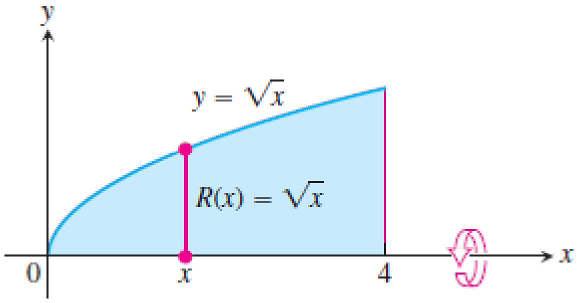
$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2$.

$$V = \int_a^b A(x)\, dx = \int_a^b \pi[R(x)]^2\, dx.$$

EXAMPLE

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid



$$V = \int_a^b \pi[R(x)]^2\,dx$$

$$= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx$$

$$= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi.$$

EXAMPLE Volume of a Sphere

The circle $x^2 + y^2 = a^2$

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

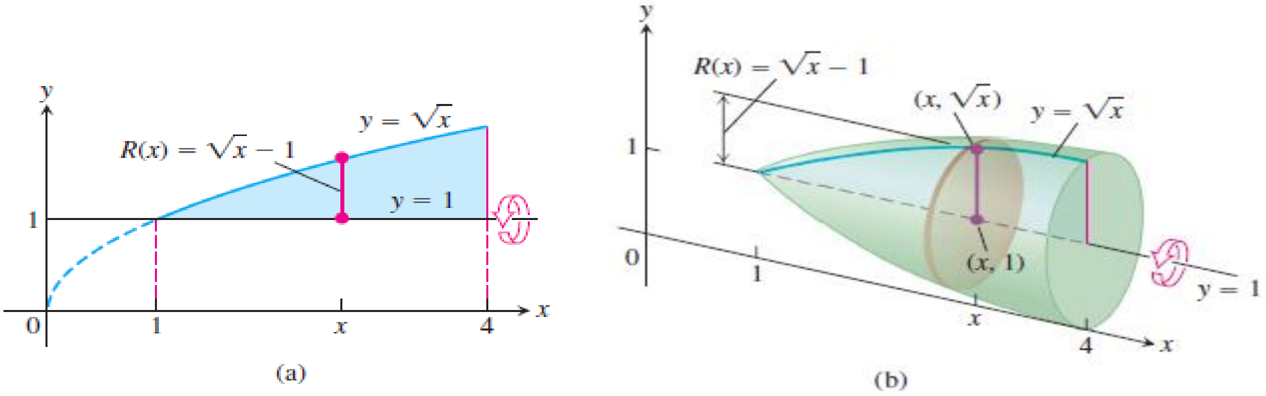
$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

EXAMPLE

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

$$V = \int_1^4 \pi[R(x)]^2\,dx \qquad = \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx$$

$$= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \quad = \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}.$$



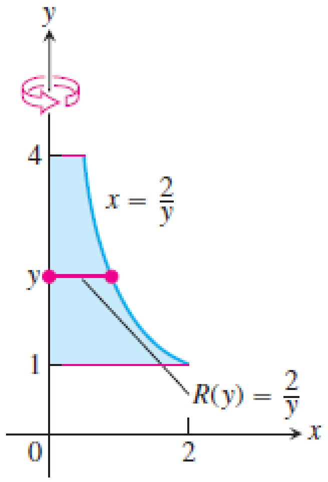
(a)

(b)

EXAMPLE

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y-axis.

Solution We draw figures showing the region, a typical radius, and the generated solid



$$V = \int_1^4 \pi[R(y)]^2\, dy$$

$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

$$= \pi\int_1^4 \frac{4}{y^2}\, dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right]$$

$$= 3\pi.$$

EXAMPLE

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

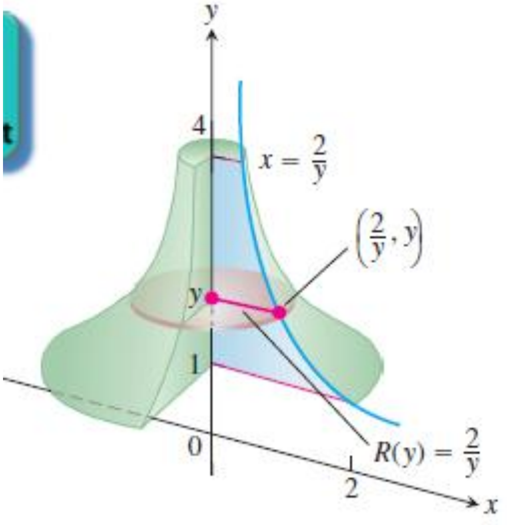
$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\, dy$$

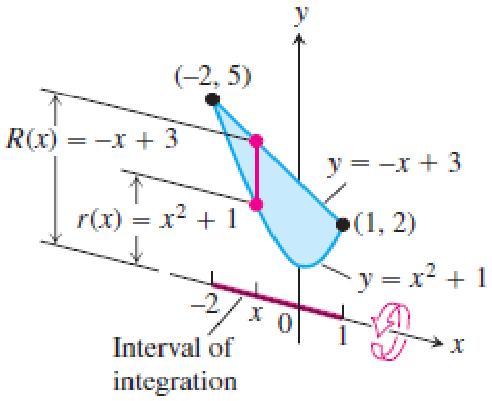
$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy$$

$$= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{64\pi\sqrt{2}}{15}.$$



Solids of Revolution: The Washer Method



Outer radius: $R(x)$

Inner radius: $r(x)$

The washer's area is $A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2)$.

$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.}$$

EXAMPLE

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution

2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

Outer radius: $R(x) = -x + 3$

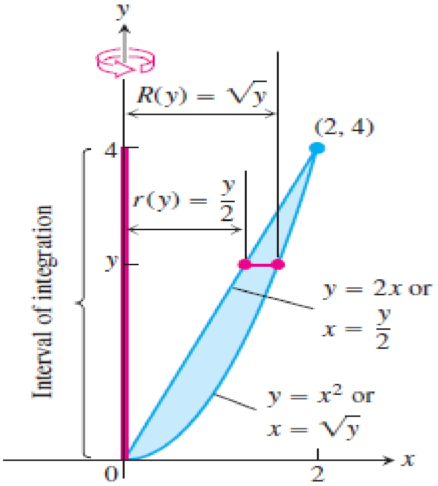
Inner radius: $r(x) = x^2 + 1$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2, \quad x &= 1 \end{aligned}$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \\ &= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\,dx \\ &= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx \\ &= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \end{aligned}$$



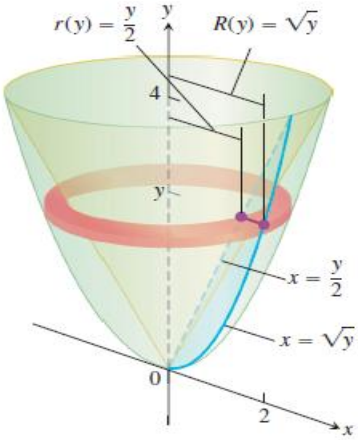
EXAMPLE

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy \\ &= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right)dy \\ &= \pi\int_0^4 \left(y - \frac{y^2}{4}\right)dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi. \end{aligned}$$



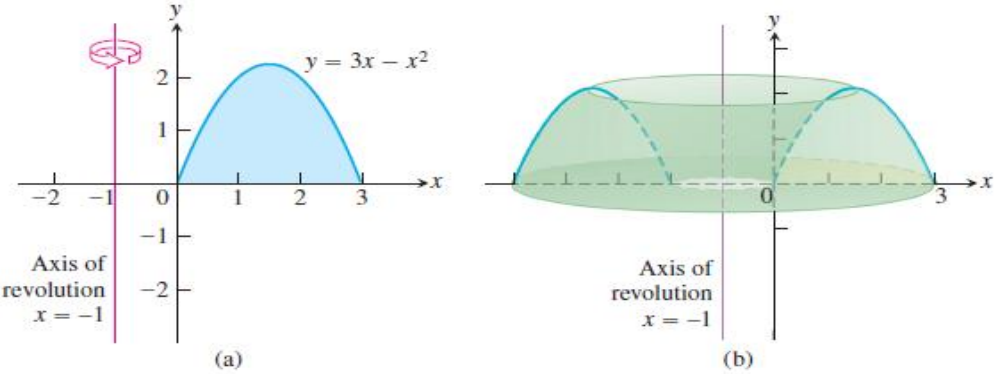
Volumes by Cylindrical Shells

we defined the volume of a solid S as the definite integral $V = \int_a^b A(x)\,dx,$

where $A(x)$ is an integrable cross-sectional area of S from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis. In this section we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way. Now we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid perpendicular to the x-axis, with the axis of the cylinder parallel to the y-axis. The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid S is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx. This allows us to apply the same integral definition for volume as before. Before describing the method in general, let's look at an example to gain some insight.

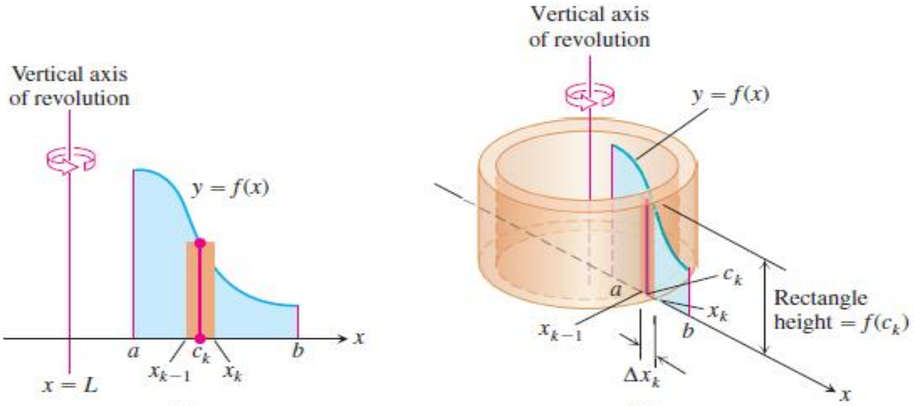
EXAMPLE

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid Find the volume of the solid.



$$
\begin{aligned}
V &= \int_0^3 2\pi(x + 1)(3x - x^2)\,dx \\
&= \int_0^3 2\pi(3x^2 + 3x - x^3 - x^2)\,dx \\
&= 2\pi\int_0^3 (2x^2 + 3x - x^3)\,dx \\
&= 2\pi\left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]_0^3 \\
&= \frac{45\pi}{2}.
\end{aligned}
$$

The Shell Method



(a) (b)

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx.$$

We refer to the variable of integration, here x, as the **thickness variable**. We use the first integral, rather than the second containing a formula for the integrand, to emphasize the *process* of the shell method. This will allow for rotations about a horizontal line L as well.

Shell Formula for Revolution About a Vertical Line

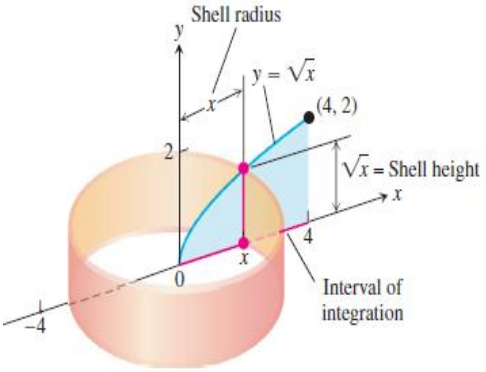
The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dx.$$

EXAMPLE

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

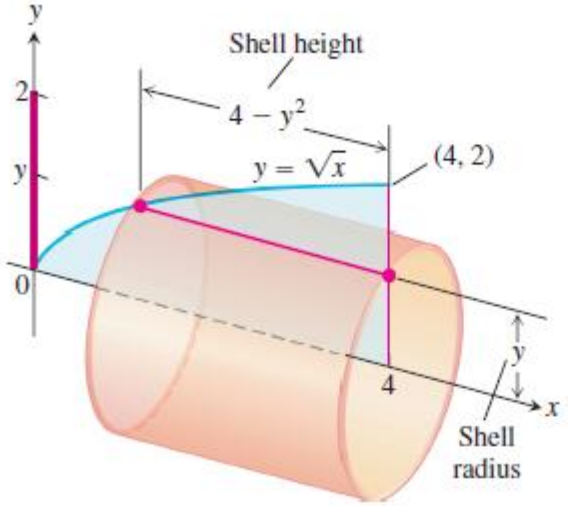
$$\begin{aligned} V &= \int_a^b 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dx \\ &= \int_0^4 2\pi(x)\left(\sqrt{x}\right)dx \\ &= 2\pi\int_0^4 x^{3/2}\,dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$



EXAMPLE

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

$$
\begin{aligned}
V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy \\
&= \int_0^2 2\pi(y)(4 - y^2)\, dy \\
&= \int_0^2 2\pi(4y - y^3)\, dy \\
&= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi .
\end{aligned}
$$



Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find* the limits of integration for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

Lengths of Plane Curves

We know what is meant by the length of a straight line segment, but without calculus, we have no precise notion of the length of a general winding curve. The idea of approximating the length of a curve running from point A to point B by subdividing the curve into many pieces and joining successive points of division by straight line segments dates back to the ancient Greeks. Archimedes used this method to approximate the circumference of a circle by inscribing a polygon of n sides and then using geometry to compute its perimeter

Length of a Parametrically Defined Curve

DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE

Find the length of the circle of radius r defined parametrically by

$$x = r\cos t \qquad \text{and} \qquad y = r\sin t, \qquad 0 \leq t \leq 2\pi.$$

Solution As t varies from 0 to 2π, the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

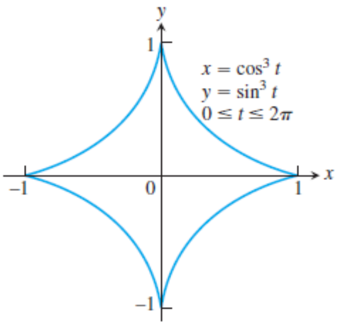
$$\frac{dx}{dt} = -r\sin t, \quad \frac{dy}{dt} = r\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r\left[t\right]_0^{2\pi} = 2\pi r.$$

EXAMPLE Find the length of the astroid $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi.$

Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have



$$x = \cos^3 t, \qquad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t(-\sin t)]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t(\cos t)]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$$= \sqrt{9\cos^2 t \sin^2 t} \quad = 3|\cos t \sin t| \quad = 3\cos t \sin t.$$

$\cos t \sin t \ge 0$ for $0 \le t \le \pi/2$

Therefore,

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t \, dt$$

$\cos t \sin t = (1/2)\sin 2t$

$$= -\frac{3}{4}\cos 2t\bigg]_0^{\pi/2} = \frac{3}{2}. \qquad = \frac{3}{2}\int_0^{\pi/2} \sin 2t \, dt$$

Formula for the Length of $y = f(x), \quad a \le x \le b$
If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{2}$$

EXAMPLE Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$

Solution

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \qquad \frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

Eq. (2) with $a = 0, b = 1$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6}.$$

Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$.

Dealing with Discontinuities in dy/dx

Formula for the Length of $x = g(y), \qquad c \le y \le d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{3}$$

EXAMPLE Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (2).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2}$$

Raise both sides to the power 3/2.

$$x = 2y^{3/2}.$$

Solve for x.

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$. The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

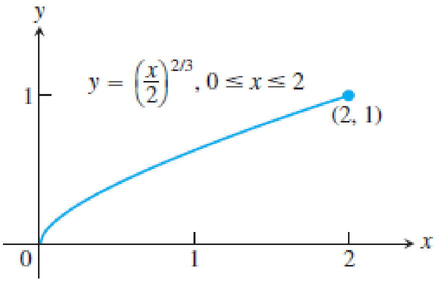
$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy$$

$$= \frac{1}{9} \cdot \frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

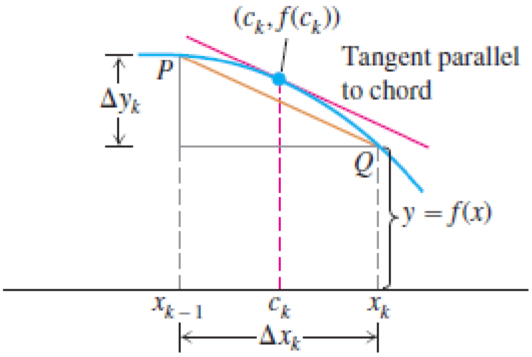
Eq. (3) with $c = 0, d = 1$.

