Experiment No.12

Impedance Element Charstristics

Object

To perform be familiar with AC circuit characteristics and their laws.

Theory

AC circuit elements consist of resistors (R), inductors (L) and capacitors(C), The symbol for an AC voltage source is

An example of an AC source is

$$V(t) = V_0 \sin \omega t$$

where the maximum value $V_\circ$ is called the amplitude. The voltage varies between $V_\circ$ and $-V_\circ$ since a sine function varies between +1 and −1. A graph of voltage as a function of time is shown in Figure 1.

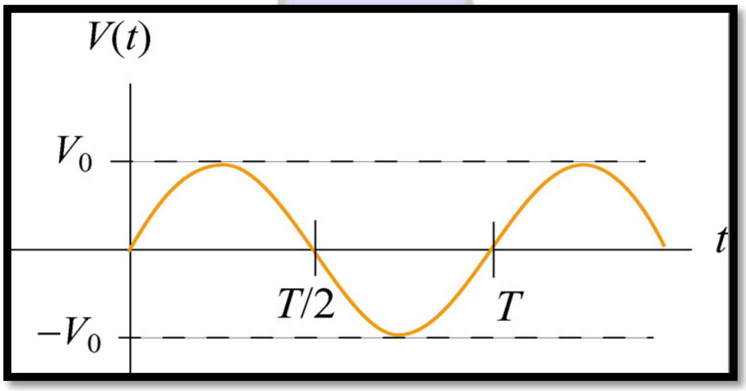

Figure 1. Sinusoidal voltage source

The sine function is periodic in time. This means that the value of the voltage at time t will be exactly the same at a later time $t' = t + T$ where T is the *period*. The *frequency*, *f*, defined as *f*=1/*T*, has the unit of inverse seconds (s^{-1}), or hertz (Hz). The angular frequency is defined to be $\omega = 2\pi f$

When a voltage source is connected to an RLC circuit, energy is provided to compensate the energy dissipation in the resistor, and the oscillation will no longer damp out. The oscillations of charge, current and potential difference are called driven or forced oscillations.

After an initial "transient time," an AC current will flow in the circuit as a response to the driving voltage source. The current, written as

$$I(t) = I_{\circ} + \sin(\omega t - \emptyset)$$

will oscillate with the same frequency as the voltage source, with an amplitude $I_{\circ}$ and phase $\emptyset$ that depends on the driving frequency.

- **Simple AC circuits**

Before examining the driven RLC circuit, let's first consider the simple cases where only one circuit element (a resistor, an inductor or a capacitor) is connected to a sinusoidal voltage source.

- **Purely Resistive load**

Consider a purely resistive circuit with a resistor connected to an AC generator, as shown in Figure 2 (As we shall see, a purely resistive circuit corresponds to infinite capacitance C=∞ zero inductance L=0)

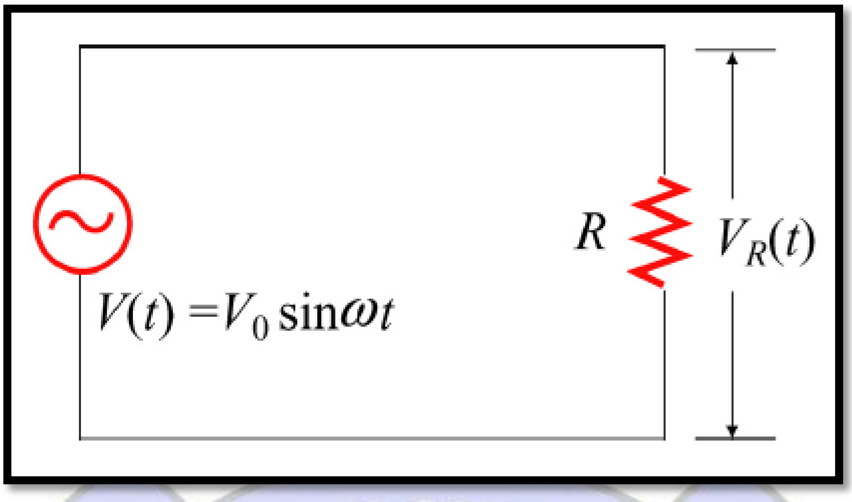

Figure 2. A purely resistive circuit

Applying Kirchhoff's loop rule yields

$$V(t) - V_R(t) = V(t) - I_R(t)R = 0 \quad \text{.........................(1)}$$

Where $V_R(t) = I_R(t).R$ is the instantaneous voltage drop across the resistor, The instantaneous current in the resistor is given by:

$$I_R(t) = \frac{V_R(t)}{R} = \frac{V_{R0}\sin\omega t}{R} = I_{R0}\sin\omega t \quad \text{....................(2)}$$

where $V_{R0} = V_0$, and $I_{R0} = V_{R0}/R$ is the maximum current. Comparing Eq. (2) with Eq. (1), we find $\emptyset = 0$, which means that $V_R(t)$ and $I_R(t)$ are in phase with each other, meaning that they reach their maximum or minimum values at the same time. The time dependence of the current and the voltage across the resistor is depicted in Figure 3(a).

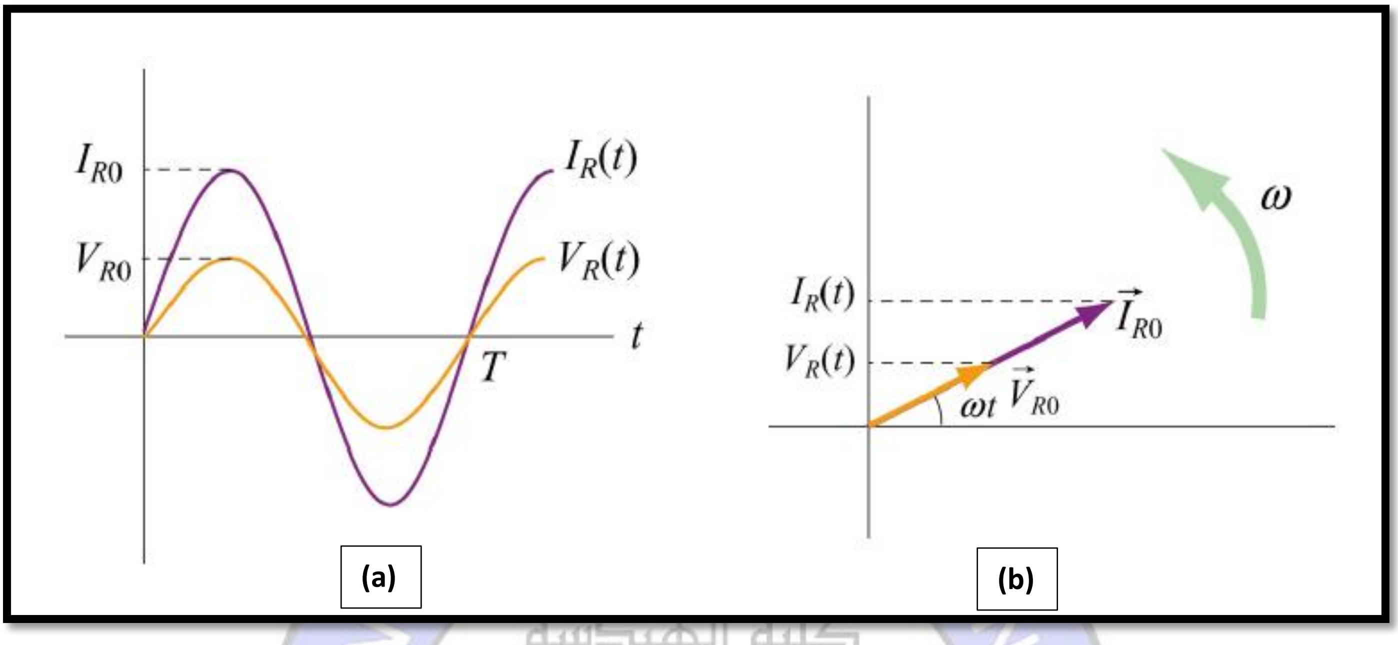

Figure 3. (a) Time dependence of $V_R(t)$ and $I_R(t)$ across the resistor. (b) Phasor diagram for the resistive circuit.

The behavior of $V_R(t)$ and $I_R(t)$ can also be represented with a phasor diagram, as shown in Figure 3(b). A phasor is a rotating vector having the following properties:

(i) length: the length corresponds to the amplitude.

(ii) angular speed: the vector rotates counterclockwise with an angular speed ω.

(iii) projection: the projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time t.

We shall denote a phasor with an arrow above it. The phasor $\overrightarrow{V_{R0}}$ has a constant magnitude of V_{R0}. Its projection along the vertical direction is $V_{R0}\sin\omega t$, which is equal to $V_R(t)$, the voltage drops across the resistor at time t. A similar interpretation applies to $\overrightarrow{I_{R0}}$ for the current passing through the resistor. From the phasor diagram, we readily see that both the current and the voltage are in phase with each other.

The average value of current over one period can be obtained as:

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)dt = \frac{1}{T}\int_0^T I_{R0}\sin\omega t\ dt = \frac{I_{R0}}{T}\int_0^T \sin\frac{2\pi t}{T}\ dt = 0$$

....(3)

This average vanishes because

$$\langle \sin\omega t\rangle = \frac{1}{T}\int_0^T \sin\omega t\ dt = 0$$

.............................(4)

Similarly, one may find the following relations useful when averaging over one period:

$$\langle \cos\omega t\rangle = \frac{1}{T}\int_0^T \cos\omega t\ dt = 0$$

$$\langle \sin\omega t\cos\omega t\rangle = \frac{1}{T}\int_0^T \sin\omega t\cos\omega t\ dt = 0$$

$$\langle \sin^2\omega t\rangle = \frac{1}{T}\int_0^T \sin^2\omega t\ dt = \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)\ dt = \frac{1}{2}$$

$$\langle \cos^2\omega t\rangle = \frac{1}{T}\int_0^T \cos^2\omega t\ dt = \frac{1}{T}\int_0^T \cos^2\left(\frac{2\pi t}{T}\right)\ dt = \frac{1}{2}$$

......(5)

From the above, we see that the average of the square of the current is non-vanishing:

$$\left\langle I_R^2(t)\right\rangle = \frac{1}{T}\int_0^T I_R^2(t)dt = \frac{1}{T}\int_0^T I_{R0}^2 \sin^2 \omega t\ dt = I_{R0}^2 \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2} I_{R0}^2$$

..(6)

It is convenient to define the root-mean-square (rms) current as

$$I_{\text{rms}} = \sqrt{\left\langle I_R^2(t)\right\rangle} = \frac{I_{R0}}{\sqrt{2}}$$

....(7)

In a similar manner, the rms voltage can be defined as

$$V_{\text{rms}} = \sqrt{\left\langle V_R^2(t)\right\rangle} = \frac{V_{R0}}{\sqrt{2}}$$

....(8)

The rms voltage supplied to the domestic wall outlets in the Iraq is $V_{rms} = 120V$ at a frequency $f = 50Hz$.

The power dissipated in the resistor is

$$P_R(t) = I_R(t)V_R(t) = I_R^2(t)R$$

.................................(9)

from which the average over one period is obtained as:

$$\langle P_R(t)\rangle = \langle I_R^2(t)R\rangle = \frac{1}{2} I_{R0}^2 R = I_{\mathrm{rms}}^2 R = I_{\mathrm{rms}} V_{\mathrm{rms}} = \frac{V_{\mathrm{rms}}^2}{R} \quad \text{........(10)}$$

- **Purely Inductive Load**

Consider now a purely inductive circuit with an inductor connected to an AC generator, as shown in Figure 4.