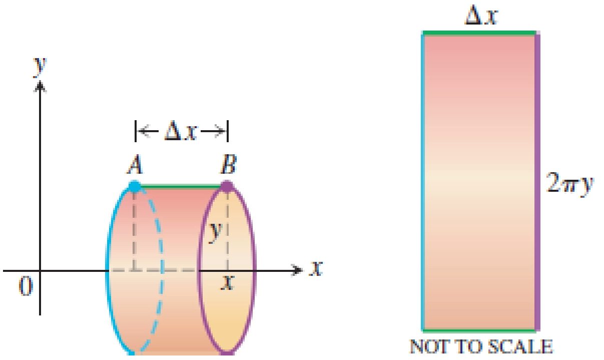
Let $u = 1 + 9y$ $du/9 = dy$, integrate, and substitute back.



Areas of Surfaces of Revolution and the Theorems of Pappus

When you jump rope, the rope sweeps out a surface in the space around you called a *surface of revolution*. The "area" of this surface depends on the length of the rope and the distance of each of its segments from the axis of revolution. In this section we define areas of surfaces of revolution.





DEFINITION **Surface Area for Revolution About the *x*-Axis**

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

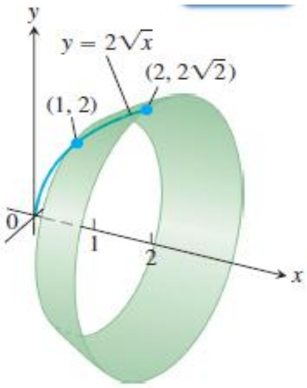
$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

EXAMPLE

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

Revolution About the y-Axis

Surface Area for Revolution About the y-Axis

If $x = g(y) \ge 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

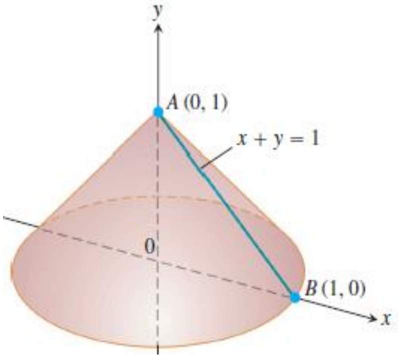
EXAMPLE

The line segment $x = 1 - y$, $0 \le y \le 1$, is revolved about the y-axis to generate the cone Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

$$c = 0, \qquad d = 1, \qquad x = 1 - y, \qquad \frac{dx}{dy} = -1,$$



$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+(-1)^2} = \sqrt{2}$$

$$\begin{aligned} S &= \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 2\pi(1-y)\sqrt{2}\,dy \\ &= 2\pi\sqrt{2}\left[y-\frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1-\frac{1}{2}\right) \\ &= \pi\sqrt{2}. \end{aligned}$$

Parametrized Curves

$$\sqrt{[f'(t)]^2+[g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.$$

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis $(y \ge 0)$:

$$S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \tag{5}$$

2. Revolution about the y-axis $(x \ge 0)$:

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \tag{6}$$

EXAMPLE

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{1}}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi\Big[t - \cos t\Big]_0^{2\pi} = 4\pi^2.$$

Inverse Functions and Their Derivatives

One-to-One Functions

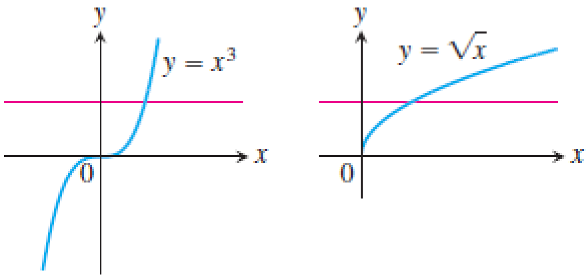
DEFINITION **One-to-One Function**
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE Domains of One-to-One Functions

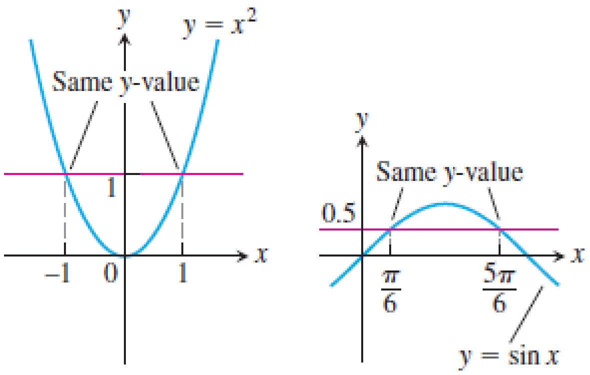
(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because it is a strictly increasing function on $[0, \pi/2]$. ■

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION **Inverse Function**

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The domains and ranges of f and f^{-1} are interchanged. The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$ we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

$(f^{-1} \circ f)(x) = x,$ for all x in the domain of f

$(f \circ f^{-1})(y) = y,$ for all y in the domain of f^{-1} (or range of f)

Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

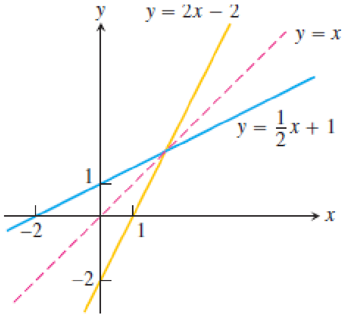
The process of passing from f to f^{-1} can be summarized as a two-step process.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution



1. *Solve for x in terms of y:*

$$\begin{aligned} y &= \frac{1}{2}x + 1 \\ 2y &= x + 2 \\ x &= 2y - 2. \end{aligned}$$

2. *Interchange x and y:* $y = 2x - 2.$

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

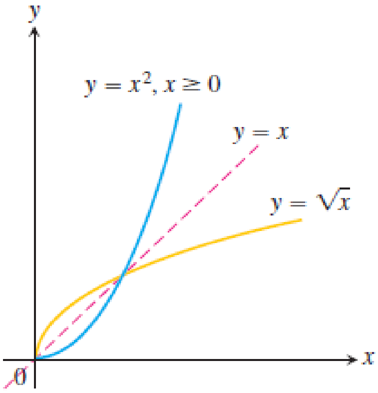
$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

EXAMPLE Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$



We then interchange x and y, obtaining $y = \sqrt{x}$.

Derivatives of Inverses of Differentiable Functions

If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

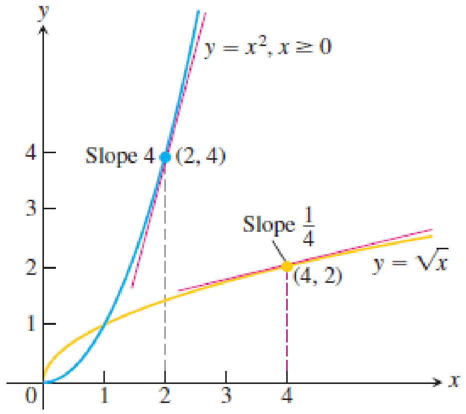
$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} \qquad (1)$$

EXAMPLE

The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} \\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$



Theorem 1 gives a derivative that agrees with our calculation using the Power Rule for the derivative of the square root function.

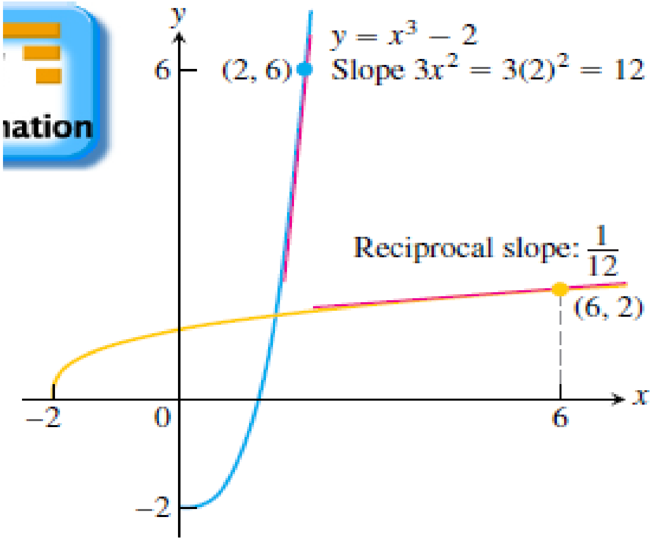
Let's examine Theorem 1 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 1 says that the derivative of f at 2, $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\bigg|_{x=2} = \frac{1}{4}.$$

EXAMPLE Finding a Value of the Inverse Derivative

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution



$$\frac{df}{dx}\bigg|_{x=2} = 3x^2\bigg|_{x=2} = 12$$

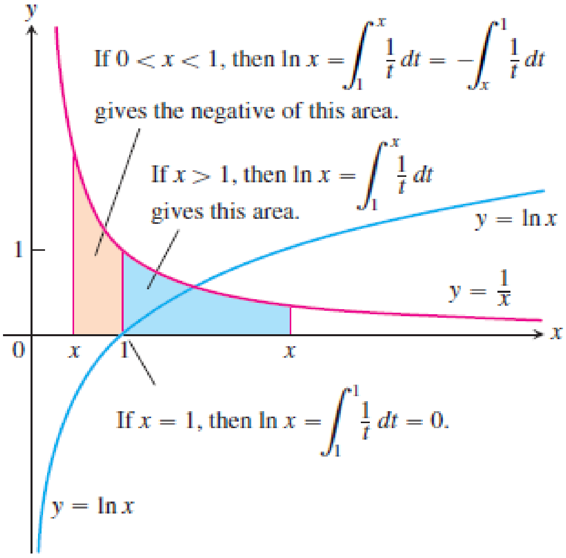
$$\frac{df^{-1}}{dx}\bigg|_{x=f(2)} = \frac{1}{\dfrac{df}{dx}\bigg|_{x=2}} = \frac{1}{12}$$

Natural Logarithms

Definition of the Natural Logarithm Function

DEFINITION **The Natural Logarithm Function**

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$



The Derivative of $y = \ln x$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}. \qquad \frac{d}{dx}\ln x = \frac{1}{x}.$$

DEFINITION **The Number *e***

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0$$

EXAMPLE Derivatives of Natural Logarithms

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$

(b) Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

Properties of Logarithms

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1.	*Product Rule*:	$\ln ax = \ln a + \ln x$	
2.	*Quotient Rule*:	$\ln\frac{a}{x} = \ln a - \ln x$	
3.	*Reciprocal Rule*:	$\ln\frac{1}{x} = -\ln x$	Rule 2 with $a = 1$
4.	*Power Rule*:	$\ln x^r = r\ln x$	r rational

EXAMPLE Interpreting the Properties of Logarithms

(a)	$\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$	Product
(b)	$\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$	Quotient
(c)	$\ln\frac{1}{8} = -\ln 8$	Reciprocal
	$= -\ln 2^3 = -3\ln 2$	Power

EXAMPLE Applying the Properties to Function Formulas

(a)	$\ln 4 + \ln\sin x = \ln(4\sin x)$	Product
(b)	$\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$	Quotient
(c)	$\ln\sec x = \ln\frac{1}{\cos x} = -\ln\cos x$	Reciprocal
(d)	$\ln\sqrt[3]{x+1} = \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)$	Power

The Integral $\int (1/u)\, du$

$$\int \frac{1}{u}\, du = \ln u + C$$

$$\int \frac{1}{u}\, du = \int \frac{1}{(-u)}\, d(-u)$$
$$= \ln(-u) + C.$$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1 \text{ and rational}$$

EXAMPLE

(a) $$\int_0^2 \frac{2x}{x^2 - 5}\, dx = \int_{-5}^{-1} \frac{du}{u} = \ln |u| \Big]_{-5}^{-1}$$
$$= \ln |-1| - \ln |-5| = \ln 1 - \ln 5 = -\ln 5$$

$u = x^2 - 5, \quad du = 2x\, dx,$
$u(0) = -5, \quad u(2) = -1$

(b) $$\int_{-\pi/2}^{\pi/2} \frac{4 \cos \theta}{3 + 2 \sin \theta}\, d\theta = \int_1^5 \frac{2}{u}\, du$$
$$= 2 \ln |u| \Big]_1^5$$
$$= 2 \ln |5| - 2 \ln |1| = 2 \ln 5$$

$u = 3 + 2 \sin \theta, \quad du = 2 \cos \theta\, d\theta,$
$u(-\pi/2) = 1, \quad u(\pi/2) = 5$

Note that $u = 3 + 2 \sin \theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Integrals of tan x and cot x

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = \int \frac{-du}{u}$$
$$= -\int \frac{du}{u} = -\ln |u| + C$$
$$= -\ln |\cos x| + C = \ln \frac{1}{|\cos x|} + C$$
$$= \ln |\sec x| + C.$$

$u = \cos x > 0$ on $(-\pi/2, \pi/2)$,
$du = -\sin x\, dx$

Reciprocal Rule

For the cotangent,

$$\int \cot x \, dx = \int \frac{\cos x \, dx}{\sin x} = \int \frac{du}{u}$$

$u = \sin x,$
$du = \cos x \, dx$

$$= \ln |u| + C = \ln |\sin x| + C = -\ln |\csc x| + C.$$

$$\int \tan u \, du = -\ln |\cos u| + C = \ln |\sec u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C = -\ln |\csc x| + C$$

EXAMPLE

$$\int_0^{\pi/6} \tan 2x \, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u \, du$$

$$= \frac{1}{2} \ln |\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Substitute $u = 2x,$
$dx = du/2,$
$u(0) = 0,$
$u(\pi/6) = \pi/3$

Logarithmic Differentiation

EXAMPLE Using Logarithmic Differentiation

Find dy/dx if $\quad y = \dfrac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\ln y = \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}$$

$$= \ln ((x^2 + 1)(x + 3)^{1/2}) - \ln (x - 1) \qquad \text{Rule 2}$$

$$= \ln (x^2 + 1) + \ln (x + 3)^{1/2} - \ln (x - 1) \qquad \text{Rule 1}$$

$$= \ln (x^2 + 1) + \frac{1}{2} \ln (x + 3) - \ln (x - 1). \qquad \text{Rule 3}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

The Exponential Function

The Inverse of ln x and the Number e

The Function $y = e^x$

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

$$\ln e^r = r \ln e = r \cdot 1 = r.$$

DEFINITION **The Natural Exponential Function**

For every real number x, $e^x = \ln^{-1} x = \exp x$.

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \qquad (2)$$

$$\ln (e^x) = x \qquad (\text{all } x) \qquad (3)$$

EXAMPLE Using the Inverse Equations

(a) $\ln e^2 = 2$

(b) $\ln e^{-1} = -1$

(c) $\ln \sqrt{e} = \frac{1}{2}$

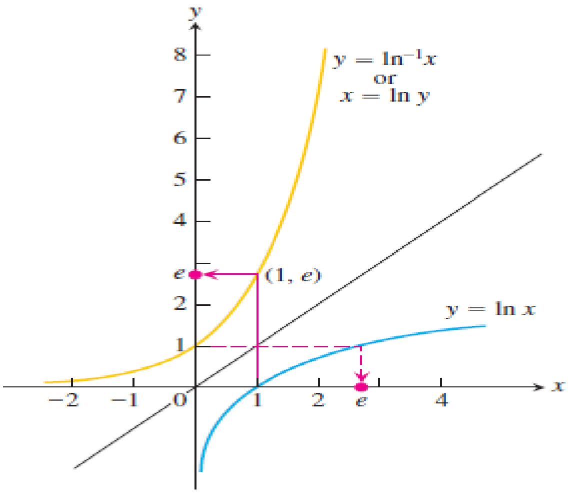
(d) $\ln e^{\sin x} = \sin x$

(e) $e^{\ln 2} = 2$

(f) $e^{\ln (x^2+1)} = x^2 + 1$

(g) $e^{3 \ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way

(h) $e^{3 \ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way



EXAMPLE Find k if $e^{2k} = 10$.

Solution

$$e^{2k} = 10$$

$$\ln e^{2k} = \ln 10$$

$$2k = \ln 10$$

$$k = \frac{1}{2} \ln 10.$$

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$.