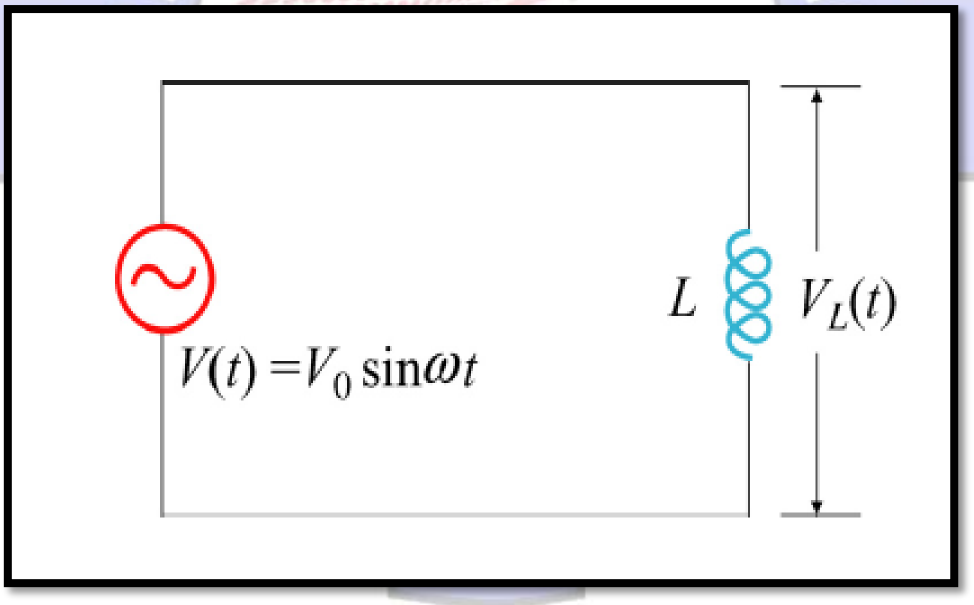

Figure 4. A purely inductive circuit

As we shall see below, a purely inductive circuit corresponds to infinite capacitance $C = \infty$ and zero resistance $R = 0$. Applying the modified Kirchhoff's rule for inductors, the circuit equation reads

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0$$ ....(11)

which implies

$$\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{L0}}{L}\sin\omega t$$ ....(12)

where $V_{L0} = V_0$. Integrating over the above equation, we find

$$I_L(t) = \int dI_L = \frac{V_{L0}}{L}\int \sin\omega t\ dt = -\left(\frac{V_{L0}}{\omega L}\right)\cos\omega t = \left(\frac{V_{L0}}{\omega L}\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$ ......(13)

where we have used the trigonometric identity

$$-\cos\omega t = \sin\left(\omega t - \frac{\pi}{2}\right)$$ ....(14)

for rewriting the last expression. Comparing Eq. (14) with Eq. (2), we see that the amplitude of the current through the inductor is

$$I_{L0} = \frac{V_{L0}}{\omega L} = \frac{V_{L0}}{X_L}$$ ....(15)

Where

$$X_L = \omega L$$ ....(16)

is called the inductive reactance. It has SI units of ohms (Ω), just like resistance. However, unlike resistance, X_L depends linearly on the angular frequency ω. Thus, the resistance to current flow increases with frequency. This is due to the fact that at higher

frequencies the current changes more rapidly than it do at lower frequencies. On the other hand, the inductive reactance vanishes as ω approaches zero.

By comparing Eq. (14) to Eq. (2), we also find the phase constant to be

$$\phi = +\frac{\pi}{2}$$ ....(17)

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 5 below.

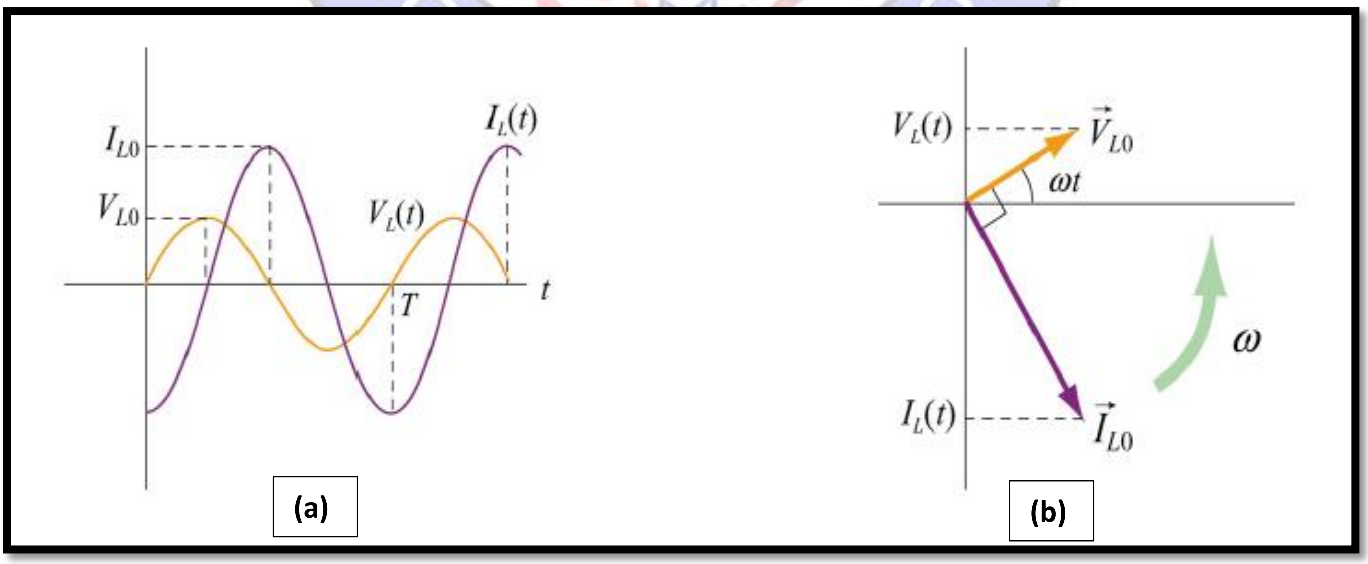

Figure 5. (a) Time dependence of $I_L(t)$ and $V_L(t)$ across the inductor. (b) Phasor diagram for the inductive circuit.

As can be seen from the figures, the current $I_L(t)$ is out of phase with $V_L(t)$ by $\emptyset = \pi/2$ it reaches its maximum value after $V_L(t)$ does by one quarter of a cycle. Thus, we say that

The current lags voltage by π / 2 in a purely inductive circuit

- **Purely Capacitive Load**

In the purely capacitive case, both resistance R and inductance L are zero. The circuit diagram is shown in Figure 6.

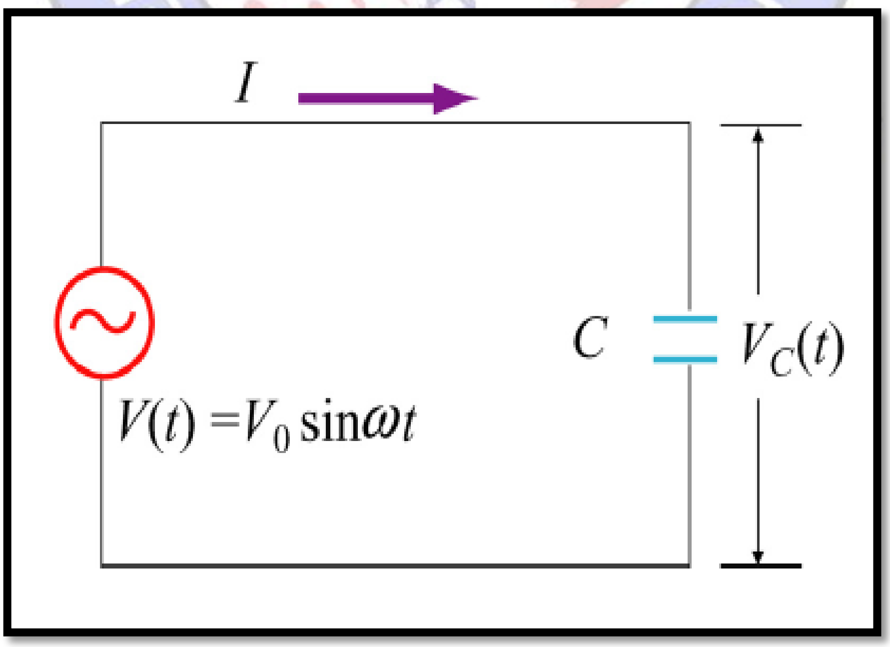

Figure 6. A purely capacitive circuit

Again, Kirchhoff's voltage rule implies

$$V(t) - V_C(t) = V(t) - \frac{Q(t)}{C} = 0$$

..............................(18)

which yields

$$Q(t) = CV(t) = CV_C(t) = CV_{C0} \sin \omega t$$

.....................(19)

where $V_{C0} = V_0$.. On the other hand, the current is

$$I_C(t) = +\frac{dQ}{dt} = \omega C V_{C0} \cos \omega t = \omega C V_{C0} \sin\left(\omega t + \frac{\pi}{2}\right)$$

......(20)

where we have used the trigonometric identity

$$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$$

..................................(21)

The above equation indicates that the maximum value of the current is

$$I_{C0} = \omega C V_{C0} = \frac{V_{C0}}{X_C}$$

...................................(22)

Where

$$X_C = \frac{1}{\omega C}$$ ....(23)

is called the capacitance reactance. It also has SI units of ohms and represents the effective resistance for a purely capacitive circuit. Note that X_C is inversely proportional to both C and ω, and diverges as ω approaches zero.