EXAMPLE Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$

Laws of Exponents

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

EXAMPLE Applying the Exponent Laws

(a) $e^{x + \ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

(b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

(c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

(d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4

The Derivative and Integral of e^x

$$f(x) = \ln x \qquad \text{and} \qquad y = e^x = \ln^{-1} x = f^{-1}(x).$$

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}(e^x) &= \frac{d}{dx}\ln^{-1} x \\
&= \frac{d}{dx} f^{-1}(x) \\
&= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{f'(e^x)} && f^{-1}(x) = e^x \\
&= \frac{1}{\left(\frac{1}{e^x}\right)} && f'(z) = \frac{1}{z} \text{ with } z = e^x \\
&= e^x.
\end{aligned}$$

$$\frac{d}{dx} e^x = e^x \tag{5}$$

EXAMPLE Differentiating an Exponential

$$\begin{aligned}
\frac{d}{dx}(5e^x) &= 5\frac{d}{dx}e^x \\
&= 5e^x
\end{aligned}$$

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{6}$$

EXAMPLE Applying the Chain Rule with Exponentials

(a) $\frac{d}{dx} e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (6) with $u = -x$

(b) $\frac{d}{dx} e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (6) with $u = \sin x$

$$\int e^u \, du = e^u + C.$$

EXAMPLE Integrating Exponentials

(a) $$\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du \qquad u = 3x,\quad \frac{1}{3}du = dx,\quad u(0) = 0,\quad u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du$$

$$= \frac{1}{3}e^u\Big]_0^{\ln 8}$$

$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

(b) $$\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2} \qquad \text{Antiderivative from Example 6}$$

$$= e^1 - e^0 = e - 1$$

The Power Rule (General Form)

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n\ln x}$. Therefore, the n in the equation $\ln x^n = n\ln x$ no longer needs to be rational—it can be any number as long as $x > 0$:

$$\ln x^n = \ln\left(e^{n\ln x}\right) = n\ln x \qquad \ln e^u = u,\ \text{any } u$$

Together, the law $a^x/a^y = a^{x-y}$ and the definition $x^n = e^{n\ln x}$ enable us to establish the Power Rule for differentiation in its final form. Differentiating x^n with respect to x gives

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} \qquad \text{Definition of } x^n,\ x > 0$$

$$= e^{n\ln x}\cdot\frac{d}{dx}(n\ln x) \qquad \text{Chain Rule for } e^u$$

$$= x^n\cdot\frac{n}{x} \qquad \text{The definition again}$$

$$= nx^{n-1}.$$

$$\frac{d}{dx}x^n = nx^{n-1}.$$

The Chain Rule extends this equation to the Power Rule's general form.

Power Rule (General Form)
If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE Using the Power Rule with Irrational Powers

(a) $\frac{d}{dx}x^{\sqrt{2}} = \sqrt{2}x^{\sqrt{2}-1} \qquad (x > 0)$

(b) $$\begin{aligned}\frac{d}{dx}(2 + \sin 3x)^{\pi} &= \pi(2 + \sin 3x)^{\pi-1}(\cos 3x) \cdot 3 \\ &= 3\pi(2 + \sin 3x)^{\pi-1}(\cos 3x).\end{aligned}$$

a^x and $\log_a x$

We have defined general exponential functions such as 2^x, 10^x, and π^x. In this section we compute their derivatives and integrals. We also define the general logarithmic functions such as $\log_2 x$, $\log_{10} x$, and $\log_{\pi} x$, and find their derivatives and integrals as well.

The Derivative of a^u

We start with the definition $a^x = e^{x \ln a}$:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx}e^u = e^u \frac{du}{dx}$$

$$= a^x \ln a.$$

If $a > 0$, then $\quad \frac{d}{dx}a^x = a^x \ln a.$

With the Chain Rule, we get a more general form.

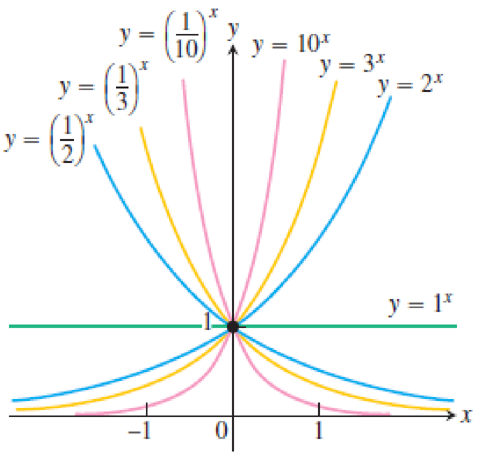
If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \qquad (1)$$

These equations show why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx}e^x = e^x \ln e = e^x.$$

EXAMPLE Differentiating General Exponential Functions



(a) $\frac{d}{dx}3^x = 3^x \ln 3$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

EXAMPLE Differentiating a General Power Function

Find dy/dx if $y = x^x$, $x > 0$.

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad a^x \text{ with } a = x.$$

Then differentiate as usual:

$$\frac{dy}{dx} = \frac{d}{dx}e^{x\ln x} \qquad = e^{x\ln x}\frac{d}{dx}(x\ln x) \qquad = x^x\left(x\cdot\frac{1}{x} + \ln x\right) = x^x(1 + \ln x).$$

The Integral of a^u

If $a \neq 1$, so that $\ln a \neq 0$, we can divide both sides of Equation (1) by $\ln a$ to obtain

$$a^u\frac{du}{dx} = \frac{1}{\ln a}\frac{d}{dx}(a^u).$$

Integrating with respect to x then gives

$$\int a^u\frac{du}{dx}\,dx = \int\frac{1}{\ln a}\frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}\int\frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}a^u + C.$$

Writing the first integral in differential form gives

$$\boxed{\int a^u\,du = \frac{a^u}{\ln a} + C. \qquad (2)}$$

EXAMPLE Integrating General Exponential Functions

(a) $$\int 2^x\,dx = \frac{2^x}{\ln 2} + C \qquad \text{Eq. (2) with } a = 2,\ u = x$$

(b) $$\begin{aligned}\int 2^{\sin x}\cos x\,dx &\\ &= \int 2^u\,du = \frac{2^u}{\ln 2} + C && u = \sin x,\ du = \cos x\,dx, \text{ and Eq. (2)}\\ &= \frac{2^{\sin x}}{\ln 2} + C && u \text{ replaced by } \sin x\end{aligned}$$

Logarithms with Base a

As we saw earlier, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of x with base a** and denote it by $\log_a x$.

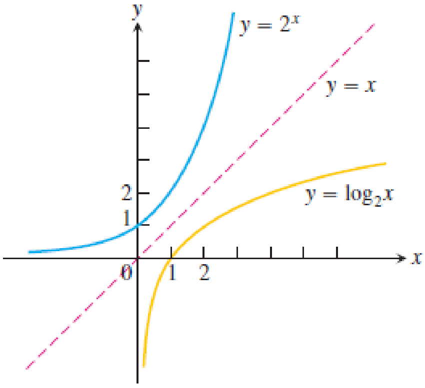
DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$$\log_a x \text{ is the inverse function of } a^x.$$

EXAMPLE Applying the Inverse Equations

$$\begin{aligned} a^{\log_a(x)} &= x && \text{Eq. (3)}\\ \ln a^{\log_a(x)} &= \ln x && \text{Take the natural logarithm of both sides.}\\ \log_a(x)\cdot\ln a &= \ln x && \text{The Power Rule in Theorem 2}\\ \log_a x &= \frac{\ln x}{\ln a} && \text{Solve for } \log_a x.\end{aligned}$$



Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \qquad (3)$$

$$\log_a(a^x) = x \qquad (\text{all } x) \qquad (4)$$

(a) $\log_2(2^5) = 5$ **(b)** $\log_{10}(10^{-7}) = -7$

(c) $2^{\log_2(3)} = 3$ **(d)** $10^{\log_{10}(4)} = 4$

$$\ln xy = \ln x + \ln y \qquad \text{Rule 1 for natural logarithms } \ldots$$

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} \qquad \ldots \text{ divided by } \ln a \ \ldots$$

$$\log_a xy = \log_a x + \log_a y. \qquad \ldots \text{ gives Rule 1 for base } a \text{ logarithms.}$$

$$\ln xy = \ln x + \ln y \qquad \text{Rule 1 for natural logarithms } \ldots$$

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} \qquad \ldots \text{ divided by } \ln a \ \ldots$$

$$\log_a xy = \log_a x + \log_a y. \qquad \ldots \text{ gives Rule 1 for base } a \text{ logarithms.}$$

Derivatives and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms.

If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}}$$

EXAMPLE

(a) $\dfrac{d}{dx}\log_{10}(3x+1) = \dfrac{1}{\ln 10}\cdot\dfrac{1}{3x+1}\dfrac{d}{dx}(3x+1) = \dfrac{3}{(\ln 10)(3x+1)}$

(b)
$$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}\,dx$$

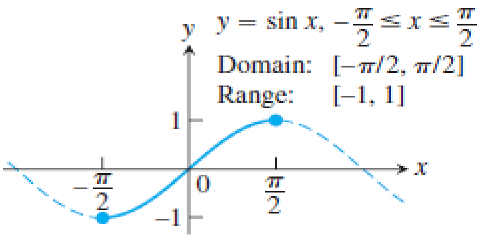
$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

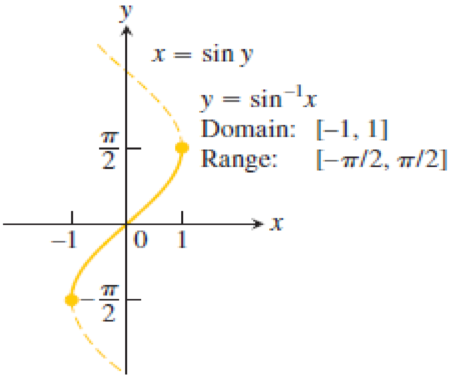
Inverse Trigonometric Functions

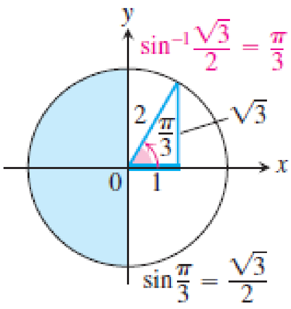
DEFINITION Arcsine and Arccosine Functions

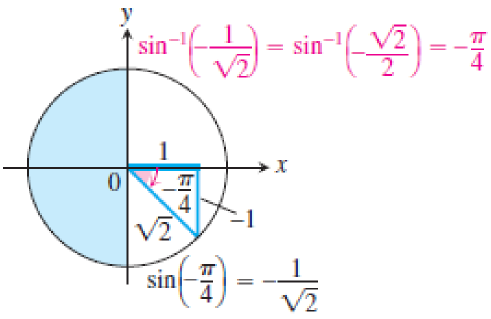
$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

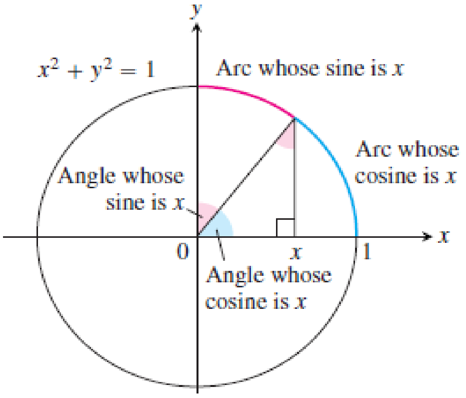
$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.











The Derivative of $y = \sin^{-1} x$

$$\sin y = x$$

$y = \sin^{-1} x \Leftrightarrow \sin y = x$

$$\frac{d}{dx}(\sin y) = 1$$

Derivative of both sides with respect to x

$$\cos y \frac{dy}{dx} = 1$$

Chain Rule

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We can divide because $\cos y > 0$ for $-\pi/2 < y < \pi/2$.

$$= \frac{1}{\sqrt{1 - x^2}}$$

$\cos y = \sqrt{1 - \sin^2 y}$

No matter which derivation we use, we have that the derivative of $y = \sin^{-1} x$ with re- spect to x is

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\boxed{\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.}$$

EXAMPLE Applying the Derivative Formula

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 1 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 1 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1}\\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u\\
&= \frac{1}{1+\tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u\\
&= \frac{1}{1+x^2} && \tan(\tan^{-1} x) = x
\end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\boxed{\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.}$$

EXAMPLE

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

Solution

$$v(t) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1+(\sqrt{t})^2} \cdot \frac{d}{dt}\sqrt{t} = \frac{1}{1+t} \cdot \frac{1}{2\sqrt{t}}$$

When $t = 16$, the velocity is $$v(16) = \frac{1}{1+16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}.$$

The Derivative of $y = \sec^{-1} u$

$$
\begin{aligned}
y &= \sec^{-1} x \\
\sec y &= x && \text{Inverse function relationship} \\
\frac{d}{dx}(\sec y) &= \frac{d}{dx} x && \text{Differentiate both sides.} \\
\sec y \tan y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\sec y \tan y} && \text{Since } |x| > 1,\ y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.
\end{aligned}
$$

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}} .$$

$$\frac{d}{dx} \sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the "±" ambiguity:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}} .$$

$$\boxed{\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad |u| > 1 .}$$

EXAMPLE Using the Formula

$$
\begin{aligned}
\frac{d}{dx} \sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}} \frac{d}{dx}(5x^4) \\
&= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) && 5x^4 > 0 \\
&= \frac{4}{x\sqrt{25x^8 - 1}}
\end{aligned}
$$

Derivatives of the Other Three

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \text{Identity}$$
$$= -\frac{d}{dx}(\sin^{-1} x)$$
$$= -\frac{1}{\sqrt{1 - x^2}} \qquad \text{Derivative of arcsine}$$

EXAMPLE Find an equation for the line tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

Solution First we note that

$$\cot^{-1}(-1) = \pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4.$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1 + x^2}\right|_{x=-1} = -\frac{1}{1 + (-1)^2} = -\frac{1}{2},$$

so the tangent line has equation $y - 3\pi/4 = (-1/2)(x + 1)$.

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$
2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$
3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1 + u^2}$
4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1 + u^2}$
5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$
6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$

Integration Formulas

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE Using the Integral Formulas

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $$\int_0^1 \frac{dx}{1 + x^2} = \tan^{-1} x\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(c) $$\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x\Big]_{2/\sqrt{3}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

EXAMPLE

(a) $$\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{dx}{\sqrt{(3)^2 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

Table 7.4 Formula 1, with $a = 3$, $u = x$

(b) $$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$$

$a = \sqrt{3}$, $u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C \qquad = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

EXAMPLE Completing the Square

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$.

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x - x^2}} &= \int \frac{dx}{\sqrt{4 - (x - 2)^2}} \\
&= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 2, u = x - 2, \text{ and } du = dx \\
&= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 7.4, Formula 1} \\
&= \sin^{-1}\left(\frac{x - 2}{2}\right) + C
\end{aligned}$$

EXAMPLE Completing the Square

Evaluate $\displaystyle\int \frac{dx}{4x^2 + 4x + 2}$.

Solution We complete the square on the binomial $4x^2 + 4x$:

$$\begin{aligned}
4x^2 + 4x + 2 = 4(x^2 + x) + 2 &= 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4} \\
&= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.
\end{aligned}$$

Then,

$$\begin{aligned}
\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1} &= \frac{1}{2}\int \frac{du}{u^2 + a^2} && a = 1, u = 2x + 1, \text{ and } du/2 = dx \\
&= \frac{1}{2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C && \text{Table 7.4, Formula 2} \\
&= \frac{1}{2} \tan^{-1}(2x + 1) + C && a = 1, u = 2x + 1
\end{aligned}$$

EXAMPLE Using Substitution

Evaluate $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$.

Solution

$$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x}-6}} &= \int \frac{du/u}{\sqrt{u^2-a^2}} && u = e^x,\ du = e^x\,dx,\ dx = du/e^x = du/u,\ a = \sqrt{6}\\ &= \int \frac{du}{u\sqrt{u^2-a^2}}\\ &= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C && \text{Table 7.4, Formula 3}\\ &= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C\end{aligned}$$

Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and they are important in applications. For instance, they are used in problems such as computing the tension in a cable suspended by its two ends, as in an electric transmission line. They also play an important role in finding solutions to differential equations. In this section, we give a brief introduction to hyperbolic functions, their graphs, how their derivatives are calculated, and why they appear as important antiderivatives.

Even and Odd Parts of the Exponential Function

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd part}}.$$

If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x+e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x-e^{-x}}{2}}_{\text{odd part}}.$$

Definitions and Identities

$$2\sinh x\cosh x = 2\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^{2x}-e^{-2x}}{2} = \sinh 2x.$$

TABLE 7.5 The six basic hyperbolic functions

Hyperbolic sine of x: $$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine of x: $$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent: $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent: $$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant: $$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant: $$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

FIGURE 7.31

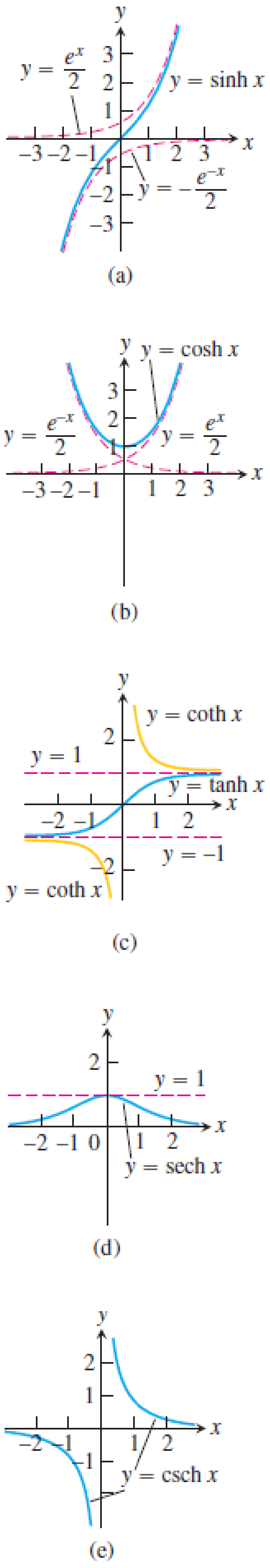


TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivatives and Integrals

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$
$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$
$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$
$$\int \cosh u \, du = \sinh u + C$$
$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$
$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$
$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$
$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

The derivative formulas are derived from the derivative of e^u:

$$\begin{aligned}\frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of } \sinh u \\ &= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u \\ &= \cosh u \frac{du}{dx} && \text{Definition of } \cosh u\end{aligned}$$

This gives the first derivative formula. The calculation

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of csch } u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of csch } u \text{ and coth } u
\end{aligned}$$

gives the last formula. The others are obtained similarly.

EXAMPLE : Finding Derivatives and Integrals

(a)
$$\begin{aligned}
\frac{d}{dt}\left(\tanh \sqrt{1+t^2}\right) &= \operatorname{sech}^2 \sqrt{1 = t^2} \cdot \frac{d}{dt}\left(\sqrt{1+t^2}\right) \\
&= \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2 \sqrt{1+t^2}
\end{aligned}$$

(b)
$$\int \coth 5x\, dx = \int \frac{\cosh 5x}{\sinh 5x}\, dx = \frac{1}{5}\int \frac{du}{u} \qquad \begin{aligned} &u = \sinh 5x, \\ &du = 5 \cosh 5x\, dx \end{aligned}$$

$$= \frac{1}{5} \ln |u| + C = \frac{1}{5} \ln |\sinh 5x| + C$$

(c)
$$\begin{aligned}
\int_0^1 \sinh^2 x\, dx &= \int_0^1 \frac{\cosh 2x - 1}{2}\, dx && \text{Table 7.6} \\
&= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\, dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1
\end{aligned}$$

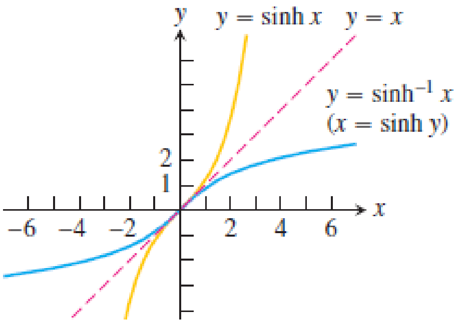
$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$

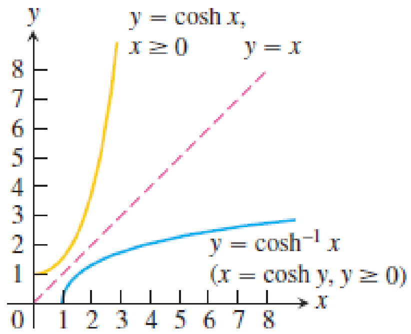
(d)
$$\begin{aligned}
\int_0^{\ln 2} 4e^x \sinh x\, dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\, dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx \\
&= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0) \\
&= 4 - 2\ln 2 - 1
\end{aligned}$$

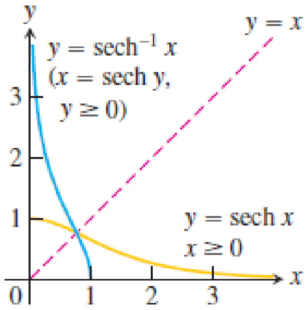
Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration. Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by

$$y = \sinh^{-1} x.$$

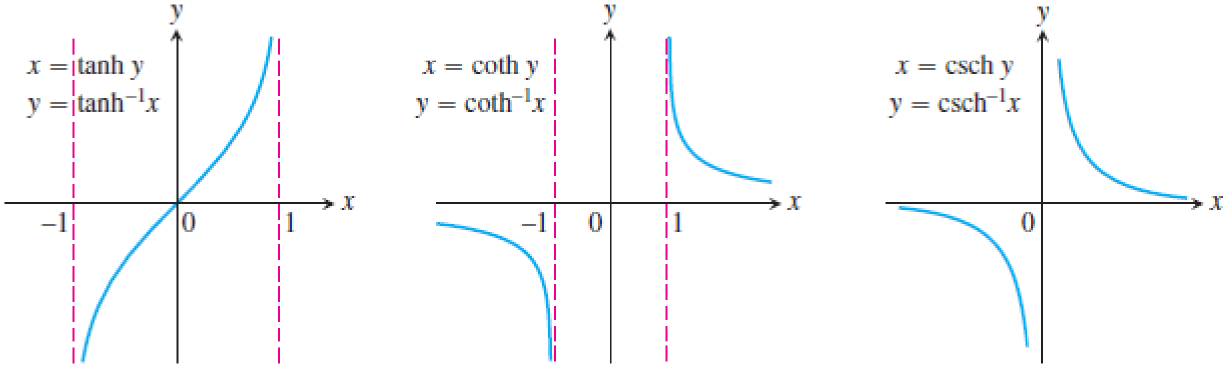






The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$



(a) (b) (c)

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$	
$\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx},$	$u > 1$
$\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx},$	$\lvert u \rvert < 1$
$\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx},$	$\lvert u \rvert > 1$
$\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1-u^2}},$	$0 < u < 1$
$\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{\lvert u \rvert\sqrt{1+u^2}},$	$u \neq 0$

EXAMPLE Derivative of the Inverse Hyperbolic Cosine

$$y = \cosh^{-1} x$$

$$x = \cosh y \qquad \text{Equivalent equation}$$

$$1 = \sinh y \frac{dy}{dx} \qquad \text{Implicit differentiation with respect to } x\text{, and the Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} \qquad \text{Since } x > 1, y > 0 \text{ and } \sinh y > 0$$

$$= \frac{1}{\sqrt{x^2 - 1}}. \qquad \cosh y = x$$

EXAMPLE : Evaluate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}$.

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3 + 4x^2}} = \int \frac{du}{\sqrt{a^2 + u^2}} \qquad u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.11}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\overline{3}}\right) - 0 \approx 0.98665.$$

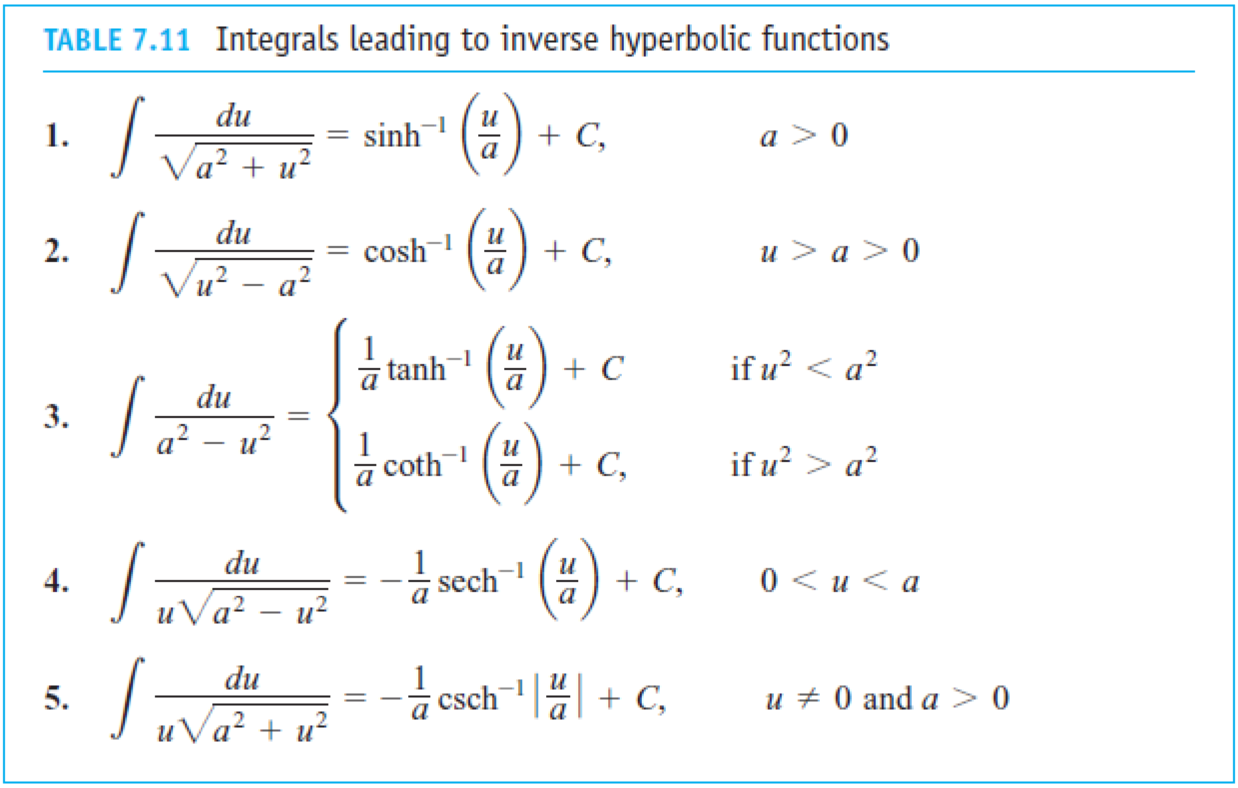


TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2 - u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

TECHNIQUES OF INTEGRATION

Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad \text{(any number } k\text{)}$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$

EXAMPLE Making a Simplifying Substitution

Evaluate $$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx.$$

Solution

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \int \frac{du}{\sqrt{u}}$$

$u = x^2 - 9x + 1,$
$du = (2x - 9)\,dx.$

$$= \int u^{-1/2}\,du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C$$

Table 8.1 Formula 4, with $n = -1/2$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$

■

EXAMPLE Completing the Square

Evaluate $$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$a = 4, u = (x - 4),$
$du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

Table 8.1, Formula 18

$$= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

EXAMPLE : Expanding a Power and Using a Trigonometric Identity

Evaluate $\displaystyle\int (\sec x + \tan x)^2\,dx.$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}\int (\sec x + \tan x)^2\,dx &= \int (\sec^2 x + 2\sec x\tan x + \sec^2 x - 1)\,dx \\ &= 2\int \sec^2 x\,dx + 2\int \sec x\tan x\,dx - \int 1\,dx \\ &= 2\tan x + 2\sec x - x + C.\end{aligned}$$

EXAMPLE Eliminating a Square Root

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx.$

Solution We use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2}\,\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx \qquad \sqrt{u^2} = |u|$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx \qquad \text{On } [0, \pi/4],\ \cos 2x \ge 0, \text{ so } |\cos 2x| = \cos 2x.$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} \qquad \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \qquad = \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.$$

EXAMPLE 5 Reducing an Improper Fraction

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx.$$

$$\begin{array}{r} x - 3 \\ 3x + 2\overline{)3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +\,6 \end{array}$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{3x + 2}\right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

EXAMPLE : Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate $\int \sec x\,dx.$

Solution

$$\int \sec x\,dx = \int (\sec x)(1)\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C. \qquad u = \tan x + \sec x,\; du = (\sec^2 x + \sec x \tan x)\,dx$$

TABLE 8.2 The secant and cosecant integrals

1. $\int \sec u \, du = \ln |\sec u + \tan u| + C$
2. $\int \csc u \, du = -\ln |\csc u + \cot u| + C$

Integration by Parts

Since

$$\int x \, dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2 \, dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx.$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u \, dv = uv - \int v \, du \tag{2}$$

This formula expresses one integral, $\int u \, dv$, in terms of a second integral, $\int v \, du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique.

EXAMPLE Using Integration by Parts

Find $\int x \cos x \, dx.$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x.$$ Simplest antiderivative of $\cos x$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

EXAMPLE Find $\int \ln x\,dx.$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated $dv = dx$ Easy to integrate

$du = \frac{1}{x}\,dx,$ $v = x.$ Simplest antiderivative

Then

$$\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

EXAMPLE Evaluate $\int x^2e^x\,dx.$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Hence,

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2\int xe^x\,dx\\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned}$$

EXAMPLE

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

Evaluating Definite Integrals by Parts

EXAMPLE

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution $\displaystyle\int_0^4 xe^{-x} \, dx.$

Let $u = x$, $dv = e^{-x} \, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x} \, dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x}) \, dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x} \, dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize

the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE Using Tabular Integration

Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

EXAMPLE Using Tabular Integration

Evaluate

$$\int x^3 \sin x\,dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \qquad ■$$

Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3},$$

which can be verified algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x + 1)(x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x - 3}{(x + 1)(x - 3)}\, dx = \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx$$

$$= 2 \ln |x + 1| + 3 \ln |x - 3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the above example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$
Do this for each distinct linear factor of $g(x)$.
2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE $\displaystyle\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx$

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A, B, and C we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

Coefficient of x^2: $A + B + C = 1$

Coefficient of x^1: $4A + 2B = 4$

Coefficient of x^0: $3A - 3B - C = 1$

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K,$$

EXAMPLE

Evaluate

$$\int \frac{6x + 7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$A = 6$ and $2A + B = 12 + B = 7$, or $A = 6$ and $B = -5$.

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$

$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$

EXAMPLE

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{rl} & 2x \\ x^2 - 2x - 3 & \overline{)\,2x^3 - 4x^2 - x - 3} \\ & \underline{2x^3 - 4x^2 - 6x} \\ & \phantom{)2x^3 - 4x^2 - {}} 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx &= \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx \\ &= \int 2x\,dx + \int \frac{2}{x + 1}\,dx + \int \frac{3}{x - 3}\,dx \\ &= x^2 + 2\ln|x + 1| + 3\ln|x - 3| + C. \end{aligned}$$

■

EXAMPLE

Evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx$$

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \qquad (2)$$

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$
$$= (A + C)x^3 + (-2A + B - C + D)x^2$$
$$+ (A - 2B + C)x + (B - C + D).$$

Equating coefficients of like terms gives

Coefficients of x^3: $0 = A + C$
Coefficients of x^2: $0 = -2A + B - C + D$
Coefficients of x^1: $-2 = A - 2B + C$
Coefficients of x^0: $4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$-4 = -2A, \quad A = 2$ Subtract fourth equation from second.
$C = -A = -2$ From the first equation
$B = 1$ $A = 2$ and $C = -2$ in third equation.
$D = 4 - B + C = 1.$ From the fourth equation

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx = \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx$$

$$= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx$$

$$= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x - 1} + C.$$

EXAMPLE

Evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned}1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A\end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \qquad C = 0, \qquad 2A + B + D = 0, \qquad C + E = 0, \qquad A = 1.$$

Solving this system gives $A = 1, \quad B = -1, \quad C = 0, \quad D = -1$, and $E = 0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2 + 1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2}\right] dx \\ &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2 + 1} - \int \frac{x\,dx}{(x^2 + 1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}\end{aligned}$$

$u = x^2 + 1,$
$du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K$$

$$= \ln|x| - \frac{1}{2}\ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} + K$$

$$= \ln\frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K.$$

EXAMPLE Find A, B, and C in the partial-fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}.$$

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,$$

$$A = 1.$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\boxed{(x-1)}\,(1-2)(1-3)} = \frac{2}{(-1)(-2)} = 1.$$

$$\underset{\text{Cover}}{\Uparrow}$$

$$B = \frac{(2)^2 + 1}{(2-1)\,\boxed{(x-2)}\,(2-3)} = \frac{5}{(1)(-1)} = -5.$$

$$\underset{\text{Cover}}{\Uparrow}$$

$$C = \frac{(3)^2 + 1}{(3-1)(3-2)\,\boxed{(x-3)}} = \frac{10}{(2)(1)} = 5.$$

$$\underset{\text{Cover}}{\Uparrow}$$

Heaviside Method

1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)}.$$

2. *Cover the factors $(x-r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1-r_2)\cdots(r_1-r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2-r_1)(r_2-r_3)\cdots(r_2-r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n-r_1)(r_n-r_2)\cdots(r_n-r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \cdots + \frac{A_n}{(x-r_n)}.$$