By comparing Eq. (21) to Eq. (2), the phase constant is given by

$$\phi = -\frac{\pi}{2}$$ ....(24)

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 7 below.

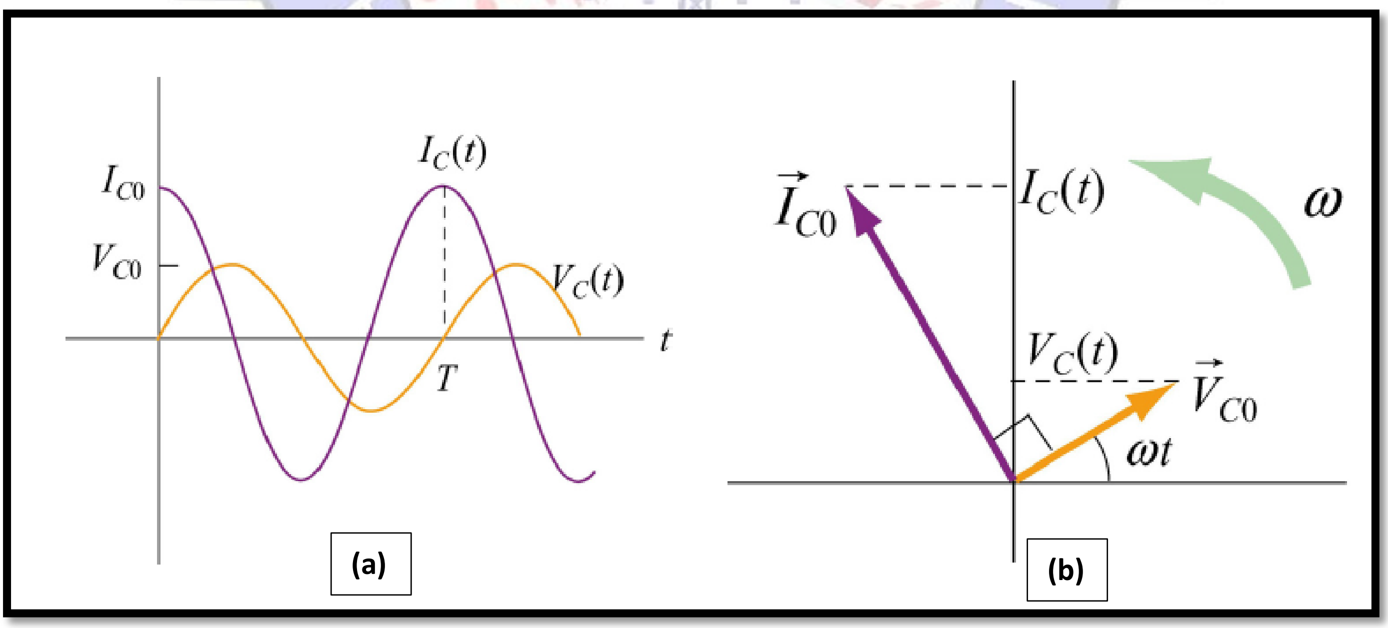

Figure 7. (a) Time dependence of $I_C(t)$ and $V_C(t)$ across the capacitor. (b) Phasor diagram for the capacitive circuit.

Notice that at t=0, the voltage across the capacitor is zero while the current in the circuit is at a maximum. In fact, $I_C(t)$ reaches its maximum before $V_C(t)$ by one quarter of a cycle ($\emptyset = \pi/2$) Thus, we say that

The current leads the voltage by π/2 in a capacitive circuit

Procedure

1. Connect the circuit as shown in Figure (8)
2. Measure I_T, V_1, V_2, and V_T, record your results in the first row of Table (1).
3. Connect the circuit as shown in Figure (9)
4. Measure I_T, V_1, V_2, and V_T, record your results in the second row of Table (2).
5. Connect the circuit as shown in Figure (10)
6. Measure V_T, I_1, I_2 and I_T, record your results in Table (2).
7. Connect the circuit as shown in Figure (11)
8. Measure V_T, I_1, I_2 and I_T, record your results in Table (2).