EXAMPLE

Evaluate

$$\int \frac{x+4}{x^3+3x^2-10x}\,dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0+4}{\underset{\underset{\text{Cover}}{\Uparrow}}{\boxed{x}}\,(0-2)(0+5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

$$A_2 = \frac{2+4}{2\,\underset{\underset{\text{Cover}}{\Uparrow}}{\boxed{(x-2)}}\,(2+5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

$$A_3 = \frac{-5+4}{(-5)(-5-2)\,\underset{\underset{\text{Cover}}{\Uparrow}}{\boxed{(x+5)}}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C.$$

Other Ways to Determine the Coefficients

EXAMPLE Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$

EXAMPLE Find A, B, and C in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution $x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$

$$\begin{aligned}
x = 1: \quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\
2 &= 2A \\
A &= 1 \\
x = 2: \quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\
5 &= -B \\
B &= -5 \\
x = 3: \quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\
10 &= 2C \\
C &= 5.
\end{aligned}$$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

Trigonometric Integrals

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

EXAMPLE

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d\,(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x \\
&= \int (u^4 - u^2) \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

EXAMPLE m is Even and n is Odd

Evaluate

$$\int \cos^5 x\, dx.$$

Solution

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x) \qquad m = 0$$

$$= \int (1 - u^2)^2\, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

EXAMPLE m and n are Both Even

Evaluate

$$\int \cos^5 x\, dx.$$

Solution

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x) \qquad m = 0$$

$$= \int (1 - u^2)^2\, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

EXAMPLE *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution

$$\begin{aligned}\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\ &= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx \\ &= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\ &= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].\end{aligned}$$

For the term involving $\cos^2 2x$ we use

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2}\int (1 + \cos 4x)\, dx \\ &= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).\end{aligned}$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx$$

$u = \sin 2x,$
$du = 2\cos 2x \, dx$

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx$$

$u = \sin 2x,$
$du = 2\cos 2x \, dx$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$ ■

Eliminating Square Roots

EXAMPLE

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \begin{array}{l}\cos 2x \geq 0 \\ \text{on } [0, \pi/4]\end{array} \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.
\end{aligned}$$

■

Integrals of Powers of tan *x* and sec *x*

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE

Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\begin{aligned}
\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx \\
&= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx
\end{aligned}$$

$$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$$
$$= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx.$$

In the first integral, we let $u = \tan x, \qquad du = \sec^2 x\,dx$

and have $\displaystyle\int u^2\,du = \frac{1}{3}u^3 + C_1.$

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

EXAMPLE

Evaluate

$$\int \sec^3 x\,dx.$$

Solution We integrate by parts, using

$$u = \sec x, \qquad dv = \sec^2 x\,dx, \qquad v = \tan x, \qquad du = \sec x \tan x\,dx.$$

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\,dx)\\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \qquad \tan^2 x = \sec^2 x - 1\\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

EXAMPLE

Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a \tan\theta$, $x = a \sin\theta$, and $x = a \sec\theta$. They come from the reference right triangles in Figure 8.2.

With $x = a \tan\theta$,

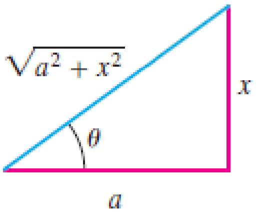
$$a^2 + x^2 = a^2 + a^2 \tan^2\theta = a^2(1 + \tan^2\theta) = a^2 \sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

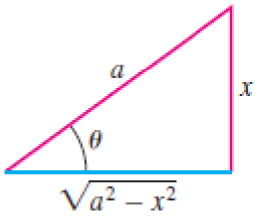
With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$



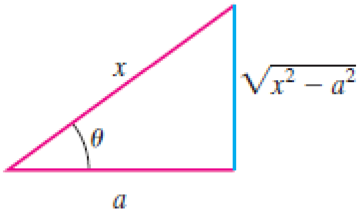
$x = a\tan\theta$

$\sqrt{a^2+x^2} = a|\sec\theta|$



$x = a\sin\theta$

$\sqrt{a^2-x^2} = a|\cos\theta|$



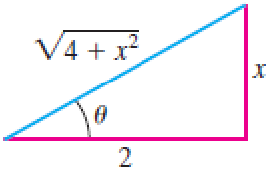
$x = a\sec\theta$

$\sqrt{x^2-a^2} = a|\tan\theta|$

EXAMPLE

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$



Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|}$$

$\sqrt{\sec^2\theta} = |\sec\theta|$

$$= \int \sec\theta\, d\theta$$

$\sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

From Fig. 8.4

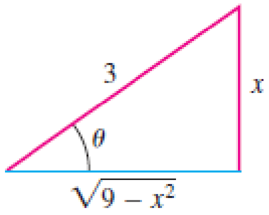
$$= \ln\left|\sqrt{4+x^2} + x\right| + C'.$$

Taking $C' = C - \ln 2$

EXAMPLE

Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$



Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta \qquad = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

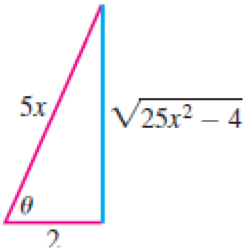
$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C. \qquad \cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

EXAMPLE

Evaluate

$$\int \frac{dx}{\sqrt{25x^2-4}}, \qquad x > \frac{2}{5}.$$



Solution We first rewrite the radical as

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$

$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$\begin{aligned} x^2 - \left(\frac{2}{5}\right)^2 &= \frac{4}{25}\sec^2\theta - \frac{4}{25} \\ &= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta \end{aligned}$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for
$0 < \theta < \pi/2$

With these substitutions, we have

$$\begin{aligned} \int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\,d\theta}{5\cdot(2/5)\tan\theta} \\ &= \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C \end{aligned}$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$

EXAMPLE

Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2 + 4)$, the x-axis, and the lines $x = 0$ and $x = 2$.

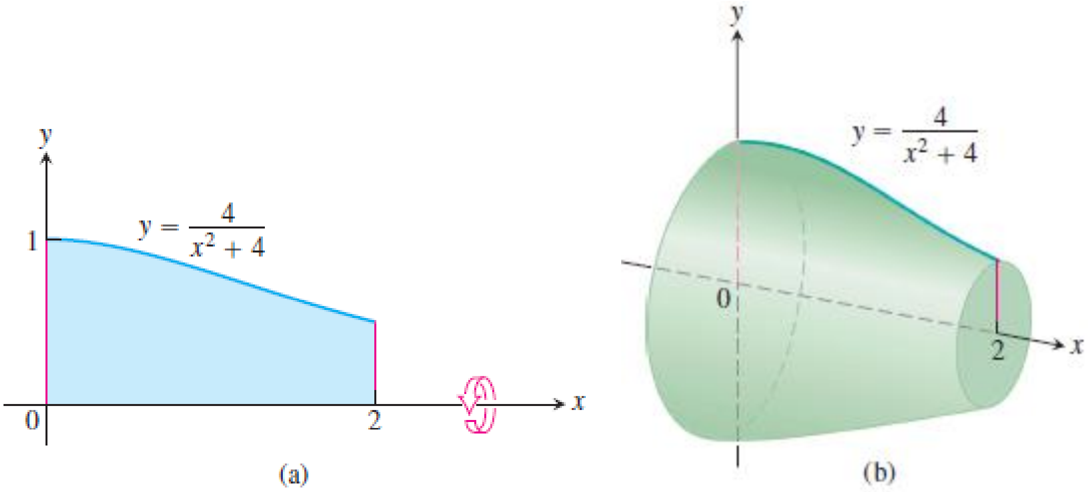
Solution

$$V = \int_0^2 \pi[R(x)]^2\,dx = 16\pi\int_0^2 \frac{dx}{(x^2 + 4)^2}.$$

$R(x) = \frac{4}{x^2 + 4}$

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\,d\theta, \qquad \theta = \tan^{-1}\frac{x}{2},$$

$$x^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$$



(a) (b)

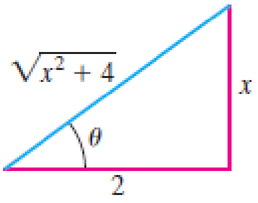
$$V = 16\pi \int_0^2 \frac{dx}{(x^2+4)^2}$$

$$= 16\pi \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{(4\sec^2\theta)^2}$$

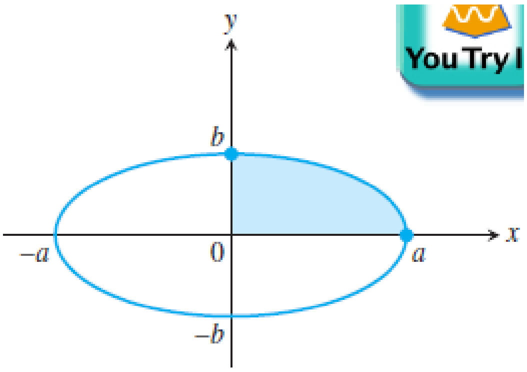
$$= 16\pi \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \pi \int_0^{\pi/4} 2\cos^2\theta\, d\theta$$

$$= \pi \int_0^{\pi/4} (1 + \cos 2\theta)\, d\theta = \pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \pi \left[\frac{\pi}{4} + \frac{1}{2}\right] \approx 4.04.$$



EXAMPLE



Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}, \qquad y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \le x \le a$$

The area of the ellipse is

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta$$

$x = a\sin\theta$, $dx = a\cos\theta\,d\theta$,
$\theta = 0$ when $x = 0$;
$\theta = \pi/2$ when $x = a$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta$$

$$= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2ab\left[\frac{\pi}{2} + 0 - 0\right] = \pi ab.$$

Improper Integrals

page619

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

EXAMPLE Evaluating an Improper Integral on $[1, \infty)$

Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

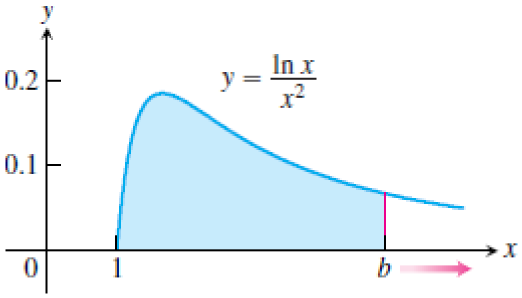
Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve . The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$.

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$



$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{\ln x}{x^2}\,dx \qquad = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1 \qquad = -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$ l'Hôpital's Rule

Thus, the improper integral converges and the area has finite value 1.

EXAMPLE Evaluating an Integral on $(-\infty, \infty)$ $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_{a}^{0} \frac{dx}{1+x^2} \qquad = \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0$$

$$= \lim_{a\to-\infty}(\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_{0}^{b} \frac{dx}{1+x^2} \qquad = \lim_{b\to\infty} \tan^{-1} x\Big]_0^b$$

$$= \lim_{b\to\infty}(\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus, $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

EXAMPLE

For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1}-1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right).$$

Thus,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b\to\infty}\frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\Big]_1^b = \lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.17b). First we find the area of the portion from a to 1

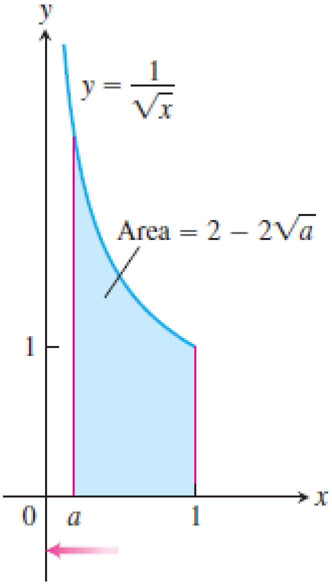
$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

The area under the curve from 0 to 1 is finite and equals

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$



DEFINITION **Type II Improper Integrals**

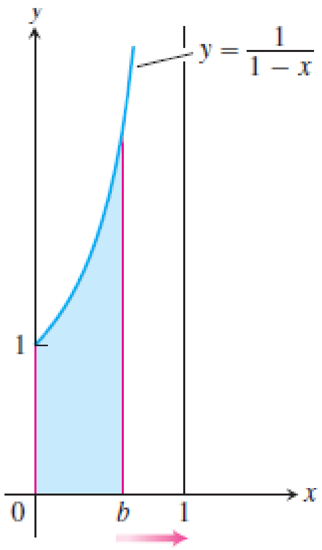
Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$
3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{1-x}\,dx.$

Solution The integrand $f(x) = 1/(1 - x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ We evaluate the integral as



$$\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-}\Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b\to 1^-}[-\ln(1-b)+0] = \infty.$$

The limit is infinite, so the integral diverges.

EXAMPLE

Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}}$$

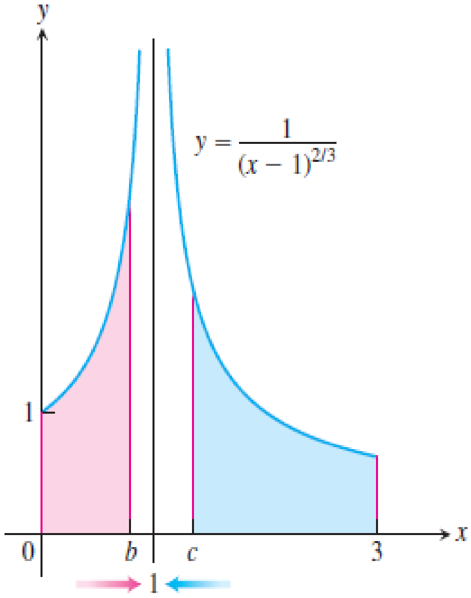
$$= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b$$

$$= \lim_{b\to 1^-}[3(b-1)^{1/3}+3] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3$$

$$= \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}$$



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

EXAMPLE

Evaluate $\displaystyle\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx.$

Solution

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\int_2^b \frac{x+3}{(x-1)(x^2+1)}\,dx$$

$$= \lim_{b\to\infty}\int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right)dx \qquad \text{Partial fractions}$$

$$= \lim_{b\to\infty}\left[2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x\right]_2^b$$

$$= \lim_{b\to\infty}\left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1}x\right]_2^b \qquad \text{Combine the logarithms.}$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1}b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1}2$$

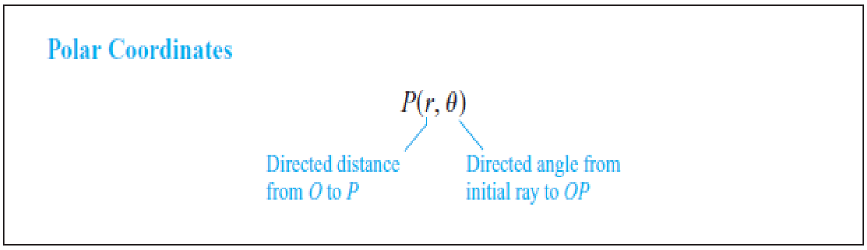
$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1}2 \approx 1.1458$$

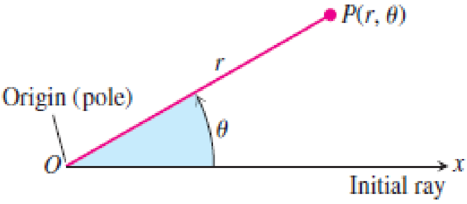
Polar Coordinates

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. This has interesting consequences for graphing, as we will see in the next section.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O. Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

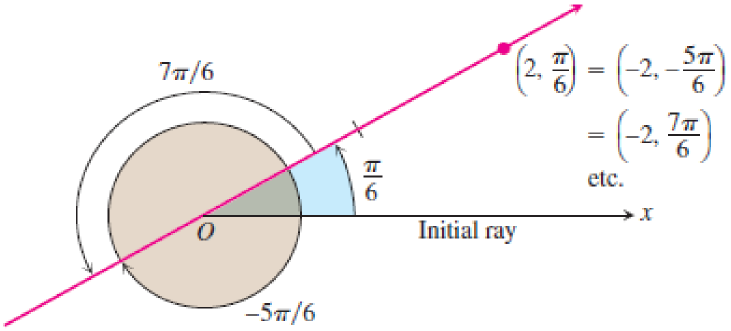




EXAMPLE Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$

We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.



For $r = 2$, the complete list of angles is $\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots.$

For $r = -2$, the angles are $-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots.$

The corresponding coordinate pairs of P are $\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$

and $$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

Polar Equations and Graphs

Equation	Graph
$r = a$	Circle radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

EXAMPLE Finding Polar Equations for Graphs

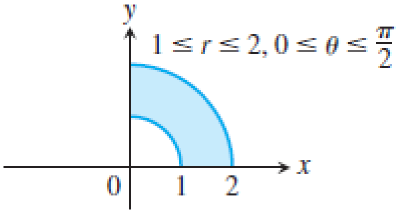
(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line

EXAMPLE Graph the sets of points whose polar coordinates satisfy the following conditions.

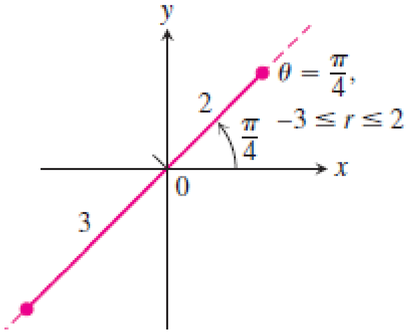
(a) $1 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le \frac{\pi}{2}$

(a)



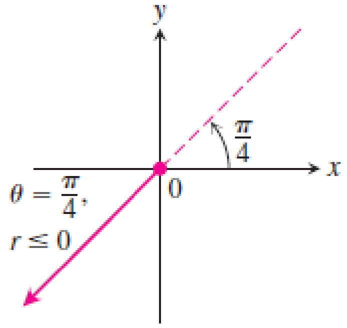
(b) $-3 \le r \le 2 \quad \text{and} \quad \theta = \frac{\pi}{4}$

(b)



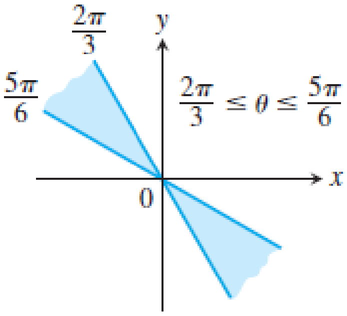
(c) $r \le 0$ and $\theta = \dfrac{\pi}{4}$

(c)



(d) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

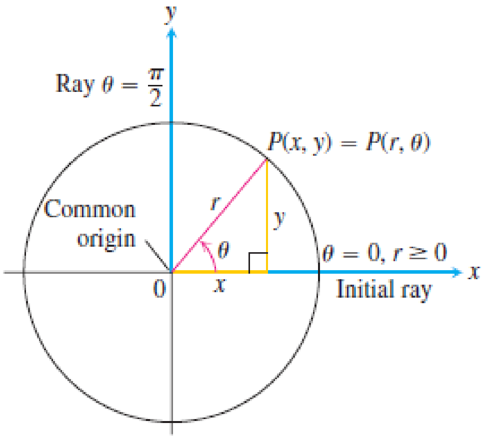
(d)



Relating Polar and Cartesian Coordinates

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$



EXAMPLE Equivalent Equations

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't. ■

EXAMPLE

Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y-3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{The 9's cancel.}$$

$$r^2 - 6r\sin\theta = 0 \qquad x^2 + y^2 = r^2$$

$$r = 0 \quad \text{or} \quad r - 6\sin\theta = 0 \qquad r = 6\sin\theta \qquad \text{Includes both possibilities}$$

EXAMPLE

Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$

$x = -4$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation: $r^2 = 4r\cos\theta$

$$x^2 + y^2 = 4x$$

$$x^2 - 4x + y^2 = 0$$

$$x^2 - 4x + 4 + y^2 = 4 \qquad \text{Completing the square}$$

$$(x - 2)^2 + y^2 = 4$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation: $r(2\cos\theta - \sin\theta) = 4$

$$2r\cos\theta - r\sin\theta = 4$$

$$2x - y = 4$$

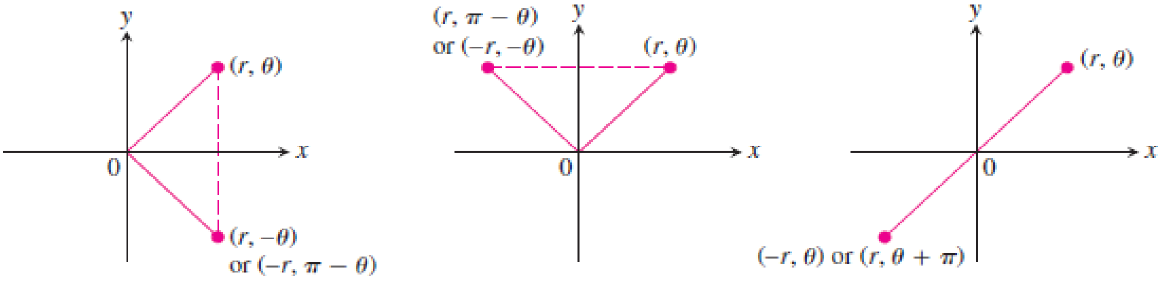
$$y = 2x - 4$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$

Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry



(a) About the x-axis (b) About the y-axis (c) About the origin

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.43a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.43c).

Slope

The slope of a polar curve $r = f(\theta)$ is given by dy/dx, not by $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

provided $dx/d\theta \neq 0$ at (r, θ).

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and the slope equation gives

$$\left.\frac{dy}{dx}\right|_{(0,\,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0.$$

If the graph of $r = f(\theta)$ passes through the origin at the value $\theta = \theta_0$, the slope of the curve there is $\tan\theta_0$. The reason we say "slope at $(0, \theta_0)$" and not just "slope at the origin" is that a polar curve may pass through the origin (or any point) more than once, with different slopes at different θ-values. This is not the case in our first example, however.

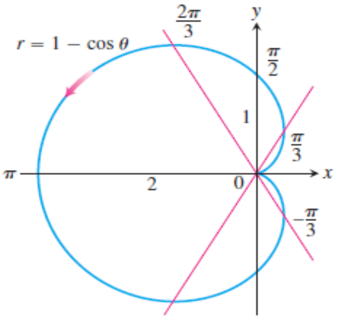
EXAMPLE Graph the curve $r = 1 - \cos\theta$.

Solution The curve is symmetric about the x-axis because

$$\begin{aligned}(r,\theta) \text{ on the graph} &\Rightarrow r = 1 - \cos\theta \\ &\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph}.\end{aligned}$$

Solution The curve is symmetric about the x-axis because

$$\begin{aligned}(r,\theta) \text{ on the graph} &\Rightarrow r = 1 - \cos\theta \\ &\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph}.\end{aligned}$$



EXAMPLE Graph the Curve $r^2 = 4\cos\theta$.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph } &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned}(r, \theta) \text{ on the graph } &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.}\end{aligned}$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

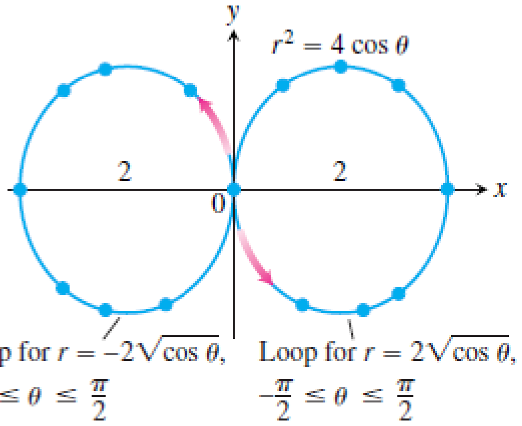
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

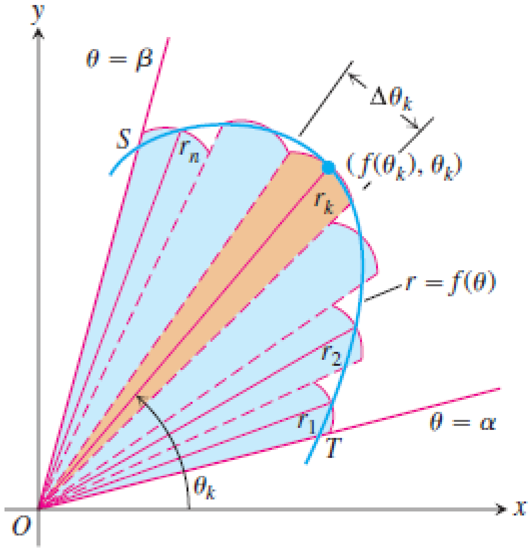
Areas and Lengths in Polar Coordinates

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta),\ \alpha \le \theta \le \beta$

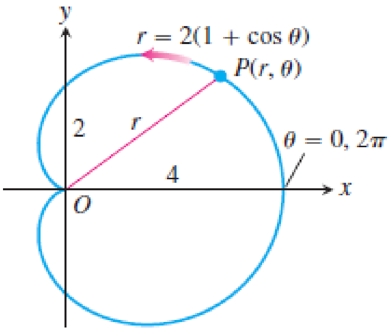
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta .$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} (f(\theta))^2 \, d\theta .$$



EXAMPLE Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.



$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

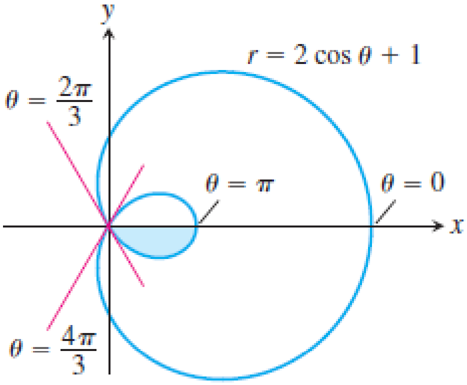
$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \quad = \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta = \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi .$$

EXAMPLE

Find the area inside the smaller loop of the limaçon $r = 2\cos\theta + 1$.

$$\begin{aligned} r^2 &= (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1 \\ &= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1 \\ &= 2 + 2\cos 2\theta + 4\cos\theta + 1 \\ &= 3 + 2\cos 2\theta + 4\cos\theta, \end{aligned}$$



$$\begin{aligned} A &= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta \\ &= \Big[3\theta + \sin 2\theta + 4\sin\theta\Big]_{2\pi/3}^{\pi} \\ &= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right) \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

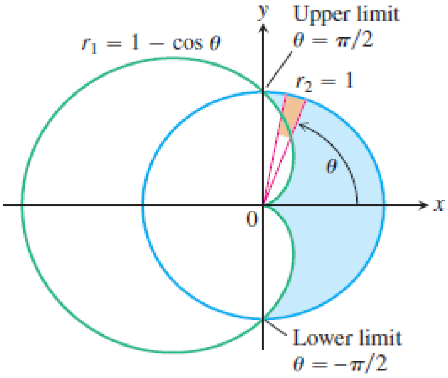
Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \qquad \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2\, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

EXAMPLE

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \\ &= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry} \\ &= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \\ &= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi/2} = 2 - \frac{\pi}{4}. \end{aligned}$$



Length of a Polar Curve

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

EXAMPLE Find the length of the cardioid $r = 1 - \cos\theta$.

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$

$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

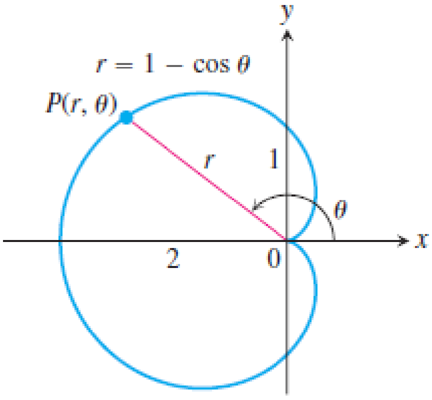
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \geq 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$



Sequences and Series

Sequences of Numbers

A ***sequence*** of numbers is a function whose domain is the set of positive integers.

Example

$0, \quad 1, \quad 2, \quad . \quad . \quad . \quad n-1, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = n -$

$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad . \quad . \quad . \quad \frac{1}{n}, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = \frac{1}{n}$

The index n is the ***domain*** of the sequence. While the numbers in the ***range*** of the sequence are called the ***terms*** of the sequence, and the number a_n being called the n^{th}-***term***, or ***the term with index*** n.

Example $\quad a_n = \frac{n+1}{n}$ $\qquad$ then the terms are

$$\begin{array}{ccccc} 1^{st} \ \ term & 2^{nd} \ \ term & 3^{rd} \ \ term & & n^{th} \ \ term \\ a_1 = 2, & a_2 = \frac{3}{2}, & a_3 = \frac{4}{3}, & . \quad . \quad . & a_n = \frac{n+1}{n}, \quad . \quad . \quad . \end{array}$$

and we use the notation $\{a_n\}$ as the sequence a_n.

Example

Find the first five terms of the following:

(a) $\left\{\frac{2n-1}{3n+2}\right\}$, $\qquad$ (b) $\left\{\frac{1-(-1)^n}{n^3}\right\}$, $\qquad$ (c) $\left\{(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}\right\}$

Solution

(a) $\frac{1}{5}, \quad \frac{3}{8}, \quad \frac{5}{11}, \quad \frac{7}{14}, \quad \frac{9}{17}$ $\qquad$ **(b)** $2, \quad 0, \quad \frac{2}{27}, \quad 0, \quad \frac{2}{125}$

(c) $x, \quad \frac{-x^3}{3!}, \quad \frac{x^5}{5!}, \quad \frac{-x^7}{7!}, \quad \frac{x^9}{9!}$

Example

Find the nth-term of the following:

(a) $0, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4},$ (b) $0, \quad \frac{\ln 2}{2}, \quad \frac{\ln 3}{3}, \quad \frac{\ln 4}{4},$ (c) $0, \quad \frac{1}{4}, \quad \frac{2}{9}, \quad \frac{3}{16},$

(d) $2, \quad 1, \quad \frac{2^3}{3^2}, \quad \frac{2^4}{4^2}, \quad \frac{2^5}{5^2}$

Solution

(a) $a_n = \frac{n-1}{n}$, **(b)** $a_n = \frac{\ln n}{n}$, **(c)** $a_n = \frac{n-1}{n^2}$, **(d)** $a_n = \frac{2^n}{n^2}$

Convergence of Sequences

The fact that $\{a_n\}$ converges to L is written as

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

and we call the limit of the sequence $\{a_n\}$. If no such limit exists, we say that $\{a_n\}$ diverges.

From that we can say that

1) $\lim_{n\to\infty} a_n = L$ (Conv.)

2) $\lim_{n\to\infty} a_n = \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)

Also, if $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ both exist and are finite, then

i) $\lim_{n\to\infty} \{a_n + b_n\} = A + B$

ii) $\lim_{n\to\infty} \{ka_n\} = kA$

iii) $\lim\limits_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

iv) $\lim\limits_{n\to\infty}\left\{\dfrac{a_n}{b_n}\right\} = \dfrac{A}{B},$ provided $B \neq 0$ and b_n is never 0

Example

Test the convergence of the following:

(a) $\left\{\dfrac{1}{n}\right\},$ (b) $\{1+(-1)^n\},$ (c) $\{n^2\},$ (d) $\{\sqrt{n+1}-\sqrt{n}\},$

(e) $\left\{\dfrac{3n^2-5n}{5n^2+2n+6}\right\},$ (f) $\left\{\dfrac{3n^2-4n}{2n-1}\right\},$ (g) $\left\{\left(\dfrac{2n-3}{3n-7}\right)^4\right\},$ (h) $\left\{\dfrac{2n^5-4n^2}{3n^7+n^2-10}\right\},$

(i) $\left\{\dfrac{2^n}{5n}\right\},$ (j) $\left\{\dfrac{\ln n}{e^n}\right\}$

Solution

(a) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right) = 0$ ***(Conv.)***

(b) $\lim\limits_{n\to\infty}\left(1+(-1)^n\right) = 1 + \lim\limits_{n\to\infty}(-1)^n = \begin{cases} 0 & n \quad odd \\ 2 & n \quad even \end{cases}$ ***(Div.)***

(c) $\lim\limits_{n\to\infty}(n^2) = \infty$ ***(Div.)***

(d) $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim\limits_{n\to\infty}\left(\left(\sqrt{n+1}-\sqrt{n}\right) \times \dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \lim\limits_{n\to\infty}\left(\dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \frac{1}{\infty+\infty} = 0$$ ***(Conv.)***

(e) $$\lim_{n\to\infty}\left(\frac{3n^2-5n}{5n^2+2n+6}\right)=\lim_{n\to\infty}\left(\frac{\frac{3n^2}{n^2}-\frac{5n}{n^2}}{\frac{5n^2}{n^2}+\frac{2n}{n^2}+\frac{6}{n^2}}\right)=\frac{3}{5}$$ ***(Conv.)***

(f) $$\lim_{n\to\infty}\left(\frac{3n^2-4n}{2n-1}\right)=\lim_{n\to\infty}\left(\frac{\frac{3n^2}{n^2}-\frac{4n}{n^2}}{\frac{2n}{n^2}-\frac{1}{n^2}}\right)=\frac{3}{0}=\infty$$ ***(Div.)***

(g) $$\lim_{n\to\infty}\left(\frac{2n-3}{3n-7}\right)^4=\left(\frac{2}{3}\right)^4=\frac{16}{81}$$ ***(Conv.)***