Figure No.	I_T	V_1	V_2	V_T
8				
9				

Table (1)

Figure No.	V_T	V_1	V_2	I_T
10				
11				

Table (2)

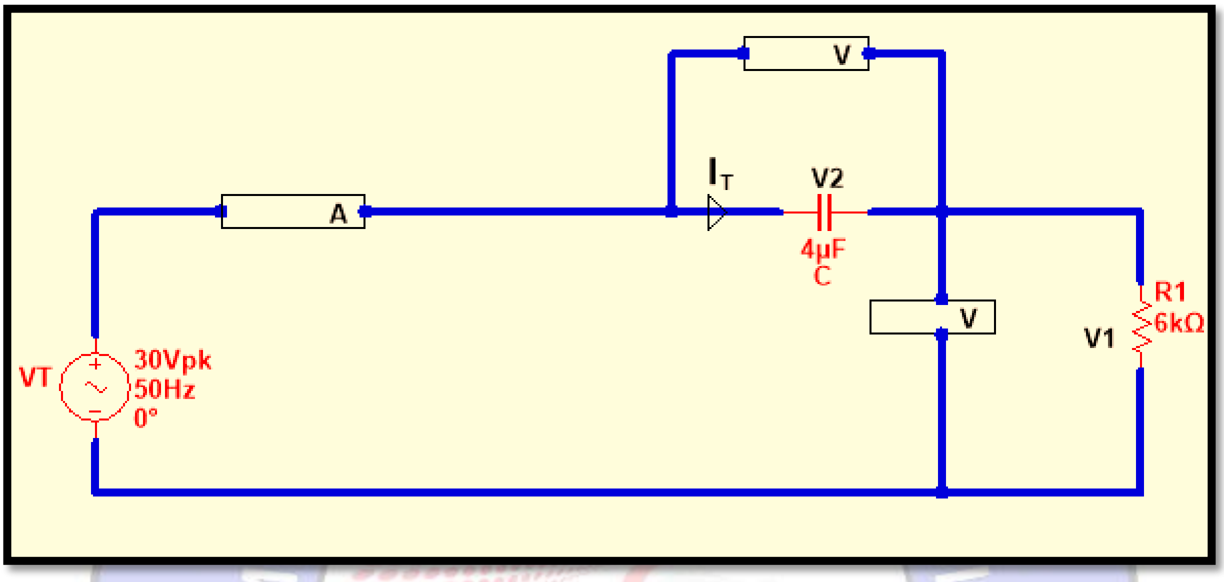

Figure 8

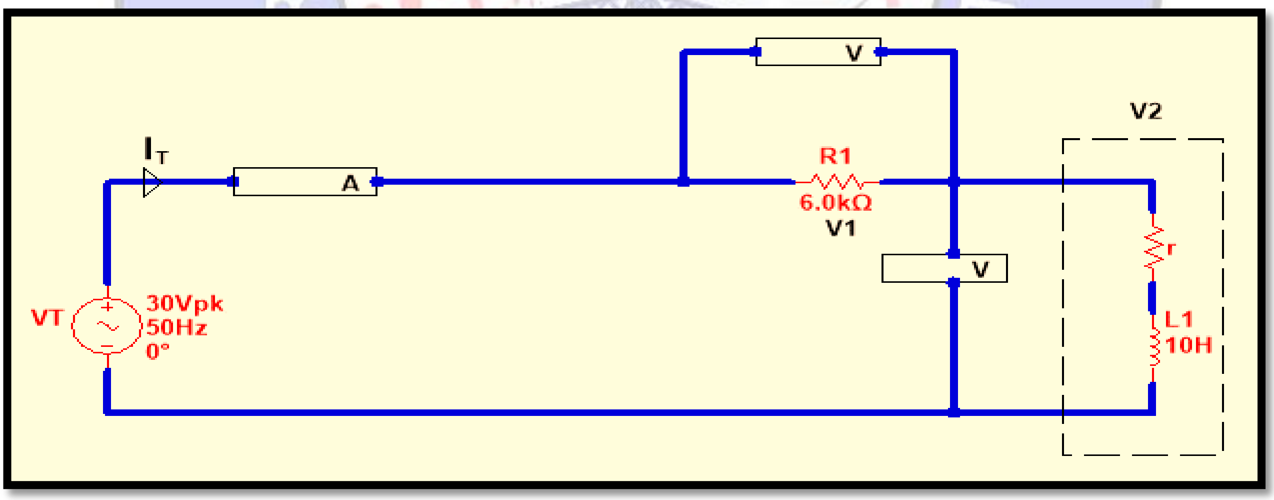

Figure 9

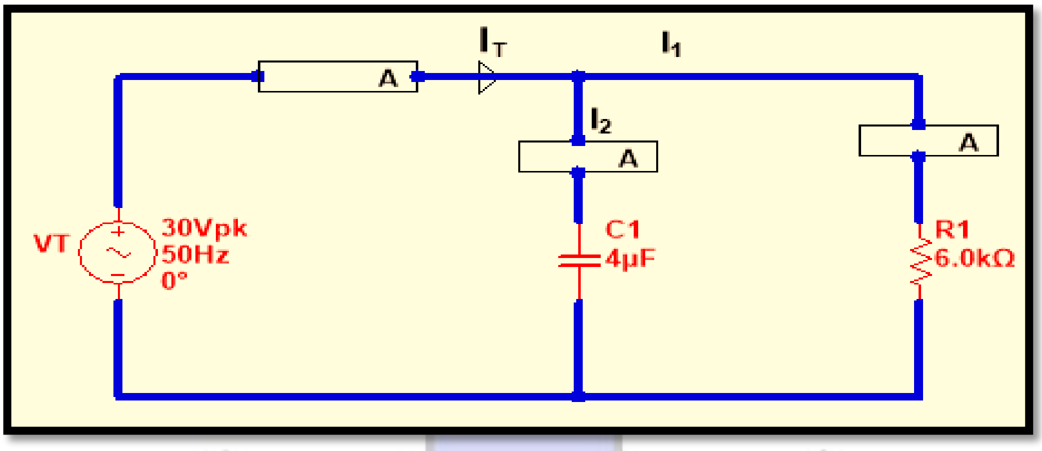

Figure 10

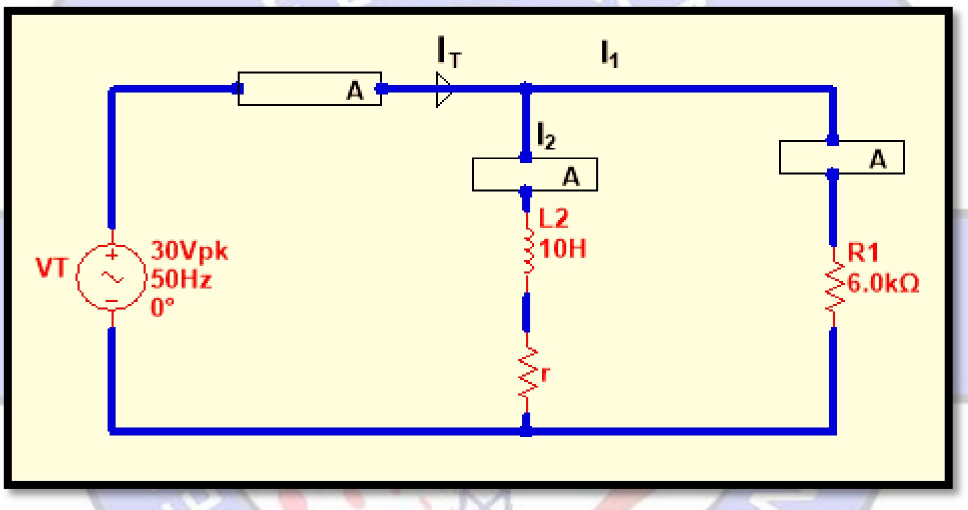

Figure 11

Discussion

1. For the Figure (8), find theoretically the value of Vc and V_R in polar form and compare them with the practical results.
2. For the Figure (10), find theoretically the value of Ic and I_R in polar form and compare them with the practical results.
3. Draw the voltage phasor diagram of the capacitive in Figure (8) and the inductive in Figure (9). Calculate the phasor angle θ.
4. Draw the current phasor diagram of the capacitive in Figure (10) and the inductive in Figure (11). Calculate the phasor angle θ.