(h) $$\lim_{n\to\infty}\left(\frac{2n^5-4n^2}{3n^7+n^2-10}\right)=\lim_{n\to\infty}\left(\frac{\frac{2}{n^2}-\frac{4}{n^5}}{3+\frac{1}{n^5}-\frac{10}{n^7}}\right)=0$$ ***(Conv.)***

(i) $$\lim_{n\to\infty}\left(\frac{2^n}{5n}\right)=\lim_{n\to\infty}\left(\frac{2^n.\ln 2}{5}\right)=\infty$$ ***(Div.)***

(j) $$\lim_{n\to\infty}\left(\frac{\ln n}{e^n}\right)=\lim_{n\to\infty}\left(\frac{1/n}{e^n}\right)=\lim_{n\to\infty}\left(\frac{1}{n.e^n}\right)=\frac{1}{\infty}=0$$ ***(Conv.)***

<u>Example</u>

Prove the following limits

(a) $\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)=0$, (b) $\lim_{n\to\infty}\left(\sqrt[n]{n}\right)=1$, (c) $\lim_{n\to\infty}\left(x^{1/n}\right)=1$ $(x>0)$,

(d) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$ (any x), (e) $\lim_{n\to\infty}\left(\frac{x^n}{n!}\right)=0$ (any x)

Solution

(a) $\lim\limits_{n\to\infty}\left(\dfrac{\ln n}{n}\right)=\lim\limits_{n\to\infty}\left(\dfrac{1/n}{1}\right)=\dfrac{0}{1}=0$

(b) Let $a_n=n^{1/n}$, then $\ln a_n=\ln n^{1/n}=\dfrac{1}{n}\ln n\to 0$,

So, $\lim\limits_{n\to\infty} n^{1/n}=e^{\ln a_n}\to e^0=1$

(c) Let $a_n=x^{1/n}$, then $\ln a_n=\ln x^{1/n}=\dfrac{1}{n}\ln x\to 0$,

So, $\lim\limits_{n\to\infty} x^{1/n}=e^{\ln a_n}\to e^0=1$

(d) Let $a_n=\left(1+\dfrac{x}{n}\right)^n$, then

$$\ln a_n=\ln\left(1+\frac{x}{n}\right)^n=n.\ln\left(1+\frac{x}{n}\right)$$

So, $\lim\limits_{n\to\infty} n.\ln\left(1+\dfrac{x}{n}\right)=\lim\limits_{n\to\infty}\dfrac{\ln(1+x/n)}{1/n}=\lim\limits_{n\to\infty}\dfrac{\left(\dfrac{1}{1+x/n}\right)\cdot\left(-\dfrac{x}{n^2}\right)}{-1/n^2}$

$$=\lim_{n\to\infty}\frac{x}{1+x/n}=x,$$

Thus, $\left(1+\dfrac{x}{n}\right)^n=a_n=e^{\ln a_n}\to e^x$

(e) $\lim\limits_{n\to\infty}\left(\dfrac{x^n}{n!}\right)=\lim\limits_{n\to\infty}\left(\dfrac{x}{1}\right)\left(\dfrac{x}{2}\right)\left(\dfrac{x}{3}\right)\ldots\left(\dfrac{x}{n}\right)=0$

Exercises on Sequences

Find the values of a_1, a_2, a_3 *and* a_4 *for the following sequences*

1) $a_n = \frac{1-n}{n^2}$

2) $a_n = \frac{1}{n!}$

3) $a_n = \frac{(-1)^{n+1}}{2n-1}$

4) $a_n = 2+(-1)^n$

5) $a_n = \frac{2^n}{2^{n+1}}$

6) $a_n = \frac{2^n - 1}{2^n}$

Find a formula for the n^{th} *term of the following sequences*

1) $1,-1,1,-1,1,....$

2) $-1,1,-1,1,-1,....$

3) $1,-4,9,-16,25,...$

4) $1,-\frac{1}{4},\frac{1}{9},-\frac{1}{16},\frac{1}{25},....$

5) $0,3,8,15,24,...$

6) $-3,-2,-1,0,1,...$

7) $1,5,9,13,17,...$

8) $2,6,10,14,18,...$

9) $1,0,1,0,1,...$

Which of the following sequences converge and which diverge?

1) $a_n = 2+(0.1)^n$ *Ans.* *Converges,* 2

2) $a_n = \frac{1-2n}{1+2n}$ *Ans.* *Converges,* -1

3) $a_n = \frac{1-5n^4}{n^4+8n^3}$ *Ans.* *Converges,* -5

4) $a_n = \frac{n^2-2n+1}{n-1}$ *Ans.* *Diverges*

5) $a_n = 1+(-1)^n$ *Ans.* *Diverges*

6) $a_n = \left(\dfrac{n+1}{2n}\right)\left(1-\dfrac{1}{n}\right)$ ***Ans.*** *Converges,* $\dfrac{1}{2}$

7) $a_n = \dfrac{(-1)^{n+1}}{2n-1}$ ***Ans.*** *Converges,* 0

8) $a_n = \sqrt{\dfrac{2n}{n+1}}$ ***Ans.*** *Converges,* $\sqrt{2}$

9) $a_n = \sin\left(\dfrac{\pi}{2}+\dfrac{1}{n}\right)$ ***Ans.*** *Converges,* 1

10) $a_n = \dfrac{\sin n}{n}$ ***Ans.*** *Converges,* 0

11) $a_n = \dfrac{n}{2^n}$ ***Ans.*** *Converges,* 0

12) $a_n = \dfrac{\ln(n+1)}{n}$ ***Ans.*** *Converges,* 0

13) $a_n = 8^{1/n}$ ***Ans.*** *Converges,* 1

14) $a_n = \left(1+\dfrac{7}{n}\right)^n$ ***Ans.*** *Converges,* e^7

15) $a_n = \sqrt[n]{10n}$ ***Ans.*** *Converges,* 1

16) $a_n = \left(\dfrac{3}{n}\right)^{1/n}$ ***Ans.*** *Converges,* 1

17) $a_n = \dfrac{\ln n}{n^{1/n}}$ ***Ans.*** *Diverges*

18) $a_n = \sqrt[n]{4^n n}$ ***Ans.*** *Converges,* 4

20) $a_n = \dfrac{n!}{10^{6n}}$ **Ans.** *Diverges*

21) $a_n = \left(\dfrac{1}{n}\right)^{1/(\ln n)}$ **Ans.** *Converges,* e^{-1}

22) $a_n = \left(\dfrac{3n+1}{3n-1}\right)^{n}$ **Ans.** *Converges,* $e^{2/3}$

23) $a_n = \left(\dfrac{x^n}{2n+1}\right)^{1/n}, \quad x > 0$ **Ans.** *Converges,* $x \ \ (x > 0)$

24) $a_n = \dfrac{3^n \times 6^n}{2^{-n} \times n!}$ **Ans.** *Converges,* 0

25) $a_n = \tanh(n)$ **Ans.** *Converges,* 1

26) $a_n = \dfrac{n^2}{2n-1}\sin\dfrac{1}{n}$ **Ans.** *Converges,* $\dfrac{1}{2}$

27) $a_n = \tan^{-1}(n)$ **Ans.** *Converges,* $\dfrac{\pi}{2}$

28) $a_n = \left(\dfrac{1}{3}\right)^{n} + \dfrac{1}{\sqrt{2^n}}$ **Ans.** *Converges,* 0

29) $a_n = \dfrac{(\ln n)^{200}}{n}$ **Ans.** *Converges,* 0

30) $a_n = n - \sqrt{n^2 - n}$ **Ans.** *Converges,* $\dfrac{1}{2}$

31) $a_n = \dfrac{1}{n}\displaystyle\int_1^n \dfrac{1}{x}\,dx$ **Ans.** *Converges,* 0

Infinite Series

Infinite series are sequences of a special kind: those in which the ***n***th-term is the sum of the first ***n*** terms of a related sequence.

Example

Suppose that we start with the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

If we denote the above sequence as a_n, and the resultant sequence of the series as s_n, then

$$s_1 = a_1 = 1,$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},$$

as the first three terms of the sequence $\{s_n\}$.

When the sequence $\{s_n\}$ is formed in this way from a given sequence $\{a_n\}$ by the rule

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

the result is called an ***Infinite Series.***

- ❖ The number $s_n = \sum_{k=1}^{n} a_k$ is called the ***n***th ***partial sum*** of the series.
- ❖ Instead of $\{s_n\}$, we usually write $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.
- ❖ The series $\sum a_n$ is said to ***converge*** to a number L if and only if

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

in which case we call L the sum of the series and write

$$\sum_{n=1}^{\infty} a_n = L \qquad \text{or} \qquad a_1 + a_2 + \ldots + a_n + \ldots = L$$

If no such limit exists, the series is said to ***diverge***.

Geometric Series

A series of the form

$$a + ar + ar^2 + ar^3 + \ldots + + ar^{n-1} + \ldots$$

is called a ***Geometric Series***. The ratio of any term to the one before it is r. If $|r| < 1$, the geometric series converges to $a/(1-r)$. If $|r| \geq 1$, the series diverges unless $a = 0$. If $a = 0$, the series converges to 0.

Example

Geometric series with $a = \dfrac{1}{9}$ and $r = \dfrac{1}{3}$.

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots = \frac{1}{9}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) = \frac{1/9}{1-(1/3)} = \frac{1}{6}$$

Geometric series with $a = 4$ and $r = -\dfrac{1}{2}$.

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \ldots = 4\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots\right)$$

$$= \frac{4}{1+(1/2)} = \frac{8}{3}$$

Example

Determine whether each series converges or diverges. If it converges, find its sum.

(a) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$, (b) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, (c) $\sum_{n=1}^{\infty}2\left(\cos\frac{\pi}{3}\right)^n$, (d) $\sum_{n=0}^{\infty}\left(\tan\frac{\pi}{4}\right)^n$, (e) $\sum_{n=1}^{\infty}\frac{5(-1)^n}{4^n}$

Solution

(a) Since the series is a geometric series with $r=\frac{2}{3}<1$, so the series is convergent with a sum of $\frac{1}{1-(2/3)}=3$

(b) Since the series is a geometric series with $r=\frac{3}{2}>1$, so the series is divergent.

(c) $\cos\pi/3=1/2$. This is a geometric series with first term $a_1=1$ and the ratio $r=1/2$; so the series converges and its sum is $1/(1-\frac{1}{2})=2$.

(d) $\tan\pi/4=1$. This is a geometric series with $r=1$, so the series diverges.

(e) This is a geometric series with first term $a_1=-5/4$ and ratio $r=-1/4$. So the series converges and its sum is $\frac{-5/4}{1+(1/4)}=-1$.

Test Convergence of Series with Non-negative Terms

1) The nth- Term Test

- If $\lim\limits_{n\to\infty}a_n\neq 0$, or if $\lim\limits_{n\to\infty}a_n$ fails to exist, then $\sum_{n=1}^{\infty}a_n$ diverges.
- If $\sum_{n=1}^{\infty}a_n$ converges, then $a_n\to 0$.
- If $\lim\limits_{n\to\infty}a_n=0$, then the test fails.

From the above, it can not be concluded that if $a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges. The series $\sum_{n=1}^{\infty} a_n$ may diverge even though $a_n \to 0$. Thus $\lim_{n\to\infty} a_n = 0$ is a necessary but not a sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ to converge.

Examples

$\sum_{n=1}^{\infty} n^2$	diverges because $n^2 \to \infty$,
$\sum_{n=1}^{\infty} \frac{n+1}{n}$	diverges because $\frac{n+1}{n} \to 1 \neq 0$,
$\sum_{n=1}^{\infty} (-1)^{n+1}$	diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist,
$\sum_{n=1}^{\infty} \frac{n}{2n+5}$	diverges because $\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \neq 0$,
$\sum_{n=1}^{\infty} \frac{1}{n}$	can not be tested by the nth-term test for divergence because $\frac{1}{n} \to 0$.

2) The Integral Test

Let the function $y = f(x)$, obtained by introducing the continuous variable x in place of the discrete variable n in the nth-term of the positive series $\sum_{n=1}^{\infty} a_n$, then

$$\int_1^{\infty} f(x)dx = \begin{cases} +\infty & Div. \\ -\infty & Div. \\ -\infty < c < \infty & Conv. \end{cases}$$

Example

Prove that, for the p-series, if p is a real constant, the series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$

converges if $p>1$ and diverges if $p\le 1$.

Solution

To prove this, let

$$f(x)=\frac{1}{x^p}$$

Then, if $p>1$, we have

$$\int_1^{\infty}x^{-p}dx=\lim_{b\to\infty}\frac{x^{-p+1}}{-p+1}\bigg|_1^b=\frac{1}{p-1}$$

which is finite. Hence, the p-series converges if $p>1$.

If $p=1$, which is called a harmonic series, we have

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots,$$

and the integral test is

$$\int_1^{\infty}x^{-1}dx=\lim_{b\to\infty}\ \ln x\Big|_1^b=+\infty$$

which diverges.

Finally, for $p<1$, then the terms of the series are greater than the corresponding terms of the divergent harmonic series. Hence the p-series diverges for $p<1$.

Thus, we have a convergence for $p>1$, but divergence for $p\le 1$.

Example

Test the convergence of

(a) $\sum_{n=1}^{\infty} \frac{1}{e^n}$, (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution

(a) $\int_1^{\infty} e^{-x} dx = -e^{-x}\Big|_1^{\infty} = -(e^{-\infty} - e^{-1}) = \frac{1}{e}$ ***(Conv.)***

(b) $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_2^{\infty} \frac{1/x}{(\ln x)^2} dx = \frac{-1}{\ln x}\Big|_2^{\infty} = \frac{-1}{\infty} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$ ***(Conv.)***

3) The Ratio Test

Let $\sum a_n$ be a series with positive terms, and suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- The series converges if $\rho < 1$,
- The series diverges if $\rho > 1$,
- The series may converge or it may diverge if $\rho = 1$. (Test fails)

The Ratio Test is often effective when the terms of the series contain factorials of expressions involving n or expressions raised to a power involving n.

Example

Test the following series for convergence or divergence , using the Ratio Test.

(a) $\sum_{n=1}^{\infty}\frac{n!n!}{(2n)!}$, (b) $\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$, (c) $\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$, (d) $\sum_{n=1}^{\infty}\frac{n!}{3^n}$, (e) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

Solution

(a) If $a_n=\frac{n!n!}{(2n)!}$, then $a_{n+1}=\frac{(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!}=\frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$=\frac{n+1}{4n+2}\to\frac{1}{4}<1 \qquad \textbf{\textit{(Conv.)}}$$

(b) If $a_n=\frac{4^n n!n!}{(2n)!}$, then $a_{n+1}=\frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n}=\frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!}\times\frac{(2n)!}{4^n n!n!}=\frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$=\frac{2(n+1)}{2n+1}\to 1 \qquad \textbf{\textit{(Test fails)}}$$

(c) If $a_n=\frac{2^n+5}{3^n}$, then $a_{n+1}=\frac{2^{n+1}+5}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n}=\frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n}=\frac{1}{3}\times\frac{2^{n+1}+5}{2^n+5}$$

$$=\frac{1}{3}\times\left(\frac{2+5\times2^{-n}}{1+5\times2^{-n}}\right)\to\frac{1}{3}\times\frac{2}{1}=\frac{2}{3}<1 \qquad \textbf{\textit{(Conv.)}}$$

(d) If $a_n = \dfrac{n!}{3^n}$, then $a_{n+1} = \dfrac{(n+1)!}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \times \frac{3^n}{n!} = \frac{n+1}{3} \to \infty > 1 \qquad \textbf{\textit{(Div.)}}$$

(e) If $a_n = \dfrac{n^n}{n!}$, then $a_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^n (n+1) n!}{(n+1) n! n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e^1 = 2.7 > 1 \qquad \textbf{\textit{(Div.)}}$$

4) The n^{th} Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n > n_0$ and suppose that

$$\sqrt[n]{a_n} \to \rho$$

Then

- ❖ The series converges if $\rho < 1$.
- ❖ The series diverges if $\rho > 1$.
- ❖ The test is not conclusive if $\rho = 1$.

Example

Test the convergence of the following series using the n^th^ Root Test.

(a) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^n}$, (b) $\sum\limits_{n=1}^{\infty} \dfrac{2^n}{n^2}$, (c) $\sum\limits_{n=1}^{\infty} \left(1 - \dfrac{1}{n}\right)^n$, (d) $\sum\limits_{n=1}^{\infty} \left(\dfrac{n}{n+1}\right)^{n^2}$, (e) $\sum\limits_{n=1}^{\infty} \left(\dfrac{2n}{n+1}\right)^n$

Solution

(a) $\sqrt[n]{\dfrac{1}{n^n}} = \dfrac{1}{n} \to 0 < 1$ *(Conv.)*

(b) $\sqrt[n]{\dfrac{2^n}{n^2}} = \dfrac{2}{\sqrt[n]{n^2}} = \dfrac{2}{\left(\sqrt[n]{n}\right)^2} \to \dfrac{2}{1^2} = 2 > 1$ *(Div.)*

(c) $\sqrt[n]{\left(1-\dfrac{1}{n}\right)^n} = \left(1-\dfrac{1}{n}\right) \to 1$ *(Test fails)*

(d) $\sqrt[n]{\left(\dfrac{n}{n+1}\right)^{n^2}} = \left(\dfrac{n}{n+1}\right)^{\frac{n^2}{n}} = \left(\dfrac{n}{n+1}\right)^n = \left(\dfrac{1}{1+1/n}\right)^n \to \dfrac{1}{e} = \dfrac{1}{2.7} < 1$ *(Conv.)*

(e) $\sqrt[n]{\left(\dfrac{2n}{n+1}\right)^n} = \dfrac{2n}{n+1} \to 2 > 1$ *(Div.)*

Exercises on Series

Find the sum of the following series

1) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$ *Ans.* $\dfrac{4}{5}$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{4^n}$ *Ans.* $\dfrac{7}{3}$

3) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$ *Ans.* $\dfrac{23}{2}$

4) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$ *Ans.* $\dfrac{17}{6}$

5) $\sum\limits_{n=1}^{\infty}\frac{4}{(4n-3)(4n+1)}$ ***Ans.*** 1

6) $\sum\limits_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2n+1)^2}$ ***Ans.*** 5

7) $\sum\limits_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ ***Ans.*** 1

8) $\sum\limits_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$ ***Ans.*** $-\frac{1}{\ln 2}$

Which of the following series converges and which diverges? Find the sum of the convergent series.

1) $\sum\limits_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ ***Ans.*** *Converges,* $2+\sqrt{2}$

2) $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$ ***Ans.*** *Converges,* 1

3) $\sum\limits_{n=0}^{\infty}\cos(n\pi)$ ***Ans.*** *Diverges*

4) $\sum\limits_{n=0}^{\infty}e^{-2n}$ ***Ans.*** *Converges,* $\frac{e^2}{e^2-1}$

5) $\sum\limits_{n=1}^{\infty}\frac{2}{10^n}$ ***Ans.*** *Converges,* $\frac{2}{9}$

6) $\sum\limits_{n=0}^{\infty}\frac{2^n-1}{3^n}$ ***Ans.*** *Converges,* $\frac{3}{2}$

7) $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$ *Ans.* *Diverges*

8) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ *Ans.* *Diverges*

9) $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$ *Ans.* *Converges,* $\frac{\pi}{\pi - e}$

Which of the following series converges and which diverges?

1) $\sum_{n=1}^{\infty} \frac{1}{10^n}$ *Ans.* ***Converges (Geometric)***

2) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ *Ans.* ***Diverges (*** n^{th}***-term test)***

3) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ *Ans.* ***Diverges (p-series)***

4) $\sum_{n=1}^{\infty} \frac{-1}{8^n}$ *Ans.* ***Converges (Geometric)***

5) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ *Ans.* ***Diverges (Integral Test)***

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ *Ans.* ***Converges (Geometric)***

7) $\sum_{n=0}^{\infty} \frac{-2}{n+1}$ *Ans.* ***Diverges (Integral Test)***

8) $\sum\limits_{n=1}^{\infty} \frac{2^n}{n+1}$ *Ans. Diverges (n^{th}-term test)*

9) $\sum\limits_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ *Ans. Diverges (n^{th}-term test)*

10) $\sum\limits_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$ *Ans. Diverges (Geometric)*

11) $\sum\limits_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n-1}}$ *Ans. Converges (Integral Test)*

12) $\sum\limits_{n=1}^{\infty} n \sin \frac{1}{n}$ *Ans. Diverges (n^{th}-term test)*

13) $\sum\limits_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ *Ans. Converges (Integral Test)*

14) $\sum\limits_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2}$ *Ans. Converges (Integral Test)*

15) $\sum\limits_{n=1}^{\infty} \frac{2n}{3n-1}$ *Ans. Diverges (n^{th}-term test)*

16) $\sum\limits_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$ *Ans. Converges (Ratio Test)*

17) $\sum\limits_{n=1}^{\infty} n!\, e^{-n}$ *Ans. Diverges (Ratio Test)*

18) $\sum\limits_{n=1}^{\infty} \frac{n^{10}}{10^n}$ *Ans. Converges (Ratio Test)*

19) $\sum\limits_{n=1}^{\infty} \left(1-\frac{3}{n}\right)^n$ *Ans. Diverges (n^{th}-term test)*

20) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$ *Ans. Converges (Ratio Test)*

21) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!3^{n}}$ *Ans. Converges (Ratio Test)*

22) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ *Ans. Converges (Ratio Test)*

23) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{n}}$ *Ans. Converges (Root Test)*

24) $\sum_{n=1}^{\infty} \frac{(n!)^{n}}{(n^{n})^{2}}$ *Ans. Diverges (Root Test)*

25) $\sum_{n=1}^{\infty} \frac{n^{n}}{2^{(n^{2})}}$ *Ans. Converges (Root Test)*

Alternating Series

A series in which the terms are alternately positive and negative.

Example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots+\frac{(-1)^{n+1}}{n}+\ldots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n}4}{2^{n}}+\ldots$$

$$1-2+3-4+5-6+\ldots+(-1)^{n+1}n+\ldots$$

The Convergence Test of Alternating Series

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-u_4+\ldots$$

converges if all three of the following conditions are satisfied:

1) The u_n's are all positive.

2) $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.

3) $u_n \to 0$.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

satisfies the three requirements of convergence; it therefore converges.

Absolute Convergence

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges, i.e.,

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Example

The geometric series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ converges absolutely because the corresponding series of absolute values $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ converges .

Conditional Convergence

A series that converges but does not converge absolutely ***converges conditionally***.

Example

The alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ does not converge absolutely. The corresponding series of absolute values $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is the divergent harmonic series.

Power Series

- A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

- A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example

The series $\sum_{n=0}^{\infty} x^n$ is a geometric series with first term 1 and ratio x. It converges to

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots \qquad \text{for } |x| < 1$$

Convergence of Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

The test of power series is done using the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \rho \begin{cases} <1 & \mathit{Conv.} \\ >1 & \mathit{Div.} \\ =1 & \mathit{Fails} \end{cases}$$

Notes:

- Use the Ratio Test to find the interval where the series converges absolutely.
- If the interval of absolute convergence is finite, test the convergence or divergence at each endpoint. Use the integral test or the Alternating Series Test for endpoints.

- If the interval of absolute convergence is $|x-a|<R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally), because the n$^{\text{th}}$-term does not approach zero for those values of x.

Example

For what values of x do the following power series converge?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,$ (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,$ (d) $\displaystyle\sum_{n=0}^{\infty}n!x^n=1+x+2!x^2+3!x^3+\ldots$

Solution

(a) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right|=\frac{n}{n+1}|x|\to|x|.$

The series converges absolutely for $|x|<1$. It diverges if $|x|>1$ because the n$^{\text{th}}$-term does not converge to zero. At $x=1$, we get the alternating harmonic series $1-1/2+1/3-1/4+\ldots$, which converges. At $x=-1$, we get $-1-1/2-1/3-1/4-\ldots$, the negative of the harmonic series; it diverges. So, the series converges for $-1<x\leq 1$ and diverges elsewhere.

(b) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{2n+1}}{2n+1}\times\frac{2n-1}{x^{2n-1}}\right|=\frac{2n-1}{2n+1}x^2\to x^2.$

The series converges absolutely for $x^2<1$. It diverges for $x^2>1$ because the n$^{\text{th}}$-term does not converge to zero. At $x=1$, the series becomes $1-1/3+1/5-1/7+\ldots$, which converges because it satisfies the three conditions of convergence of alternating series. It also converges at $x=-1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x=-1$ is the negative of the value at $x=1$. So, the series converges for $-1 \le x \le 1$ and diverges elsewhere.

(c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$ for every x.

The series converges absolutely for all x.

(d) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \to \infty$ unless $x=0$.

The series diverges for all values of x except $x=0$.

Exercises on Alternating & Power Series

Which of the following series converges and which diverges?

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ *Ans. Converges*

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$ *Ans. Diverges,* $a_n \to \infty$

3) $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$ *Ans. Converges*

4) $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln(n^2)}$ *Ans. Diverges,* $a_n \to \frac{1}{2}$

5) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$ *Ans. Converges*

Which of the following series converges absolutely, conditionally, and which diverges?

1) $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$ ***Ans.*** *Converges absolutely*

2) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ ***Ans.*** *Converges conditionally*

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$ ***Ans.*** *Diverges,* $a_n \to 1$

4) $\sum_{n=1}^{\infty}(-1)^n n^2\left(\frac{2}{3}\right)^n$ ***Ans.*** *Converges absolutely*

5) $\sum_{n=1}^{\infty}(-1)^n\frac{\tan^{-1} n}{n^2+1}$ ***Ans.*** *Converges absolutely*

6) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$ ***Ans.*** *Diverges,* $a_n \to 1$

7) $\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$ ***Ans.*** *Converges absolutely*

8) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n\sqrt{n}}$ ***Ans.*** *Converges absolutely*

9) $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^n}{(2n)^n}$ ***Ans.*** *Converges absolutely*

10) $\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{2^n n! n}$ ***Ans.*** *Diverges,* $a_n \to \infty$

Find the interval of convergence for the following series

1) $\sum_{n=0}^{\infty} x^n$ ***Ans.*** $-1 < x < 1$

2) $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ ***Ans.*** $-\frac{1}{2} < x < 0$

3) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ ***Ans.*** $-8 < x < 12$

4) $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$ ***Ans.*** $-1 < x < 1$

5) $\sum_{n=1}^{\infty} \frac{x^n}{3^n n\sqrt{n}}$ ***Ans.*** $-3 \leq x \leq 3$

6) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ ***Ans.*** *For all* x

7) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ ***Ans.*** *For all* x

8) $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$ ***Ans.*** $-1 \leq x < 1$

9) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ ***Ans.*** $-8 < x < 2$

10) $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$ ***Ans.*** $-3 < x < 3$

11) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$ ***Ans.*** $-1 < x < 1$

12) $\sum\limits_{n=1}^{\infty} n^n x^n$ *Ans.* $x = 0$

13) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{2^n n}$ *Ans.* $-4 < x \leq 0$

Find the interval of convergence and the sum within this interval for the following series

1) $\sum\limits_{n=0}^{\infty} \frac{(x-1)^{2n}}{4n}$ *Ans.* $-1 < x < 3, \frac{4}{3+2x-x^2}$

2) $\sum\limits_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1\right)^n$ *Ans.* $0 < x < 16, \frac{2}{4-\sqrt{x}}$

3) $\sum\limits_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ *Ans.* $-\sqrt{2} < x < \sqrt{2}, \frac{3}{2-x^2}$

Taylor Series & Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the ***Taylor Series*** generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

The ***Maclaurin Series*** generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

which is a Taylor series generated by f at $x = 0$.

Example

Find the Taylor series and the interval of convergence for the following functions (a) $f(x) = 1/x$ at $x = 2$, (b) $f(x) = \ln(x)$ at $x = 1$.

Solution

(a) We need to find $f(2), f'(2), f''(2), \ldots.$ Taking derivatives we get

$$f(x) = x^{-1}, \qquad f(2) = \frac{1}{2},$$

$$f'(x) = -x^{-2}, \qquad f'(2) = -\frac{1}{2^2},$$

$$f''(x) = 2!x^{-3}, \qquad \frac{f''(2)}{2!} = \frac{1}{2^3},$$

$$f'''(x) = -3!x^{-4}, \qquad \frac{f'''(2)}{3!} = -\frac{1}{2^4},$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)}, \qquad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \ldots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \ldots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \ldots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \ldots$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ or $0 < x < 4$.

(b) $f(x) = \ln(x)$, $\qquad f(1) = 0$,

$f'(x) = \frac{1}{x}$, $\qquad f'(1) = 1$,

$f''(x) = -\frac{1}{x^2}$, $\qquad f''(1) = -1$,

$f'''(x) = \frac{2}{x^3}$, $\qquad f'''(1) = 2$,

$f^{(4)}(x) = \frac{-6}{x^4}$, $\qquad f^{(4)}(1) = -6$,

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}, \qquad\qquad f^{(n)}(1) = (-1)^{n+1}(n-1)!,$$

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \ldots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \ldots$$

$$\ln(x) = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \ldots + \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n + \ldots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots + \frac{(-1)^{n+1}}{n}(x-1)^n + \ldots$$

$$\ln(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$$

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\times\frac{n}{(x-1)^n}\right| = \lim_{n\to\infty}\left|(x-1)\frac{n}{n+1}\right| = |x-1| < 1$$

So, $-1 < x-1 < 1$ or $0 < x < 2$. For $x = 0$, we get $-1-1/2-1/3-1/4-\ldots$ which diverges because it is the negative of the harmonic series. While, for $x = 2$, we get $1-1/2+1/3-1/4+\ldots$ which converges because it is an alternating series that satisfies the three conditions of convergence of alternating series. So, the region of convergence will be $0 < x \le 2$.

Example

Find the Maclaurin series generated by the following functions

(a) e^x, (b) $\cosh(x)$, (c) $\sinh(x)$

Solution

(a) $f(x) = e^x \quad \Rightarrow \quad f(0) = 1,$

$$f'(x) = e^x \quad \Rightarrow \quad f'(0) = 1,$$

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1,$$

$$f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = 1,$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find the interval of convergence, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1.$$

So, the series is convergent for all values of x.

(b) $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right]$$

$$= \frac{1}{2}\left[(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) + (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots)\right]$$

$$= \frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots\right]$$

$$= \frac{1}{2} \times 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

To find the interval of convergence, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!} \times \frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(n+1)}\right| = 0 < 1.$$

So, the series is convergent for all values of x.

(c) $\sinh(x) = f'(\cosh(x))$

$$= f'\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!}\right)$$

$$= 0 + \frac{2x}{2!} + \frac{4x^3}{4!} + \frac{6x^5}{6!} + \dots + \frac{2nx^{2n-1}}{(2n)!}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n-1}}{(2n-1)!}$$

$$\sinh(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}, \quad \text{or} \quad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

To find the interval of convergence, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{(2n+1)!} \times \frac{(2n-1)!}{x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n)}\right| = 0 < 1.$$

So, the series is convergent for all values of x.

Exercises on Taylor Series

Find Maclaurin series for the following functions

1) e^{-x} *Ans.* $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

2) $\frac{1}{1+x}$ *Ans.* $\sum_{n=0}^{\infty} (-1)^n x^n$

3) $\sin(3x)$ *Ans.* $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$

4) $7\cos(-x)$ *Ans.* $7\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

5) $x^4 - 2x^3 - 5x + 4$ *Ans.* $x^4 - 2x^3 - 5x + 4$

Find the Taylor series generated by f at $x = a$ for the following functions

1) $f(x) = x^3 - 2x + 4$, $a = 2$ *Ans.* $8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$

2) $f(x) = x^4 + x^2 + 1$, $a = -2$ *Ans.* $21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

3) $f(x) = \frac{1}{x^2}$, $a = 1$ *Ans.* $\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

4) $f(x) = e^x$, $a = 2$ *Ans.* $\sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

Find Maclaurin series for the following functions

1) e^{-5x} *Ans.* $\displaystyle\sum_{n=0}^{\infty}\frac{(-5x)^n}{n!}$

2) $5\sin(-x)$ *Ans.* $\displaystyle\sum_{n=0}^{\infty}\frac{5(-1)^n(-x)^{2n+1}}{(2n+1)!}$

3) $\cos\sqrt{x+1}$ *Ans.* $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x+1)^n}{(2n)!}$

4) xe^x *Ans.* $\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$

5) $\dfrac{x^2}{2}-1+\cos x$ *Ans.* $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

6) $x\cos(\pi\cdot x)$ *Ans.* $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}x^{2n+1}}{(2n)!}$

7) $\cos^2(x)$ *Ans.* $\displaystyle 1+\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{2\cdot(2n)!}$

8) $\dfrac{x^2}{1-2x}$ *Ans.* $\displaystyle x^2\sum_{n=0}^{\infty}(2x)^n$

9) $\dfrac{1}{(1-x)^2}$ *Ans.* $\displaystyle\sum_{n=1}^{\infty}nx^{n-1}$